Mean curvature estimator on digital surfaces using a Varifold formulation

Romain Negro

Université Savoie Mont Blanc, CNRS, LAMA, F-73000 Chambéry, France
romain.negro@univ-smb.fr

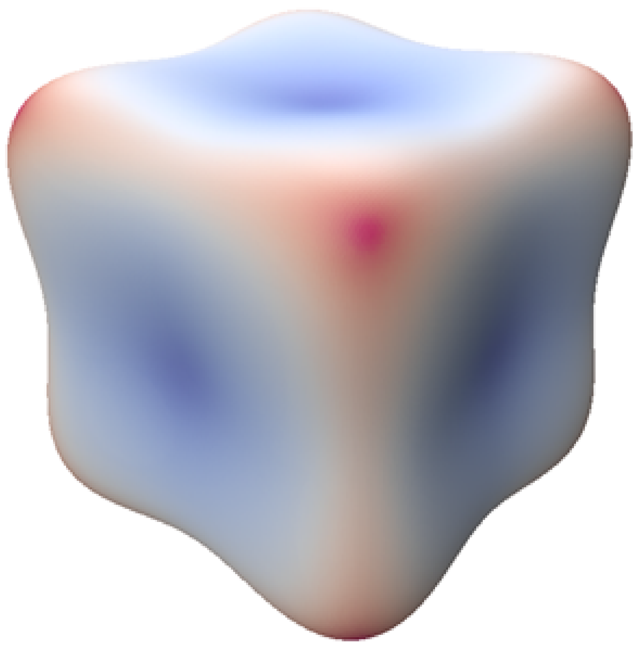
Mean curvature in differential geometry

Mean curvature formula

$H(x) = \frac{k_1(x)+k_2(x)}{2}$ where k_1 and k_2 are the principal curvatures at x.
$\vec{H}(x) = H(x)\vec{N}(x)$ where $\vec{N}(x)$ is the normal vector at x.

Problems

- Undefined on surfaces that are not C^2
- Very sensitive to noise



Other methods

- Normal cycle method [3]
- Fitting methods [4]
- Integral invariants (II) [5]
- Corrected Normal Current (CNC) method [6]
- Voronoi-based methods [7]

Varifold formulation

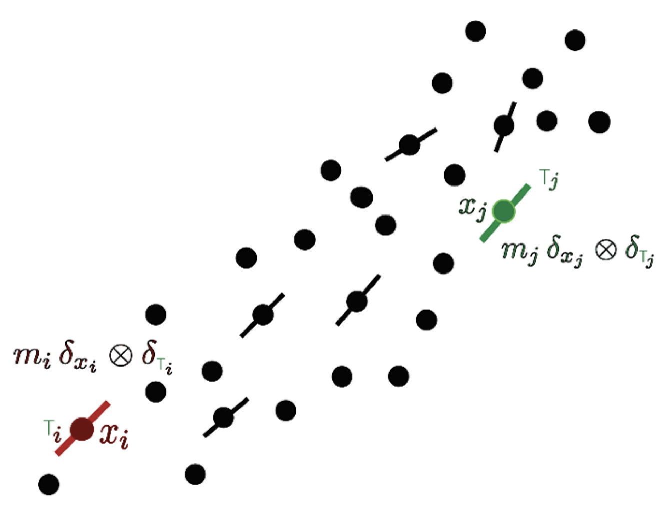
Varifold definition

A varifold V is a Radon measure on $\mathbb{R}^n \times G_{d,n}$ where $G_{d,n}$ is the Grassmannian of d-planes in $\mathbb{R}^n$.

Divergence operator

- $T \in G_{d,n}$
- $X = (X_1, X_2, \ldots, X_n) \in C_c^1(\Omega, \mathbb{R}^n)$
- $(e_1, e_2, \ldots, e_n)$ canonical basis of $\mathbb{R}^n$

$$\mathrm{div}_T X(x) = \sum_{i=1}^{n} \langle \Pi_T(\nabla X_i(x)), e_i \rangle$$



Mean curvature vector formula

Let $V = (v_1, v_2, \ldots, v_n)$ be a C^2 vector field on a submanifold M of $\mathbb{R}^n$ that locally forms an orthonormal basis of $(T_x M)^\perp$. The mean curvature vector of V at x is defined by

$$\vec{H}(x) = -\sum_{i=1}^{n-d} \left(\mathrm{div}_{T_x M} v_i(x)\right) v_i(x)$$

- Converges to the mean curvature vector of the submanifold M when V represents a smooth surface

Digital surfaces

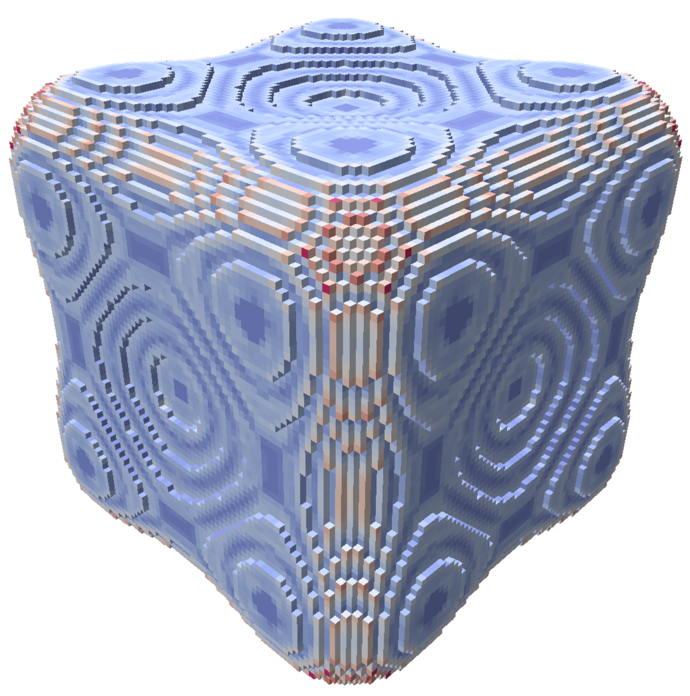
Mean curvature formula

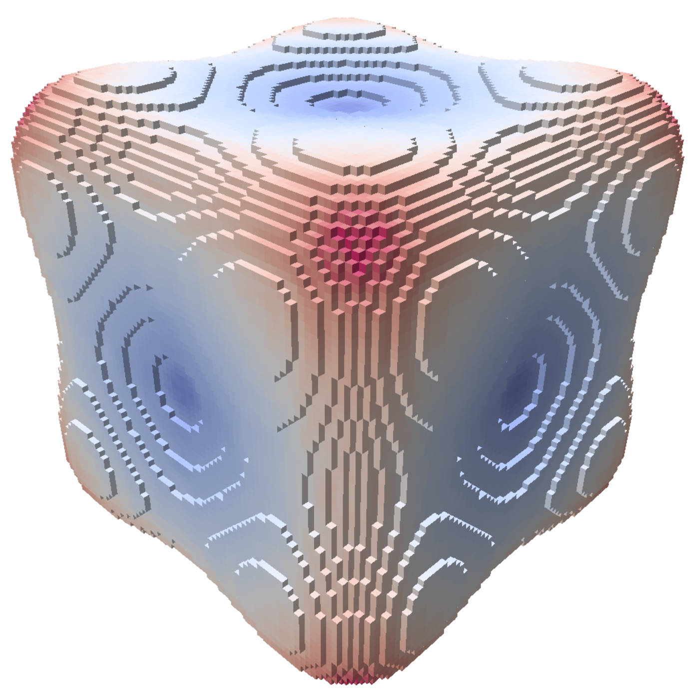
- P list of positions
- N list of normals
- r kernel radius
- $W_k : \mathbb{R} \to \mathbb{R} \in C^1$ kernel associated to element k
- $\Pi_n : \mathbb{R}^3 \to \mathbb{R}^3$ orthogonal projection on the plane normal to n

$$\vec{H}(x) = -\frac{\sum_{i=1}^{n} \frac{W_k'\left(\frac{||P(i)-P(k)||}{r}\right) \cdot \Pi_{N(i)}(P(i)-P(k))}{||P(i)-P(k)||}}{r \cdot \sum_{i=1}^{n} W_k\left(\frac{||P(i)-P(k)||}{r}\right)}$$

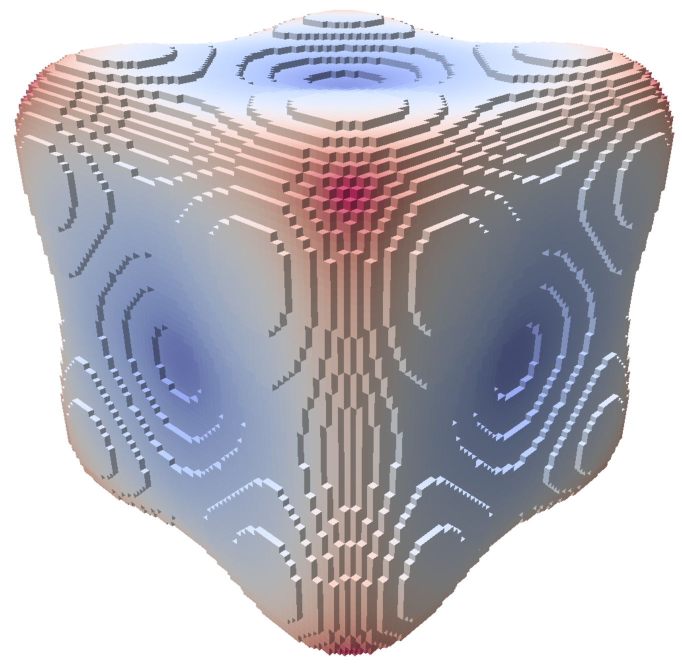
Normals management

- Normals are not well-defined on digital surfaces (only 6 possible directions)
- Correction of normals is required to get meaningful results

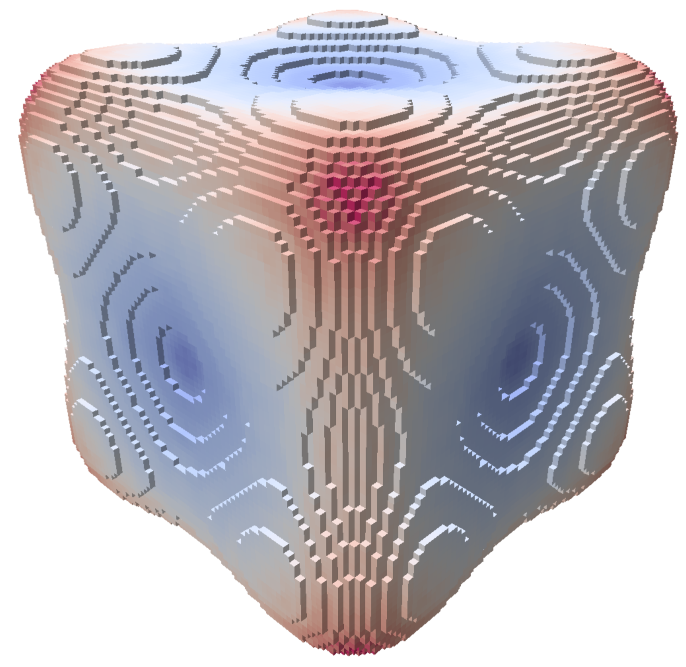




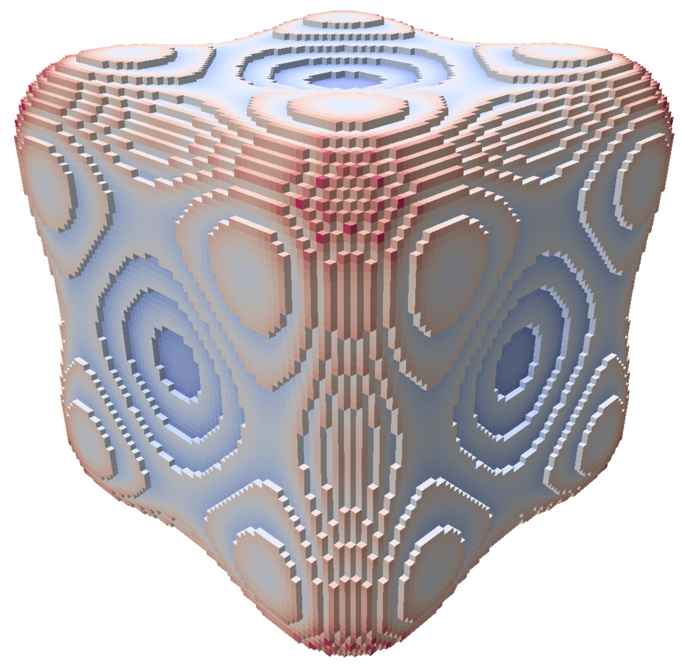
Comparison with other methods



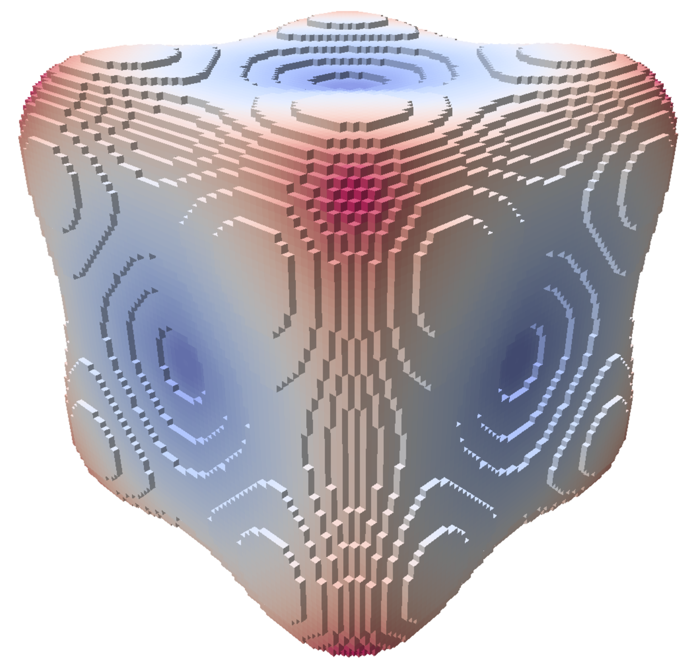
True mean curvature



Varifolds method



Integral invariants (II) method

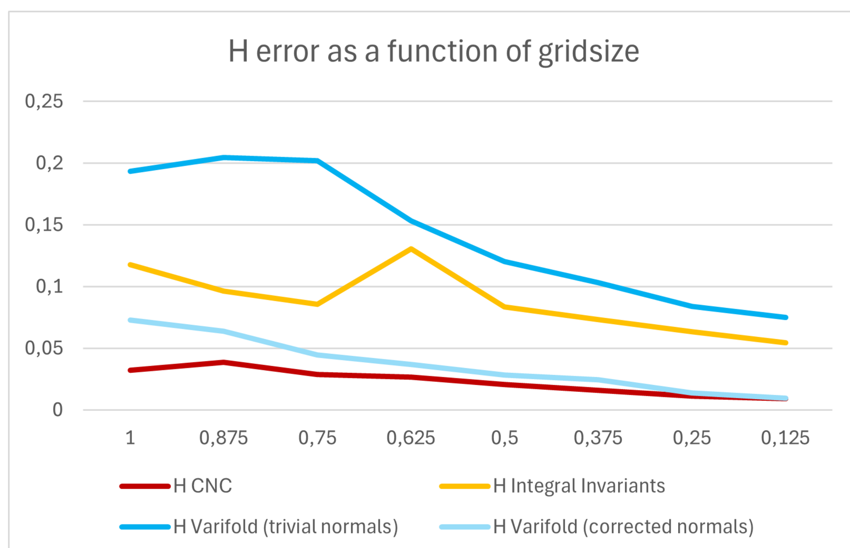


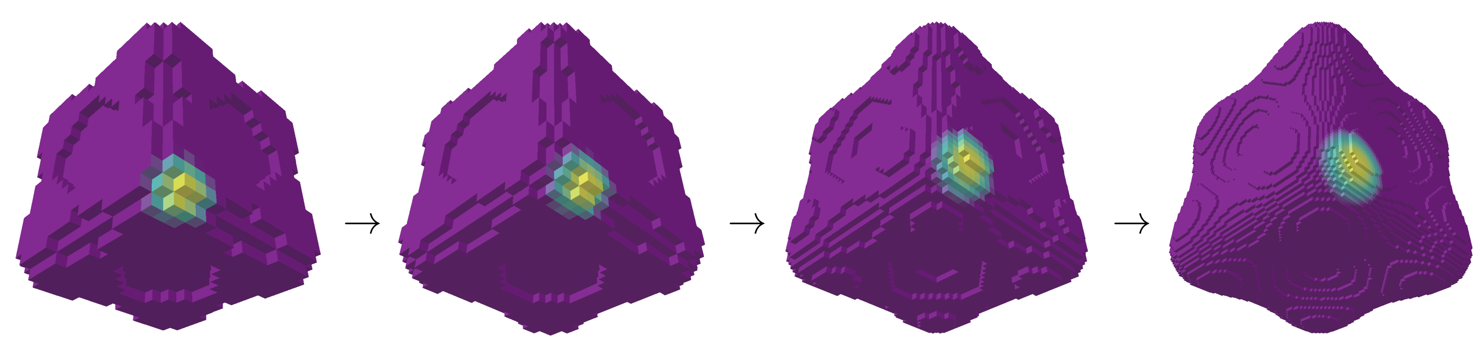
Corrected Normal Current (CNC) method

Numerical comparison with other methods

Error expression

The error is defined as the L^2 norm of the difference between the mean curvature vector projected from the differential surface and the mean curvature vector computed on the digital surface.





Gridsize 1 → **Gridsize 0.75** → **Gridsize 0.5** → **Gridsize 0.25**

References

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