On the Twin-Width of Smooth Manifolds [arXiv:2407.10174]

Édouard Bonnet* and Kristóf Huszár†

* CNRS, ENS de Lyon, Laboratoire de l'Informatique du Parallélisme, † Graz University of Technology, Institute of Geometry

Context, Motivation and Guiding Questions

I. What are "good" triangulations of manifolds for computational purposes? [The answer depends on who you ask.]

- Several algorithms are efficient (i.e. FPT) on 3-manifold triangulations with dual graph of small ***treewidth***: [Burton–Spreer, 2013], [Burton–Petterson, 2014], [Burton–Lewiner–Paixão–Spreer, 2016], [Burton–Downey, 2017], [Burton–Maria–Spreer, 2018].
- Introduced in 2020 ([Bonnet–Kim–Thomassé–Watrigant, 2022]), the ***twin-width*** is a graph parameter of emerging interest with many applications, e.g. in *first-order model checking* and *approximation algorithms* in graph optimization problems, cf. [Bonnet, 2024].
- Anticipating algorithmic applications in computational topology, we aim at exhibiting triangulations of (triangulable) manifolds of any dimension with dual graph of twin-width as small as possible.

II. Can such triangulations be efficiently constructed? → Currently only partial answers known (different for treewidth and twin-width).

III. What are some algorithmic applications of triangulations with dual graph of small twin-width? → Currently wide open.

Our Main Results

Theorem 1: Triangulations of bounded twin-width

Any compact d-dimensional smooth manifold admits a triangulation with dual graph of twin-width $\leqslant d^{O(d)}$.

A corollary of Theorem 1. There is a universal constant $C > 0$, such that every 3-manifold admits a triangulation with dual graph of twin-width $\leqslant C$.

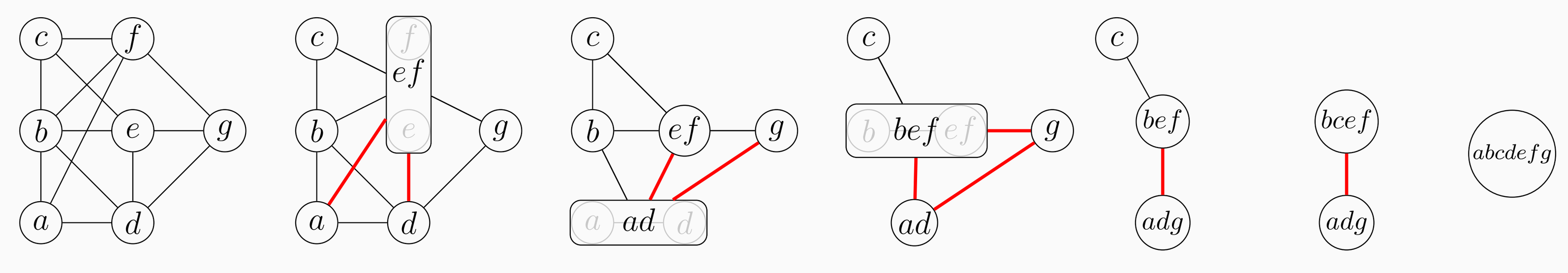
Theorem 2: Triangulations of large twin-width

For every compact d-dimensional ($d \geqslant 3$) piecewise-linear manifold $\mathcal{M}$ and $n \in \mathbb{N}$, there is a triangulation $\mathcal{T}$ of $\mathcal{M}$ with dual graph of twin-width $\mathrm{tww}(\Gamma(\mathcal{T})) \geqslant n$.

Theorem 1 does not hold for treewidth. For every $n \in \mathbb{N}$, there are 3-manifolds $\mathcal{M}$, such that the treewidth $\mathrm{tw}(\Gamma(\mathcal{T}))$ of the dual graph of any triangulation $\mathcal{T}$ of $\mathcal{M}$ is at least n. [H–Spreer–Wagner, 2019], [H–Spreer, 2023]

Contraction Sequences and the Twin-Width of a Graph

We consider **contraction sequences** of graphs over which some edges become **red**.



The *width* of a contraction sequence is the maximum red degree in the sequence. The **twin-width** $\mathrm{tww}(G)$ of a graph is the smallest width of any contraction sequence of G.

The above example shows that the twin-width of the leftmost graph is at most two. Figure source: [Bonnet, 2024]

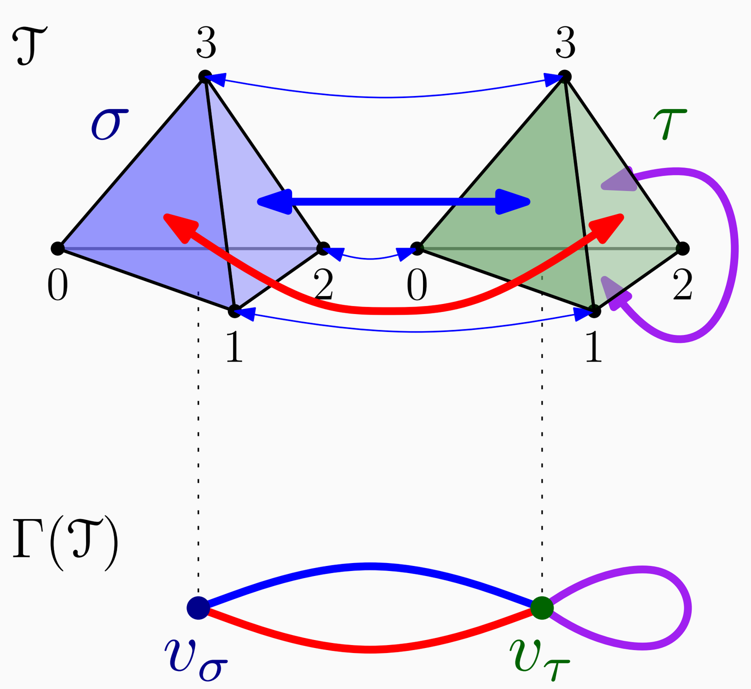
Triangulations of Manifolds and their Dual Graphs

Triangulations...

Every compact smooth d-manifold admits a **triangulation**, i.e. finitely many **d-simplices** glued together along pairs of **$(d-1)$-faces.**

...and their Dual Graphs

The **dual graph** $\Gamma(\mathcal{T})$ of a triangulation $\mathcal{T}$ is a multigraph whose vertices correspond to the d-simplices of $\mathcal{T}$ and edges to the face gluings.



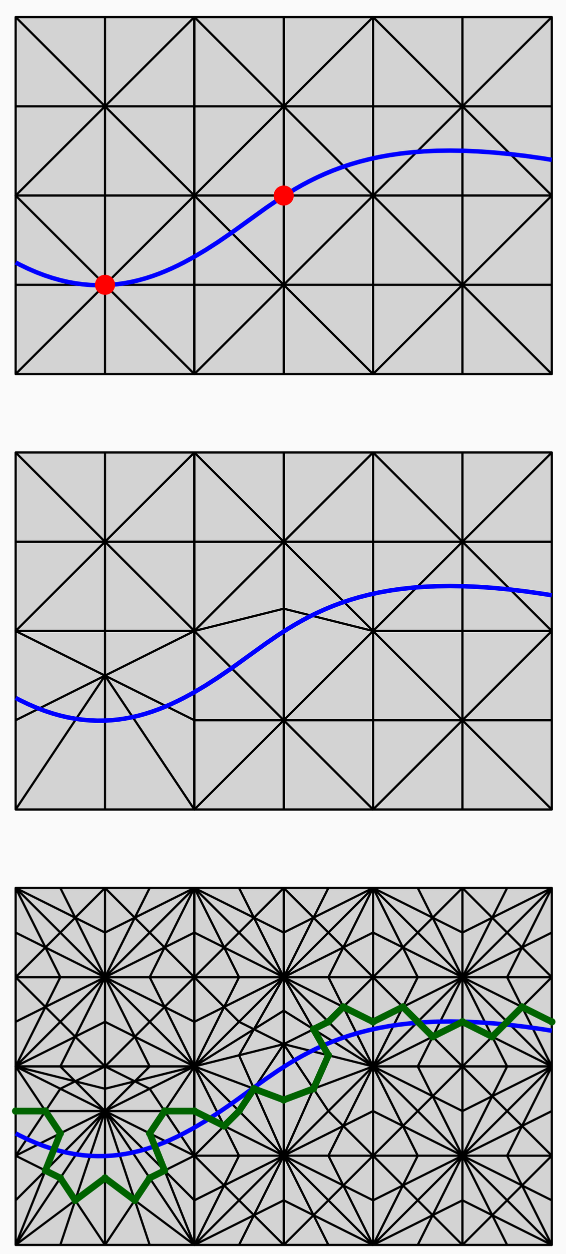
Fact: More than 13 thousand 3-manifolds can be triangulated with $\leqslant 11$ tetrahedra.

The notion of the dual graph can naturally be extended to *pure complexes* as well.

Whitney's Method

Strong Embedding Theorem. Any smooth d-manifold $\mathcal{M}$ can be smoothly embedded into $\mathbb{R}^{2d}$.

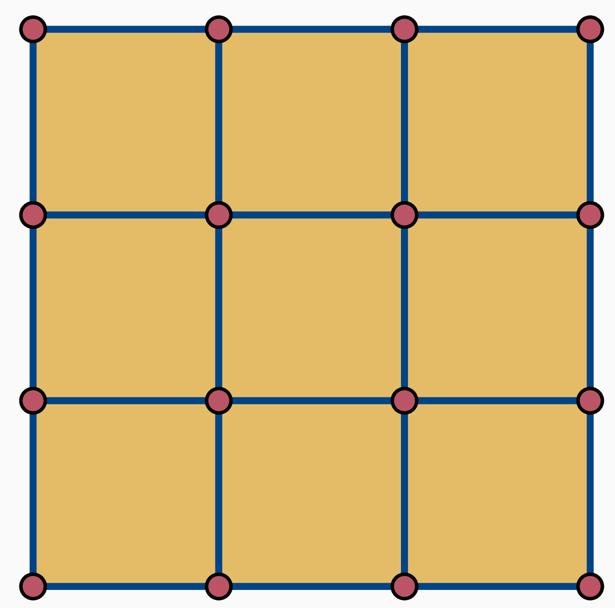
Given a smooth embedding of $\mathcal{M}$ and a triangulation of $\mathbb{R}^{2d}$, a triangulation of $\mathcal{M}$ (called a **Whitney triangulation**) may be constructed [Whitney, 1944], cf. [Boissonnat–Kachanovich–Wintraecken, 2021.].



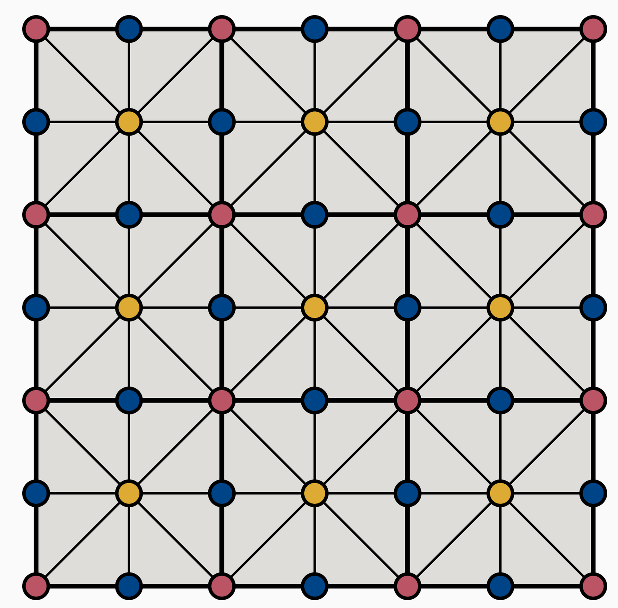
Bounding the Twin-Width of $G_{d,n}$ for the Proof of Theorem 1

$\mathbf{H}^{2d,n}$ is the $2d$-dimensional **hypercubic honeycomb** that decomposes $[1,n]^{2d}$ into $(n-1)^{2d}$ geometric cubes. $G_{d,n}$ is the dual graph of the d-skeleton of the second barycentric subdivision of $\mathbf{H}^{2d,n}$. In the figures below $d = 1$ and $n = 4$.

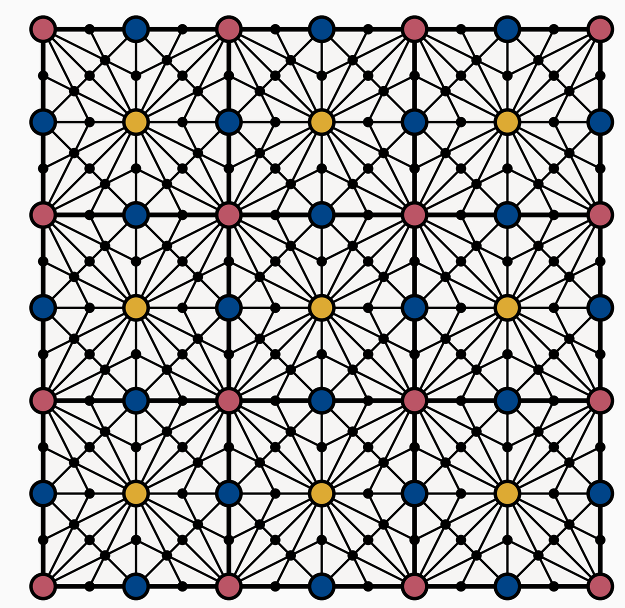
Fact. For any Whitney triangulation $\mathcal{T}$ of a smooth d-manifold, $\Gamma(\mathcal{T})$ is an *induced subgraph* of $G_{d,n}$ for a sufficiently large n.



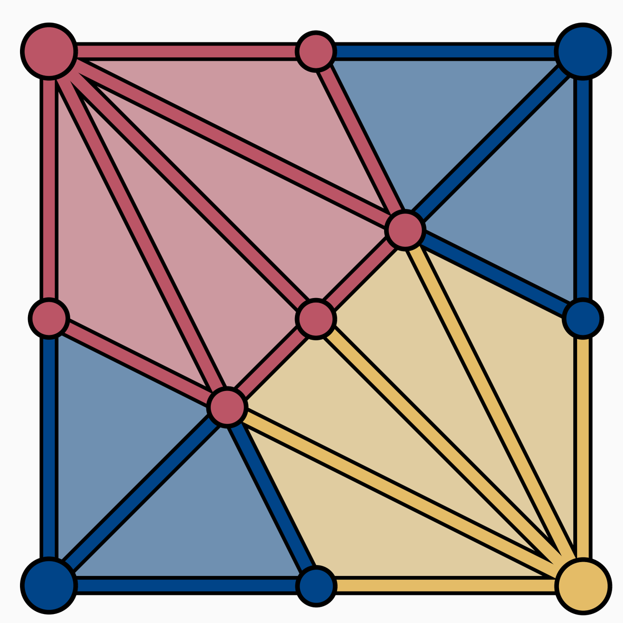
The hypercubic honeycomb $\mathbf{H}^{2d,n}$; its cells are colored by their dimension.



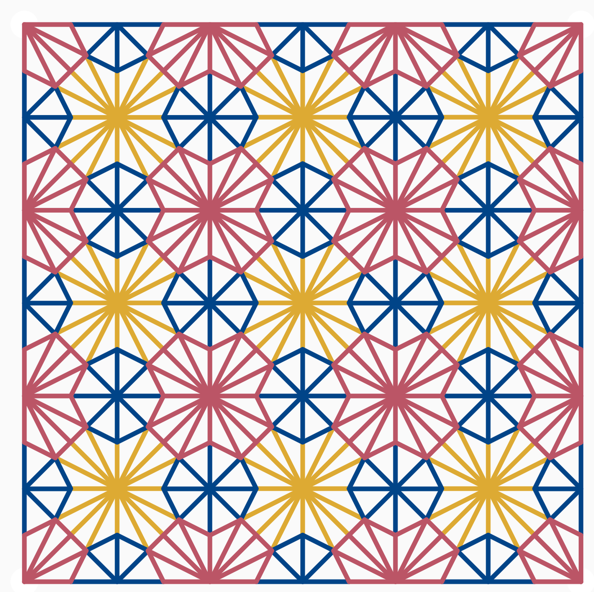
The barycentric subdivision $(\mathbf{H}^{2d,n})'$ with its vertices colored as their corresponding cells in $\mathbf{H}^{2d,n}$.



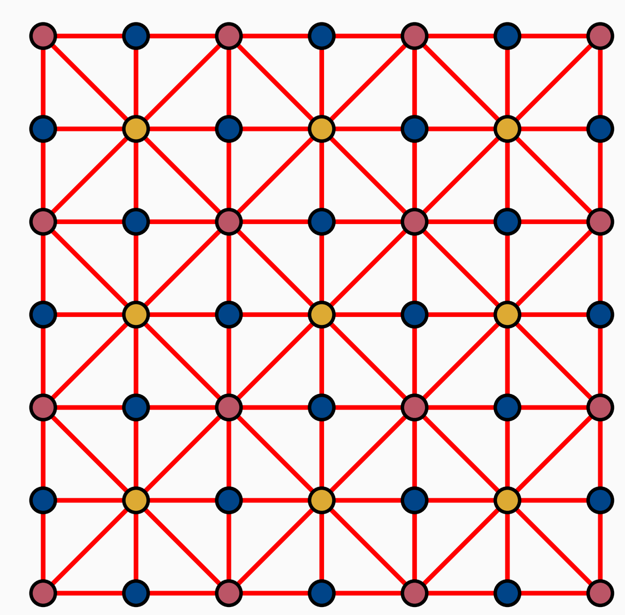
The second barycentric subdivision $(\mathbf{H}^{2d,n})''$. Vertices corresponding to those in $(\mathbf{H}^{2d,n})'$ are colored as before.



An extension of the coloring to all simplices of $(\mathbf{H}^{2d,n})''$ (shown near the top left corner).



The extended coloring restricted to the d-simplices of $(\mathbf{H}^{2d,n})''$ gives a partition of the d-simplices into connected monochromatic components.



By contracting each monochromatic component to a single node, we obtain a graph $G^*_{d,n}$ with all edges red. It is known that $\mathrm{tww}(G^*_{d,n}) \leqslant 3^{2d+1}$.