# Computing the volume of the Combinahedron


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## 1.The combinahedron

The combinahedron $C(r,n)$ is the graph such that:

- the nodes are rearrangements of the word $(\underbrace{1,\dots,1}_{r},2,\dots,n)$
- two nodes are connected if and only they differ by two adjacent entries.

It is a generalization of the well-known permutahedron. We can embed the combinahedron [1] into a hyperplane of $\mathbb{R}^n$ such that every edge has length $\sqrt{2}$:

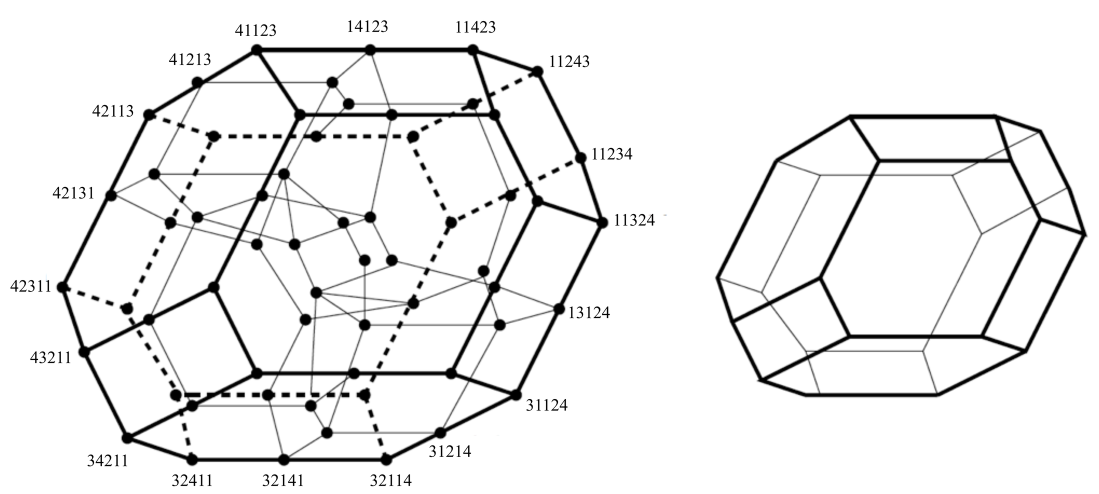


Figure 1:The embedding of $C(2,4)$, from [1]

The polytope $P_{C(r,n)}$ obtained as the convex hull of the embedding is a zonotope: it is the image of the $\binom{n}{2}$-dimensional hypercube into $\mathbb{R}^n$ by a linear map induced by

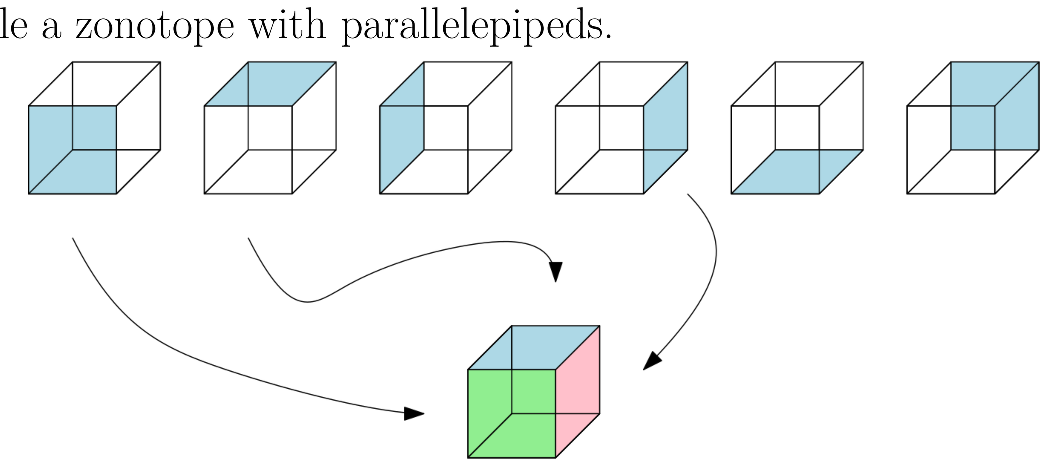
$$u: f_{\{i,j\}} \mapsto \begin{cases} e_i - e_j & \text{if } 1 < i < j \leq n \\ r(e_1 - e_j) & \text{if } i = 1 \text{ and } 1 < j \leq n \end{cases}$$

with $(f_{i,j})_{\{i,j\}\in\binom{n}{2}}$ denoting the canonical base of $\mathbb{R}^{\binom{n}{2}}$ and $(e_i)_{i\in[n]}$ the canonical base of $\mathbb{R}^n$.

**Theorem (B+R.A. 2024):** *The polytope $P_{C(r,n)}$ tiles the hyperplane it lives in.*

## 2.Decomposition into parallelepipeds

We can select some faces of the $\binom{n}{2}$-dimensional hypercube to tile a zonotope with parallelepipeds.



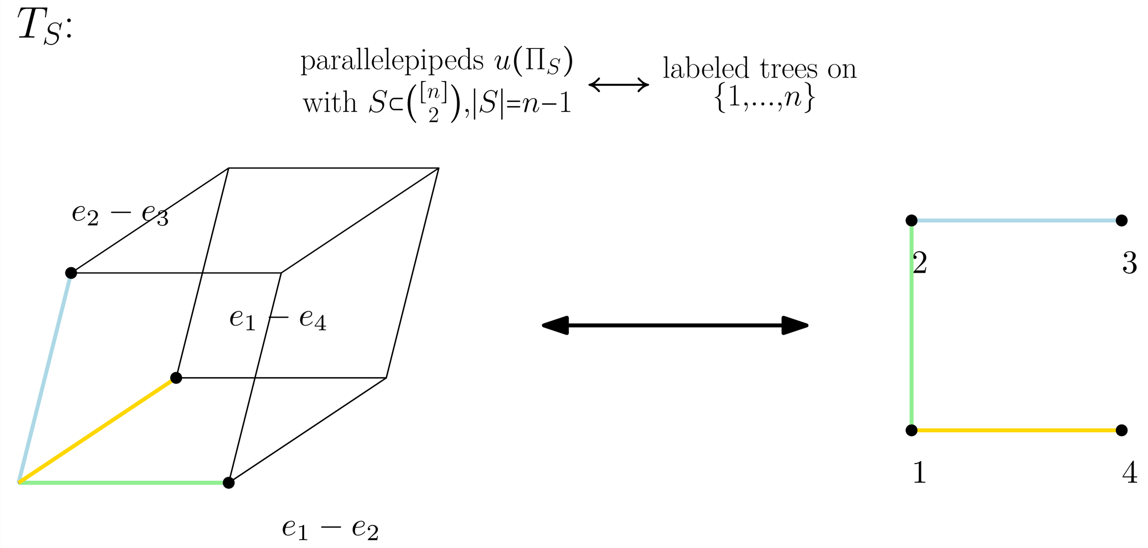
$$P_{C(r,n)} = \bigsqcup_{\substack{S\subset\binom{[n]}{2},\ |S|=n-1 \\ u(S) \text{ independant set}}} u(\Pi_S)$$

with $\Pi_S$ denoting the $(n-1)$-dimensional cube inside $\mathbb{R}^{\binom{n}{2}}$ along the directions given by $S$. The volume is [2]:

$$\mathsf{Vol}(P_{C(r,n)}) = \sum_{\substack{S\subset\binom{[n]}{2},\ |S|=n-1 \\ u(S) \text{ independant set}}} \mathsf{Vol}(u(\Pi_S))$$

## 3.Correspondance with labeled trees

To an independant set $S\subset\binom{[n]}{2}$, we can associate a labeled tree $T_S$:




We have $\mathsf{Vol}(u(\Pi_S)) = r^{\deg_{T_S}(1)}$.

## 4.Prüfer sequences

Prüfer sequences [3] define a correspondance:

set of labeled trees on $\{1,\dots,n\}$ $\longleftrightarrow$ set $\{1,\dots,n\}^{n-2}$,

## 5.Computing the volume

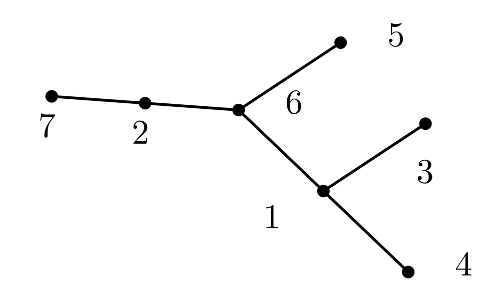
We want to compute $\mathsf{Vol}(P_{C(r,n)}) = \sum_{T \text{ labeled tree}} r^{\deg_T(1)}$.

We can relate the degree of vertex 1 in the associated tree $T$ to the number of occurrences of the symbol 1 in the Prüfer sequence $\omega$:

$$|\omega|_1 + 1 = \deg_T(1)$$

If we consider the monomial $X_{\omega_1}\times\dots\times X_{\omega_{n-2}}$ associated to the word $\omega = \omega_1\dots\omega_{n-2}\in\{1,\dots,n\}^{n-2}$, replacing $X_1$ by $r$ and $X_i$ by 1 for $i>1$ yields

$$r^{\deg_{T_\omega}(1)} = rX_{\omega_1}\times\dots\times X_{\omega_{n-2}}(r,1,\dots,1)$$




$$\begin{array}{lccccc} \omega_T = & 1 & 1 & 6 & 6 & 2 \\ r\ \cdot & X_1 & X_1 & X_6 & X_6 & X_2 \\ r\ \cdot & r & r & 1 & 1 & 1 \end{array}$$

Thus,

$$\sum_T r^{\deg_T(1)} = r\sum_{\omega\in\{1,\dots,n\}^{n-2}} X_{\omega_1}\times\dots\times X_{\omega_{n-2}}(r,1,\dots,1)$$
$$= r(X_1+\dots+X_n)^{n-2}(r,1,\dots,1)$$

## 6.Volume of the combinahedron

**Theorem (B+R.A. 2024):** *The volume of the combinahedron is* $\mathsf{Vol}(P_{C(r,n)}) = r(r+n-1)^{n-2}$.

## 7.The volume of another zonotope

We can compute the volume of the zonotope $\widehat{P}(r_1,\dots,r_n)$ defined in [4] as the Minkozski sum $\sum_{i<j} r_j[e_i,e_j]$: it is the image of the $\binom{n}{2}$-dimensional hypercube by the linear map induced by $u: f_{\{i,j\}} \mapsto r_j(e_i - e_j)$ if i<j.

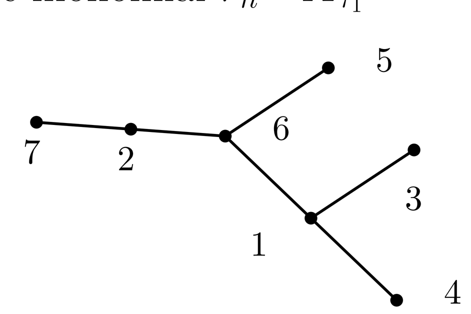
In a similar way, we get $Vol(\widehat{P}(r_1,\dots,r_n)) = \sum_{T \text{ labeled tree}} r_1^{\widehat{\deg}_T(1)}\dots r_n^{\widehat{\deg}_T(n)}$ where $\widehat{\deg}_T(i) = |\mathsf{Vois}_T(i)\cap[i]|$

We introduce another bijection similar to Prüfer sequences, called the dominant tree code:

set of labeled trees on $\{1,\dots,n\}$ $\longleftrightarrow$ set $\{1,\dots,n\}^{n-2}$

The word $\tau_T$ associated to the tree $T$ satisfies a new property:

Evaluating the monomial $r_n\times X^{(1)}_{\tau_1}\times\dots\times X^{(n-2)}_{\tau_{n-2}}$ associated to $\tau_T$ on $(X_i^{(j)} \leftarrow r_{\max(i,j+1)})_{i,j}$ yields $r_1^{\widehat{\deg}_T(1)}\dots r_n^{\widehat{\deg}_T(n)}$.




$$\begin{array}{lccccc} \omega_T = & 6 & 1 & 1 & 6 & 2 \\ r_7\ \cdot & X_6^{(1)} & X_1^{(2)} & X_1^{(3)} & X_6^{(4)} & X_2^{(5)} \\ r_7\ \cdot & r_6 & r_3 & r_4 & r_6 & r_6 \end{array}$$

We get

$$\sum_{T \text{ labeled tree}} r_1^{\widehat{\deg}_T(1)}\dots r_n^{\widehat{\deg}_T(n)} = r_n\sum_{\tau\in\{1,\dots,n\}^{n-2}} X^{(1)}_{\tau_1}\times\dots\times X^{(n-2)}_{\tau_{n-2}}(X_i^{(j)}\leftarrow r_{\max(i,j+1)})$$
$$= r_n(X_1^{(1)}+\dots+X_n^{(1)})\times\dots\times(X_1^{(n-2)}+\dots X_n^{(n-2)})(X_i^{(j)}\leftarrow r_{\max(i,j+1)})$$

**Theorem (B+R.A. 2024):** $\mathsf{Vol}(\widehat{P}(r_1,\dots,r_n)) = r_n(2r_2+r_3+\dots+r_n)(3r_3+r_4+\dots+r_n)\dots((n-1)r_{n-1}+r_n)$

## 8.New tree coding: symbols-edges correspondance

Both encodings yield a correspondance between the characters of the sequence and the edges of the tree.

$$\phi_P:\ \text{letters of } \omega_T\cdot n \longleftrightarrow \text{edges of } T$$
$$\phi_D:\ \text{letters of } \tau_T\cdot n \longleftrightarrow \text{edges of } T$$

such that:

- The oriented edge $\phi_P(i)$ in $\widetilde{T_P}$ originates from node $(\omega_T\cdot n)_i$
- The oriented edge $\phi_D(i)$ in $\widetilde{T_D}$ originates from the node labeled $\max(i+1,(\omega_T\cdot n)_i)$

with two different orientations $\widetilde{T_P},\widetilde{T_D}$ on the tree $T$.

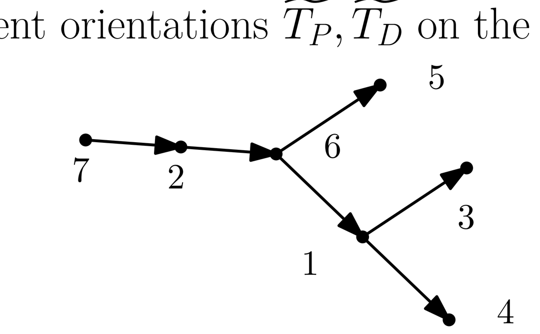



Figure 2:Orientation $\widetilde{T_P}$ of $T$ as a rooted tree with root 7.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $(\omega_T\cdot n)_i$ | 1 | 1 | 6 | 6 | 2 | 7 |
| $\phi_P(i)$ | $(1,3)$ | $(1,4)$ | $(6,1)$ | $(6,5)$ | $(2,6)$ | $(7,2)$ |

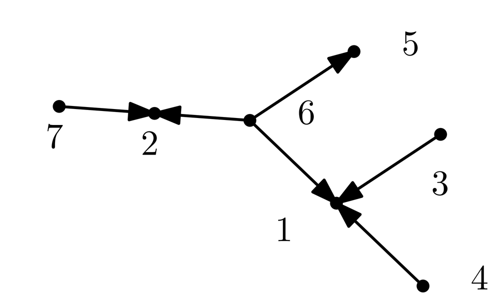



Figure 3:Orientation $\widetilde{T_D}$ of $T$ such that $i\to j$ if $i>j$

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $(\tau_T\cdot n)_i$ | 6 | 1 | 1 | 6 | 2 | 7 |
| $\phi_D(i)$ | $(6,1)$ | $(3,1)$ | $(4,1)$ | $(6,5)$ | $(6,2)$ | $(7,2)$ |

These properties are the key to getting the previous results.

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