Using Delaunay triangulation to enumerate saddle connections

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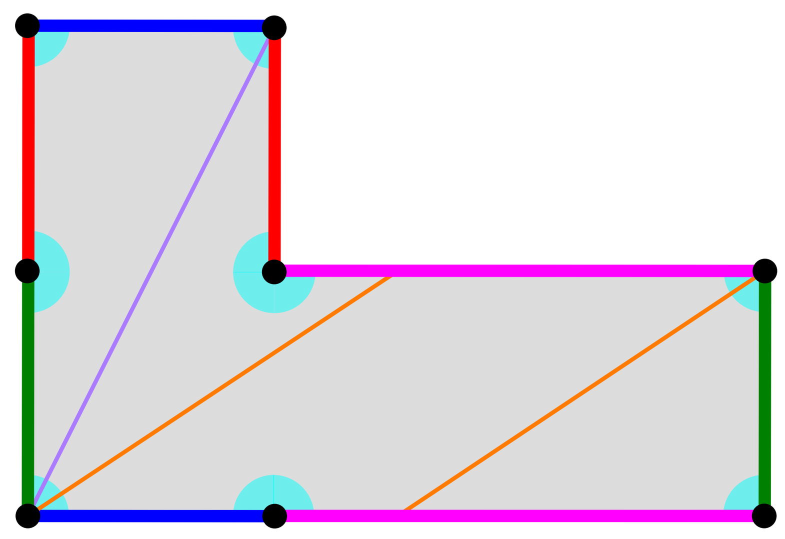
Translation surfaces



L-shape : Genus 2 translation surface with 6π singularity

Translation surfaces : polygons glued by translations.
Singularity : point not locally isometric to the Euclidian plane.
Saddle connection : line segment between singularities.
Goal : Enumerate all saddle connections.
Motivations : systole computing, length spectra, dynamics...

Results

Infinitely many saddle connections but finitely many of them of length at most R.

Theorem : Masur 1990

Let M a translation surface and $R > 0$. Let $SC(M, R)$ the set of all saddle connections of length at most R.

$$c_1 R^2 - b \leq |SC(M, R)| \leq c_2 R^2 + b$$

where $c_1 > 0$, $c_2 > 0$ and b are constant depending only of M.

Two flavors of enumeration of saddle connections :

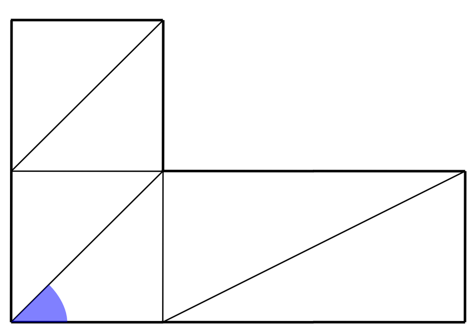
- Semi-algorithm which enumerate all saddle connections.
- Algorithm which enumerate all saddle connections with length at most R.

Theorem : V. Delecroix and O.F. 2024

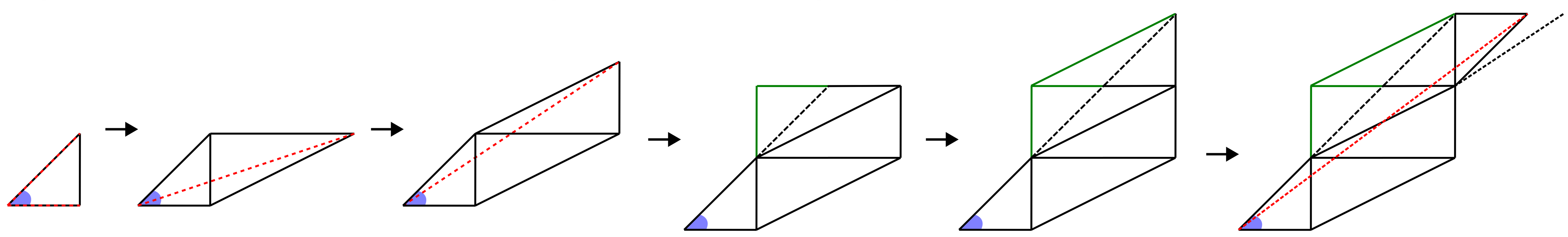
- Enumeration of saddle connections with length at most R in $\Theta(R^3)$. (Unfolding algorithm)
- Enumeration of saddle connections with $O(1)$ delay. (iso-Delaunay enumeration)
- If the surface is a Veech surface, enumeration of saddle connections with length at most R in $\Theta(R^2)$. (iso-Delaunay enumeration)

Unfolding algorithm

Idea of the algorithm : Given a triangulation of a surface, explore all the triangles visible from a singularity and enumerate the discovered saddle connections.



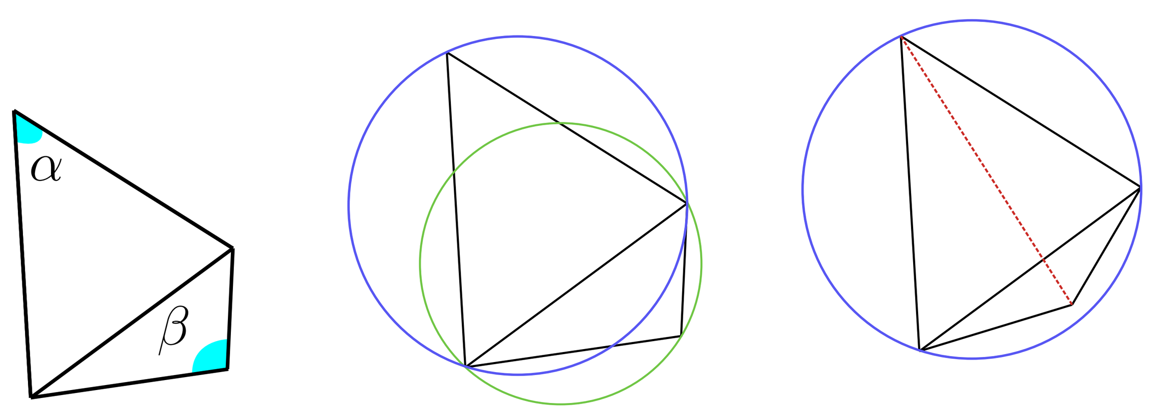
Triangulation of the L-shape



Different steps of the naive algorithm on the L-shape

Delaunay triangulation

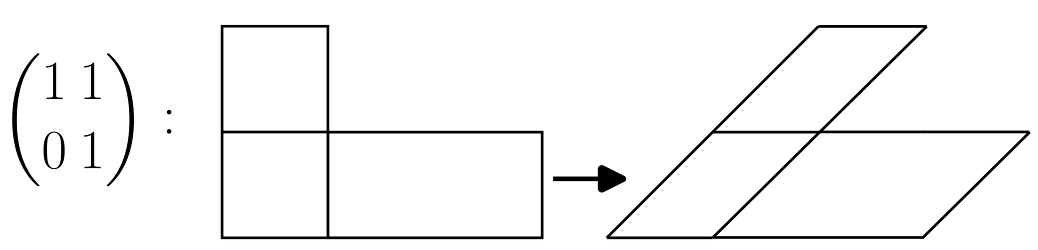
A triangulation is a Delaunay triangulation if it satisfy the following local property :



$\alpha + \beta \leq \pi$

$SL_2(\mathbb{R})$-action and the hyperbolic plane

$SL_2(\mathbb{R})$ acts on the space of translation surface by applying a matrix on each of the polygons forming a translation surface.



The $SL_2(\mathbb{R})$-orbit of a translation surface M corresponds to $\mathbb{H}^2 = \{u + iv \in \mathbb{C} | v > 0\} \simeq SO_2(\mathbb{R})\backslash SL_2(\mathbb{R})$ the hyperbolic plane by

$$u + iv \in \mathbb{H}^2 \mapsto \frac{1}{\sqrt{v}} \begin{pmatrix} 1 & u \\ 0 & v \end{pmatrix} \cdot M$$

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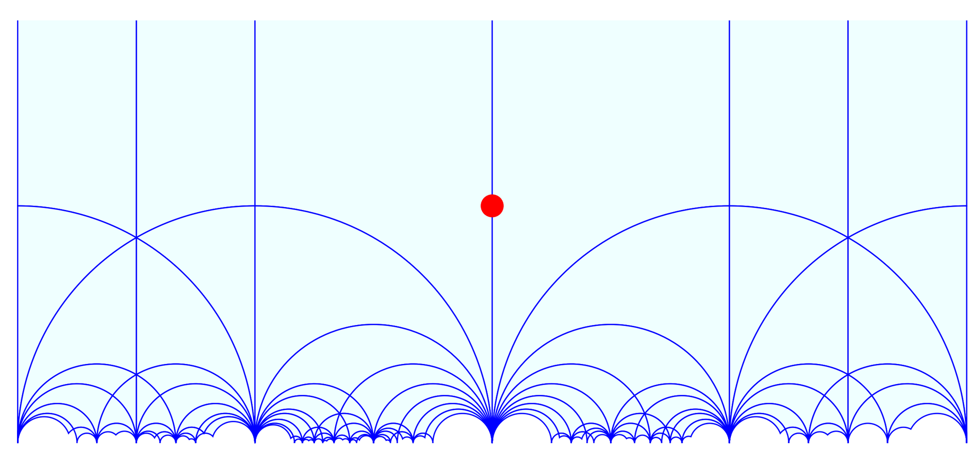
Iso-Delaunay cell

Let τ be a triangulation of M. The iso-Delaunay cell of τ is the set of point of $\mathbb{H}^2$ admitting τ as a Delaunay triangulation.



Some iso-Delaunay cells for the L-shape

Theorem : V. Delecroix and O.F. 2024

For each saddle connection γ, there is at least one triangulation τ with an non-empty iso-Delaunay cell containing γ. (In fact, infinitely many of them)

Idea of the iso-Delaunay enumeration : explore all the iso-Delaunay cells in the orbit of a translation surface and note every saddle connections discovered.

References

H. Masur, 1990 : The growth rate of trajectories of a quadratic differential.
J. Bowman, 2008 : Teichmüller geodesics, Delaunay triangulations and Veech groups.
W. Veech, 2011 : Bicuspid F-structures and Hecke groups.

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