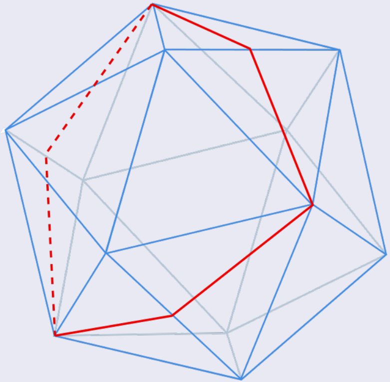
Sweeping spheres
Quasigeodesics and width of knots

Arnaud de Mesmay (CNRS, LIGM, Université Gustave Eiffel, Paris)

Geometry and Computing, CIRM, October 23rd, 2024

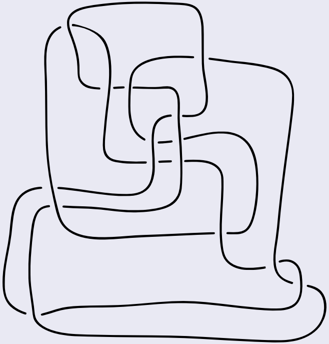
Gist of the presentation

Finding quasigeodesics on polyhedra



[Chartier, dM DCG '24]

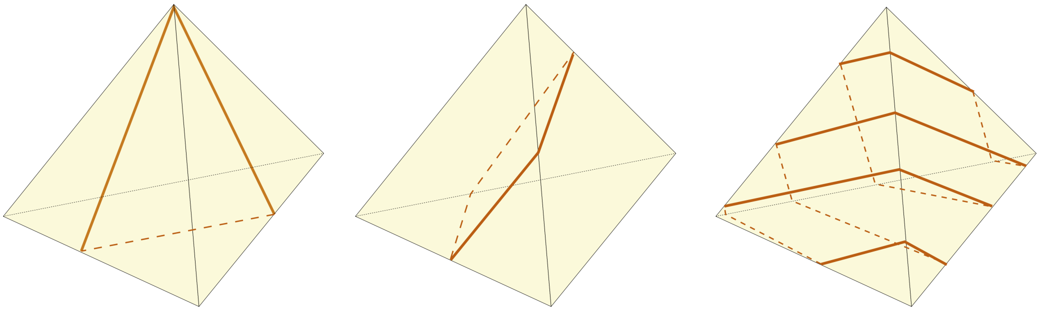
Treewidth of knot diagrams



[dM, Purcell, Schleimer, Sedgwick JoCG '19], [Lunel, dM SOCG '23]

These two seemingly very different problems boil down to understanding the best way to *sweep a sphere* using *lower-dimensional spheres*.

1. Closed (quasi-)geodesics on polyhedra

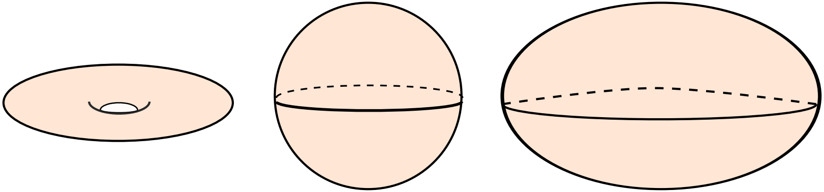


Geodesics on smooth manifolds

- A *Riemannian metric* is a smooth metric on a manifold.
  → **Example**: Submanifolds of $\mathbb{R}^3$ are endowed with the metric induced by the ambient space.
- A *geodesic* on a Riemannian manifold is a curve that is locally straight at each point (equivalently, a curve that locally realizes shortest paths).

Very old question in differential geometry

Does every compact Riemannian manifold contain a closed geodesic?



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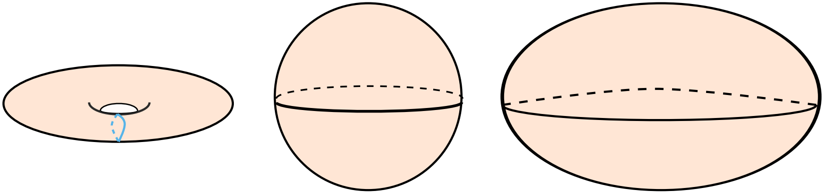

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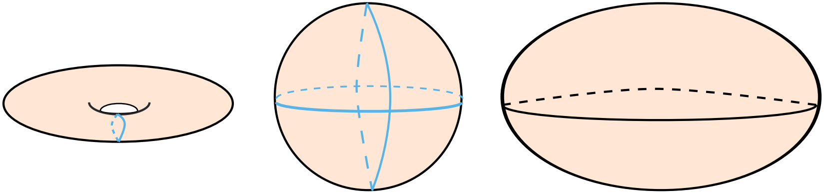

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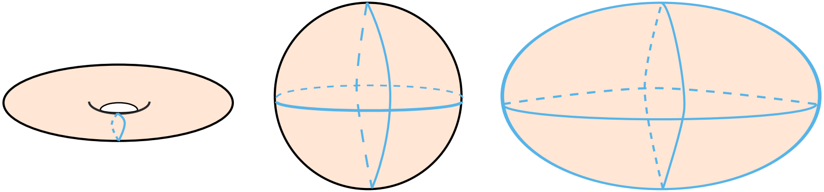

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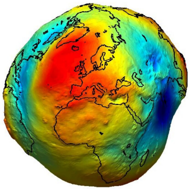

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Theorem (Lyusternik-Fet '51)

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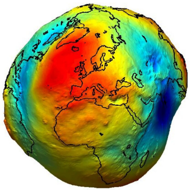

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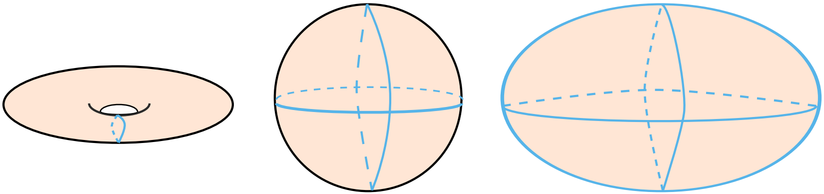
- A curve is *simple* if it does not self-intersect.

Poincaré '1905

Does every Riemannian *two-dimensional sphere* contain three simple closed geodesics?



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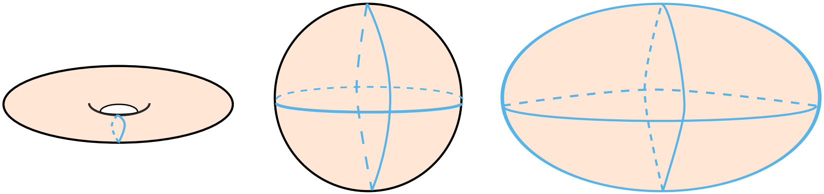
Theorem (Lyusternik-Schnirrelmann '29, Ballmann '78, Grayson '89)

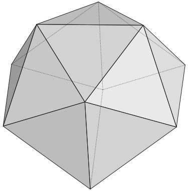
Every Riemannian two-dimensional sphere contains at least three simple closed geodesics.



Geodesics on polyhedra

Are there polyhedral versions of this theorem?

What is a geodesic on a convex polyhedron in $\mathbb{R}^3$? It goes straight within faces and when crossing edges and



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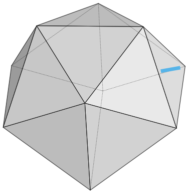

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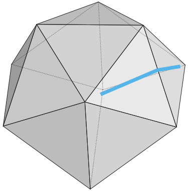

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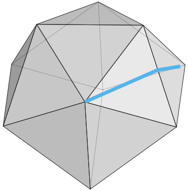

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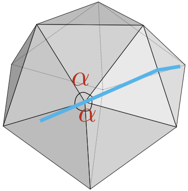

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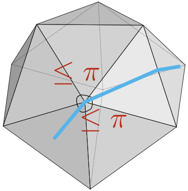
3. Hits vertices with angles at most π on both sides? $\rightarrow$ *quasigeodesic*

Theorem (Pogorelov '1949)

Any convex polyhedron in $\mathbb{R}^3$ admits at least three simple closed quasigeodesics.



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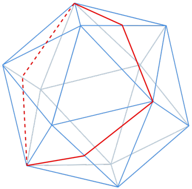
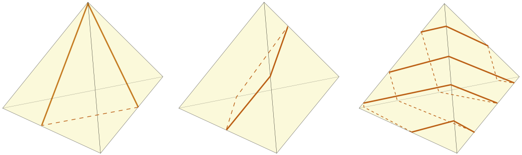


Computation of quasigeodesics

Open problem: O'Rourke Wyman '90, Demaine O'Rourke '07

Can we compute such a simple closed quasigeodesic (in polynomial time)?

- Motivation: unfolding polyhedra, since quasigeodesics develop to straight lines.
- Pogorelov's proof proceeds by Riemannian approximation and is non-constructive.
- Issue: how to control the combinatorics of such a quasigeodesic?



Open problem

Does there exist a universal constant c so that there always exists a simple closed quasigeodesic crossing at most c times each edge?

State of the art and our results

Theorem (Demaine, Hesterberg, Ku '2020)

There is an algorithm running in pseudo-polynomial time to find a closed quasigeodesic on any convex polyhedron, not necessarily simple.*

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Theorem (Simplified version of [Chartier, dM DCG 24])

Any polyhedron admits a weakly ***simple*** *closed quasigeodesic of length $L(\gamma) \leq M$, which crosses or uses $O(dM/h)$ times the edges or the vertices of the sphere.*

M: sum of edge lengths of a triangulation.

d: maximum degree of vertices.

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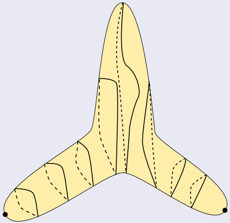
Corollary

There is an algorithm to find such a quasigeodesic in pseudo-exponential time.

From geodesics to sweepouts

How to find (quasi-)geodesics on (polyhedral) spheres? [Birkhoff '17]

1. We sweep the sphere with a continuous family of simple closed curves.
2. Such a *sweepout* is tightened by an operation that continuously shortens the lengths of fibers.
3. By iterating this process, one fiber converges to a quasigeodesic.



- Our theorem also applies to non-convex polyhedral spheres.
- In the polyhedral case, specific techniques are needed for the tightening [Chartier, dM].
- The length of the quasigeodesic is at most the best possible *width* of a sweepout:

$$width(S) = \inf_{f:S\to[0,1]} \sup_{t\in[0,1]} ||f^{-1}(t)||$$

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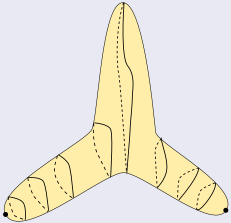

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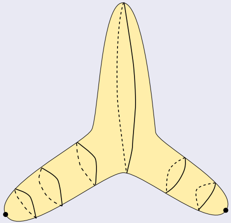

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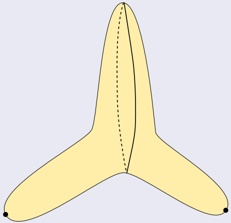

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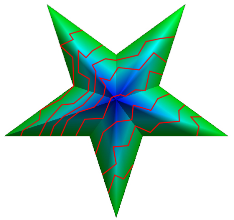

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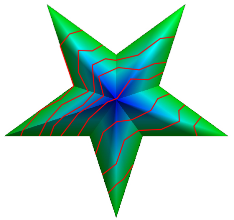

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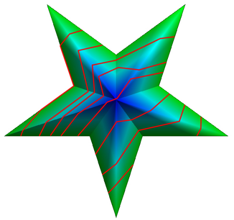

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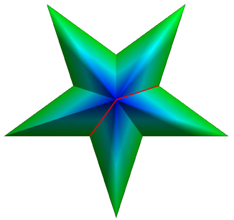

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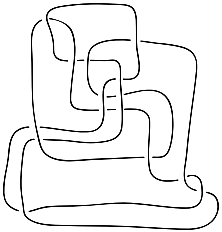
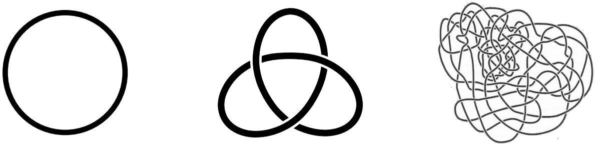
2. (Tree-)width of knot diagrams



Knots and algorithms

- A *knot* is a closed curve in $\mathbb{R}^3$ (or $\mathbb{S}^3$).



- Two knots are considered equivalent if one can be deformed into the other one without self-crossings (*isotopy*).
- A *knot diagram* is a projection of a knot in the plane.

Deciding whether two knots are equivalent is hard!

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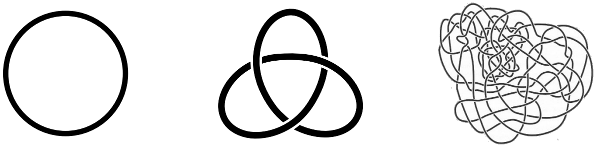

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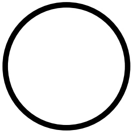
Knot invariants

- So in order to tell knots apart, a common way is to use *invariants*.
- For example, the *Jones polynomial* allows us to distinguish the previous examples:



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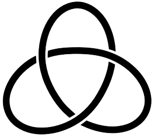
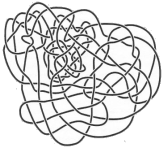


$t + t^3 - t^4$



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Computing invariants is also hard!

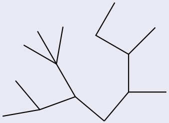
For example, computing the Jones invariant of a knot is #P-hard [Jaeger, Vertigan, Welsh '90],[Kuperberg '15].

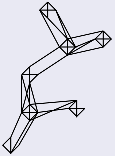
So what can we do? Look for low-width diagrams

The *treewidth* of a graph aims to measure "how close" a graph is to a tree.

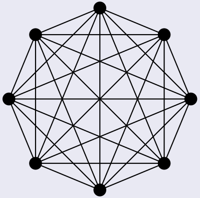
Some examples

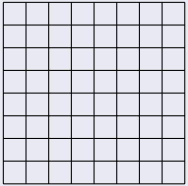
Small treewidth





High treewidth

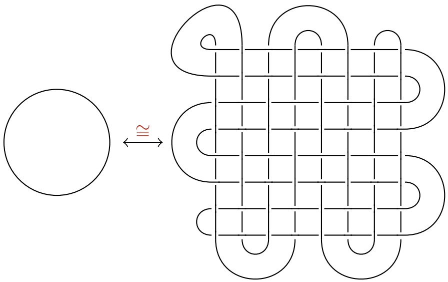




- When a knot diagram has *small* treewidth, one can use dynamic programming to compute the Jones polynomial [Makowsky and Mariño, 2003] (and many other invariants) efficiently.

Knots of high treewidth?

- Any knot has infinitely many diagrams, of arbitrarily high treewidth.



Question from [Makowsky and Mariño, 2003] and [Burton, 2016]

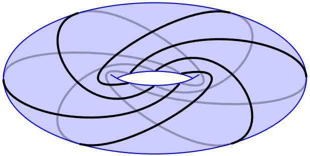
Are there knots for which *all* diagrams have high treewidth?

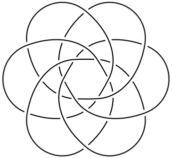
Yes!

Theorem (Simplified version of [dM, Purcell, Schleimer, Sedgwick JoCG 2019][Lunel, dM SoCG 2023])

Any diagram $D_{p,q}$ of a torus knot $T_{p,q}$ has high treewidth, i.e.,

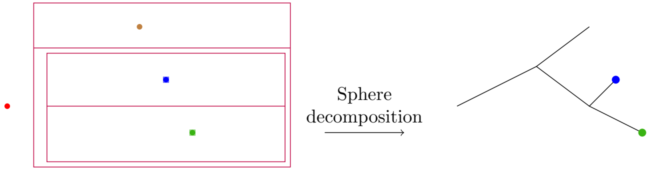
$$\text{treewidth}(D_{p,q}) = \Omega(\min(p, q)).$$





The proofs revolve around understanding how to sweep $\mathbb{S}^3$ with 2-dimensional spheres.

Sphere decompositions

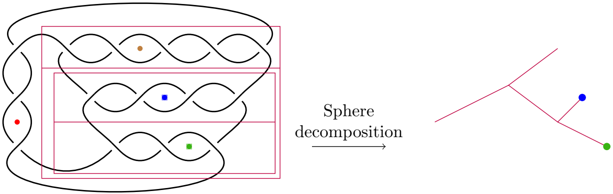


Sphere decomposition

A **sphere decomposition** of $\mathbb{S}^3$ is a continuous map $f : \mathbb{S}^3 \to T$ where T is a trivalent tree such that:

$$f^{-1} : \begin{cases} \text{leaf} & \mapsto \quad \text{point} \\ \text{inner vertex} & \mapsto \quad \text{double bubble} \\ \text{point interior to an edge} & \mapsto \quad \text{sphere} \end{cases}$$

Spherewidth



Spherewidth

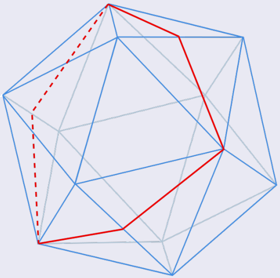
The spherewidth of K is :

$$\text{sw}(K) = \inf_f \sup_{e \in E(T), x \in \mathring{e}} |f^{-1}(x) \cap K|.$$

- It is not very hard to prove that $\text{sw(K)} \leq 2 \min_{D \text{ diagram of } K} \text{treewidth}(D)$.
- Our key technical contribution is to lower bound the spherewidth using tools from knot theory in [dM, Schleimer, Sedgwick, Purcell] and from structural graph theory in [Lunel, dM].

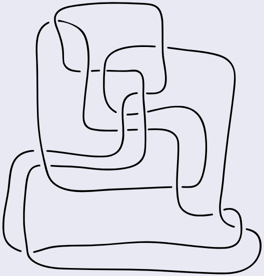
Sweeping spheres

Finding quasigeodesics on polyhedra



$$\text{width}(S) = \inf_{f:S\to[0,1]} \sup_{t\in[0,1]} ||f^{-1}(t)||$$

Treewidth of knot diagrams



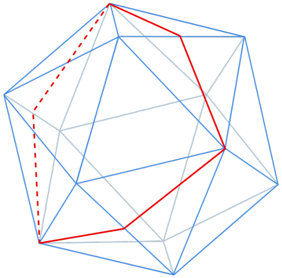
$$\text{sw}\,(K) = \inf_{f:\mathbb{S}^3\to T} \sup_{e\in E(T),x\in\mathring{e}} |f^{-1}(x)\cap K|.$$

The two problems revolve around understanding very similar width quantities (see also *pathwidth*, *branchwidth*, etc. in graph theory).

Open problems

Open problem from O'Rourke Wyman '90, Demaine O'Rourke '07

Given a polyhedron, what is the complexity of computing a quasigeodesic? Can one do it in polynomial time? What about the second and third one?

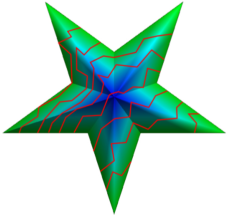


Open problems

Computing the width of polyhedra

Given a polyhedron, or more generally a polyhedral sphere, what is the complexity of computing its width?

$$\mathrm{width}(S) = \inf_{f:S\to[0,1]} \sup_{t\in[0,1]} ||f^{-1}(t)||$$



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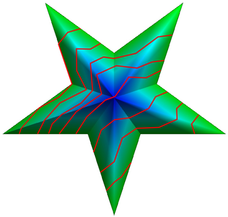

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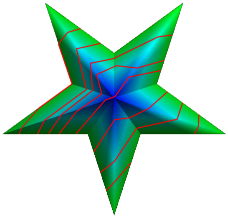

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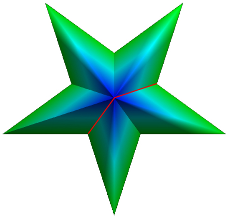
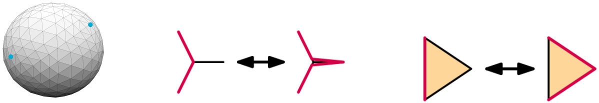


Open problems

Computing discrete versions of the width

Given a triangulated sphere, what is the complexity of computing its width when sweeping using elementary discrete operations?



- Related to *Homotopy Height* and *Homotopic Fréchet Distance*,
- Known to be in **NP** [Chambers, Chambers, dM, Ophelders, Rotman J. Diff. Geom. '21],[Chambers, dM, Ophelders SODA '18], but unknown to be polynomial or **NP**-hard.

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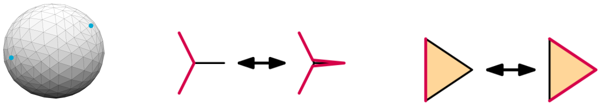

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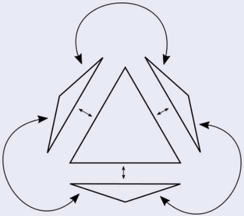
Thank you! Questions?

Polyhedral spheres

Definition

A polyhedral sphere is a set of polygonal Euclidean faces glued together along their edges. Convex polyhedra are examples of polyhedral spheres.

Make sure that the glued edges are the same length and that genus is zero.



The adopted point of view is intrinsic.

Polyhedral spheres

Polyhedral spheres have two types of vertices :

- Around a *convex vertex*, the sum of the angles is at most 2π.
- Around a *reflex vertex*, the sum of the angles is at least 2π.

Quasigeodesic

- Passing a convex vertex, a quasigeodesic forms two angles at most π.
- Passing a reflex vertex, a quasigeodesic forms two angles at least π.

A quasigeodesic is said to be weakly simple if it is the limit of a sequence of simple curves.