# Hopf Arborescent Links, Minor Theory, and Decidability of the Genus Defect

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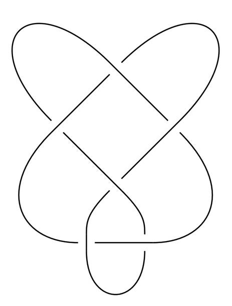
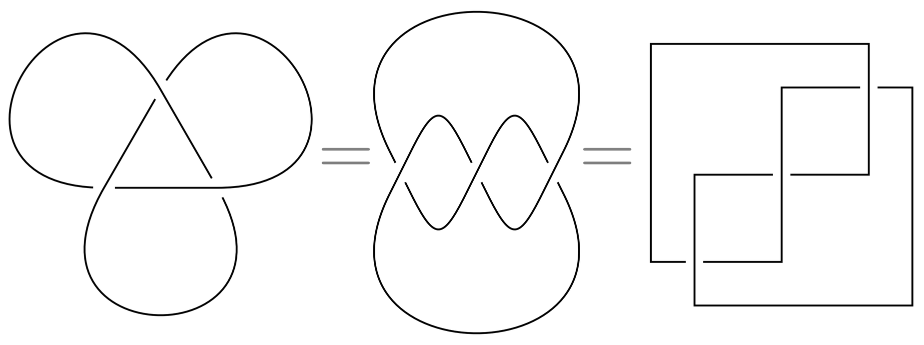
## Knot:

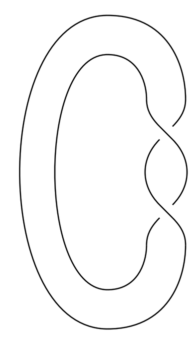
A **knot** is a polygonal embedding $\mathbb{S}^1 \to \mathbb{S}^3$ considered up to ambient isotopy, i.e. continuous deformation without self-intersections. A link is a disjoint union of knots.



Bowline knot.



Equivalent representations of the trefoil knot.



Hopf link.

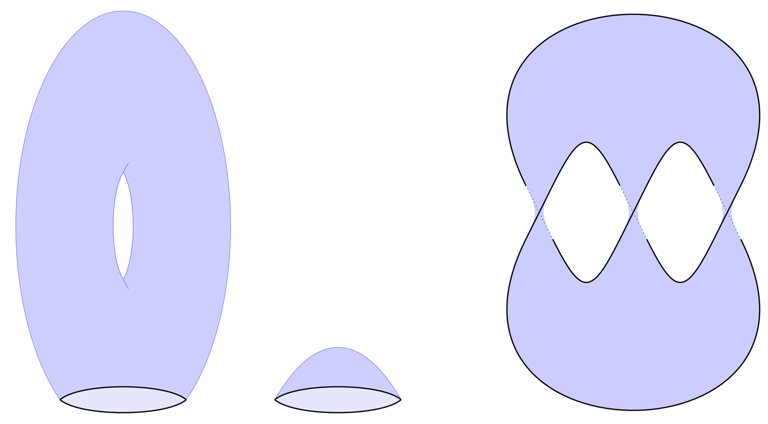
## Fundamental question: are two given knots equivalent?

It is difficult, both theoretically and in practice, to give a positive answer to this question. However, **invariants**, which are quantities remaining the same for all representations of a knot, allow us to distinguish two knots with different invariants.

## Genus & Seifert surfaces:

For a given knot $K$, a **Seifert surface** is an orientable surface embedded in $\mathbb{S}^3$ which admits $K$ as boundary. The **genus** $g(K)$ of $K$ is the minimal genus among its Seifert surfaces.

The trivial knot has genus 0.



The trefoil knot has genus 1.

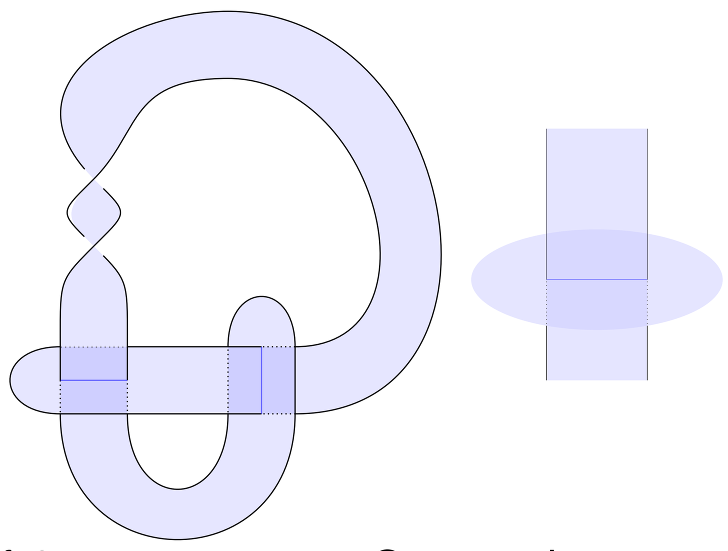
Deciding if a knot has genus at most $k$ is in **NP** $\cap$ **co-NP** [1, 2]. Does there exist a polynomial algorithm?

The genus is a well-known knot invariant that can be extended to oriented links. It can also be extended further to dimension 4.

## 4-dimensional genus & genus defect:

Considering $\mathbb{S}^3$ as the boundary of the 4-dimensional ball $\mathbb{B}^4$, the **4-genus** $g_4(K)$ of $K \subset \mathbb{S}^3$ is the minimal genus among of an orientable surface embedded* in $\mathbb{B}^4$ admitting $K$ as its boundary.

The **genus defect** is the difference between the classical genus and the 4-genus $\Delta_g(K) = g(K) - g_4(K)$.



Such notions are hard to picture. On the left, a ribbon knot can help us understand them. These knots can be obtained as the boundary of a disc admitting ribbon intersections (self-intersections as pictured next to the knot). Such intersections can be pushed into $\mathbb{B}^4$ so that they disappear. Hence, ribbon knots have 4-genus 0.

*In dimension 4 smooth and flat embeddings are different so that two notions of 4-genus appear. Our results are valid in both classes, hence we will write $g_4$ to designate either notion.

## What is known about computational topology in dimension 4?

No algorithmic framework at all is known to attack topological problems in 4-dimensional topology. Many of these problems are known to be undecidable, like homeomorphism between 4-manifolds [3]. No algorithm is known to decide the 4-genus of a knot or even to decide whether it is 0. However an analogue of the 4-genus for links, the 4-ball Euler characteristic, is **NP**-hard to compute [4], and it is also not known to be decidable.

## Our contributions [5, Dehornoy, L, de Mesmay]:

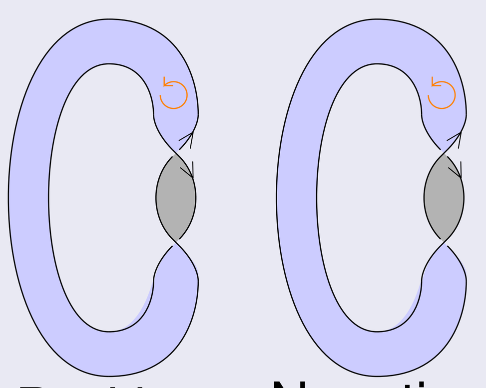
We define a family of knots and links called **Hopf arborescent links** and prove that the genus defect is decidable on this family. Furthermore we prove that surfaces associated to these links are well-quasi-ordered.

**Theorem 1:**
For any fixed $k$, there exists an algorithm deciding whether a Hopf arborescent link $L$ has genus defect at most $k$.

**Theorem 2:**
The class of Seifert surfaces associated to Hopf arborescent links is well-quasi-ordered by a surface containment relation called **surface-minor**.

## Hopf bands & Hopf arborescent surfaces:

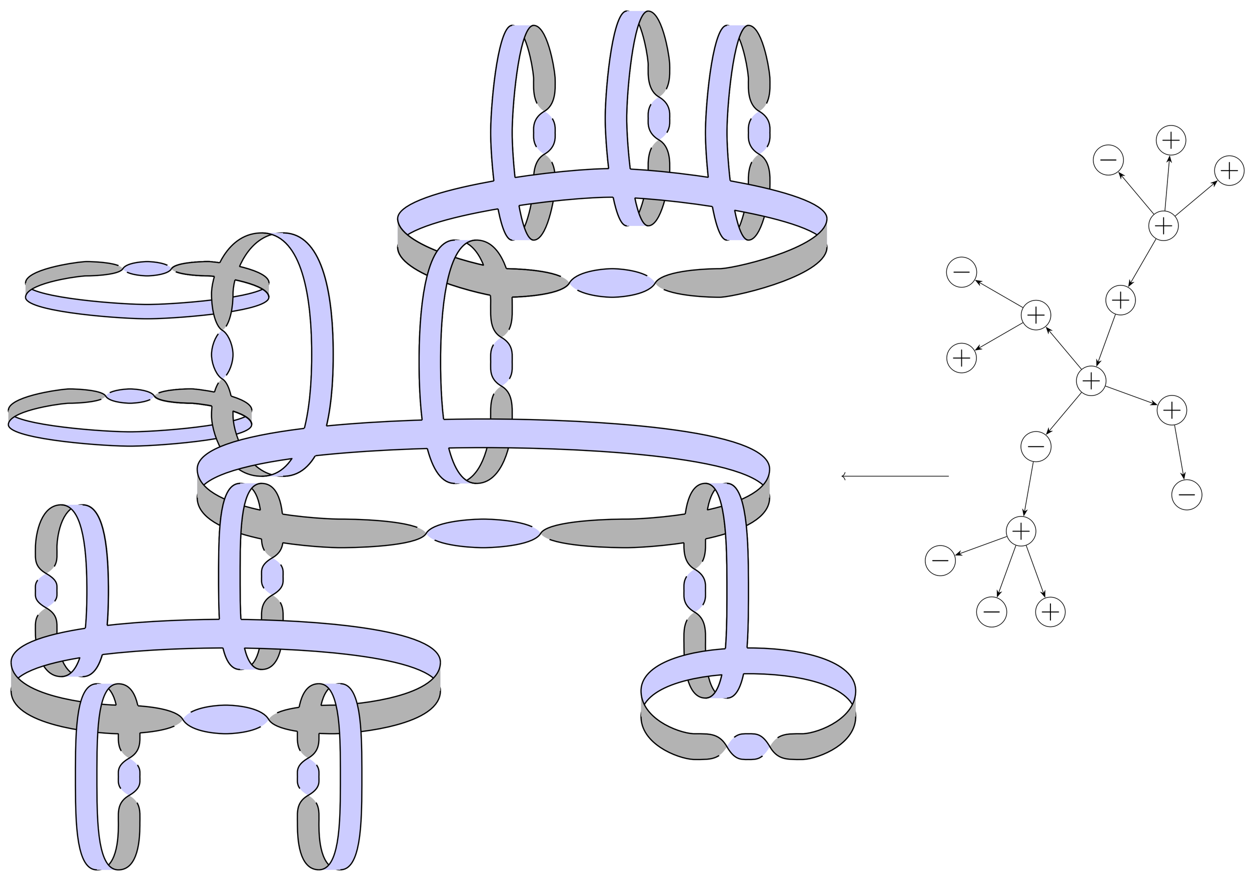
A **Hopf band** is an annulus whose boundary is a Hopf link.




A **plumbing** allows to merge a Hopf band to a surface with boundary:

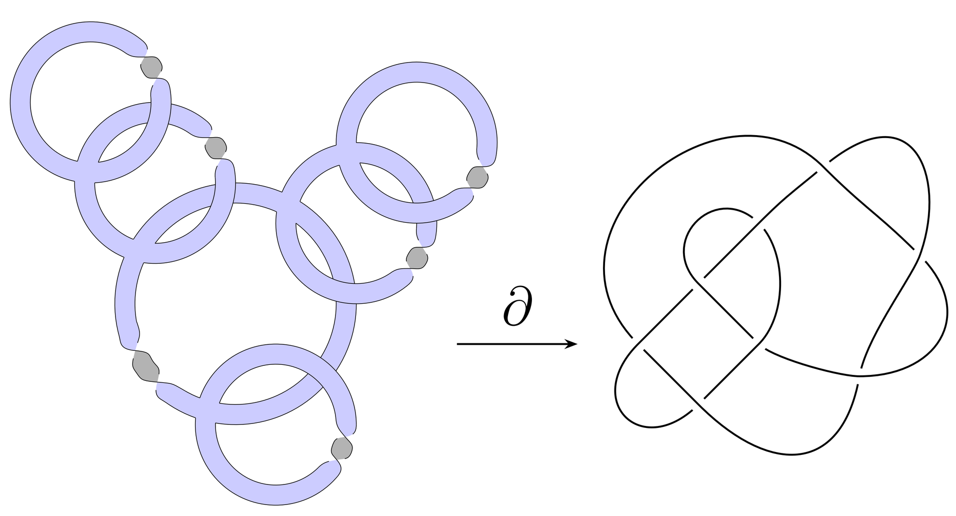

A **Hopf arborescent surface** is a surface that is built by iteratively plumbing Hopf bands along a tree.

We associate to each plane tree a Hopf arborescent surface:



## Hopf arborescent links:

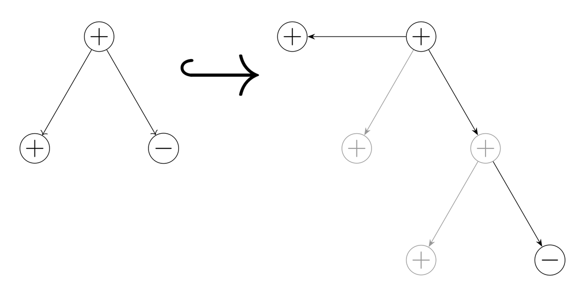
A **Hopf arborescent link** is a link that can be obtained as boundary of a Hopf arborescent surface.

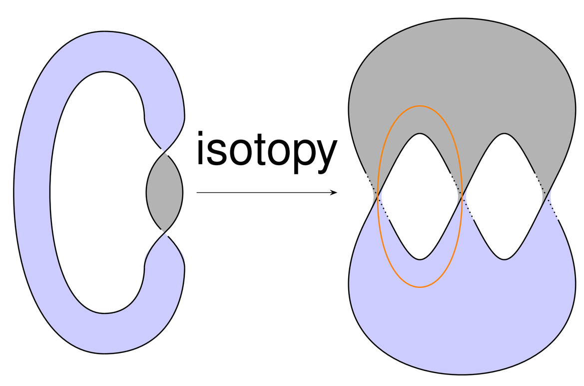



## Well-quasi-order:

An order $\leq$ is a **well-quasi-order** on $X$, if every infinite sequence of $X$ admits at least two comparable elements by $\leq$.

An order is defined on each family of objects: plane trees, surfaces, and links. We exploit the relations between these orders to prove that our surfaces and links are well-quasi-ordered by their minor relation. We prove that the order defined on links is decidable, this allows to prove Theorem 1.







[1] Joel Hass, Jeffrey C. Lagarias, and Nicholas Pippenger. The computational complexity of knot and link problems. *Journal of the ACM (JACM)*, 46(2):185–211, 1999.
[2] Marc Lackenby. The efficient certification of knottedness and Thurston norm. *Advances in Mathematics*, 387:107796, 2021.
[3] Andrei Andreevich Markov. The insolubility of the problem of homeomorphy. In *Doklady Akademii Nauk*, volume 121, pages 218–220. Russian Academy of Sciences, 1958.
[4] Arnaud de Mesmay, Yo'av Rieck, Eric Sedgwick, and Martin Tancer. The unbearable hardness of unknotting. *Advances in Mathematics*, 381:107648, 2021.
[5] Pierre Dehornoy, Corentin Lunel, and Arnaud de Mesmay. Hopf arborescent links, minor theory, and decidability of the genus defect. *Proceedings of the 40th International Symposium on Computational Geometry*, 2024.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