The Unfinished Epic of Discrete Tomography

Yan Gerard *(yan.gerard@uca.fr)*

Geometry and Computing

October 24th 2024, ***CIRM, Luminy***

A 1h talk in 30 minutes!

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UNIVERSITÉ
Clermont
Auvergne

LIMOS

What are we talking about ?

A 1h talk in 30 minutes!

0 Discrete Tomography in brief

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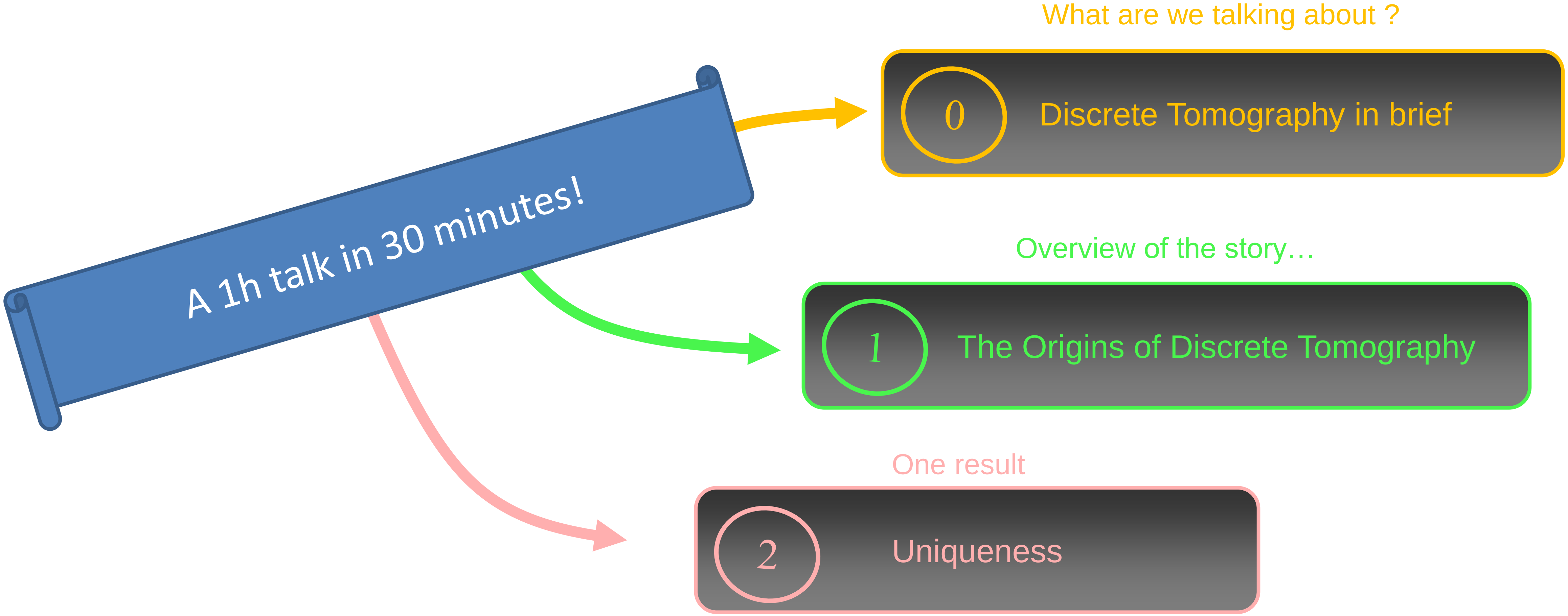
0 Discrete Tomography in brief

Overview of the story…

1 The Origins of Discrete Tomography



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1 The Origins of Discrete Tomography

One result

2 Uniqueness

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One open problem

3 Alon’s open problem

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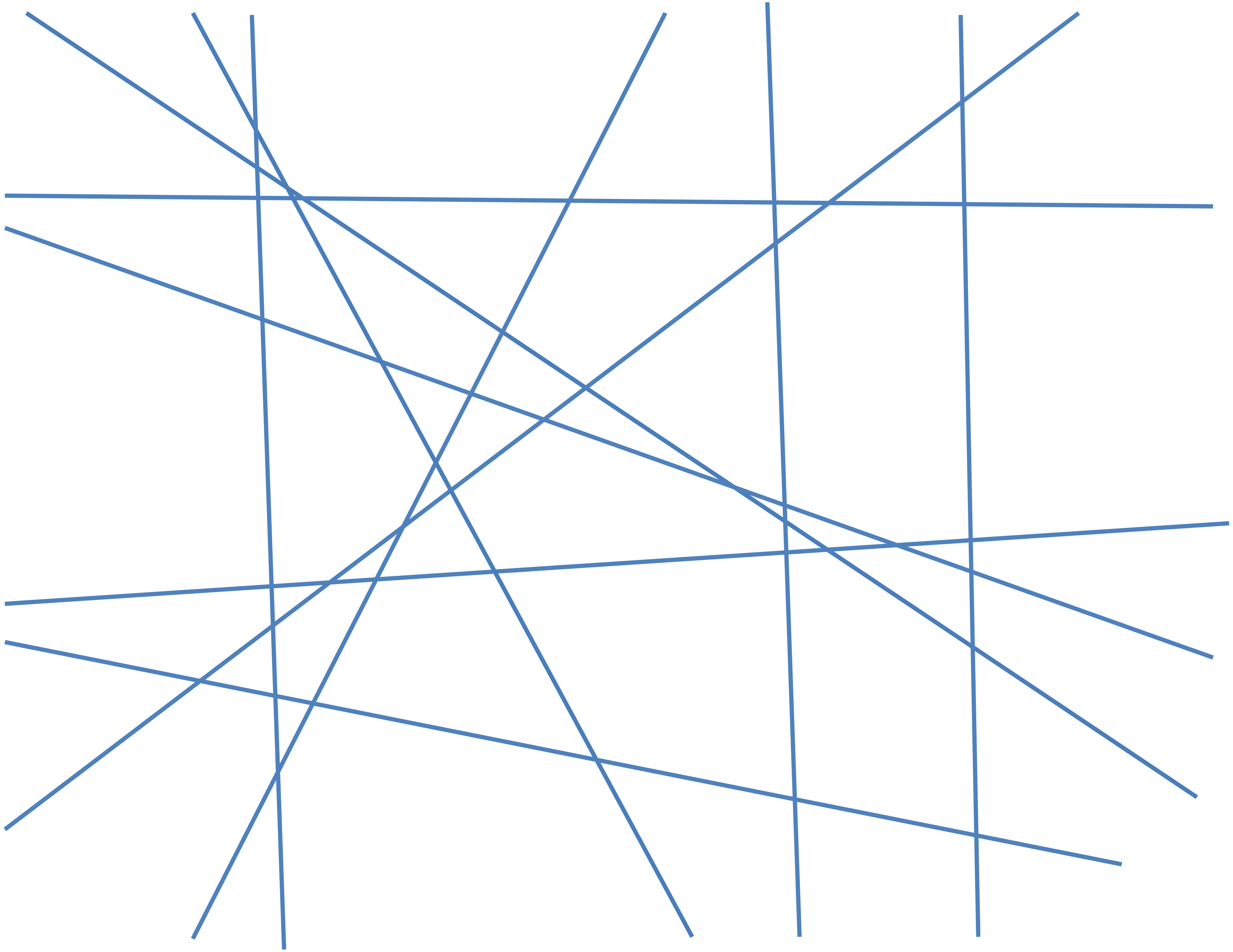

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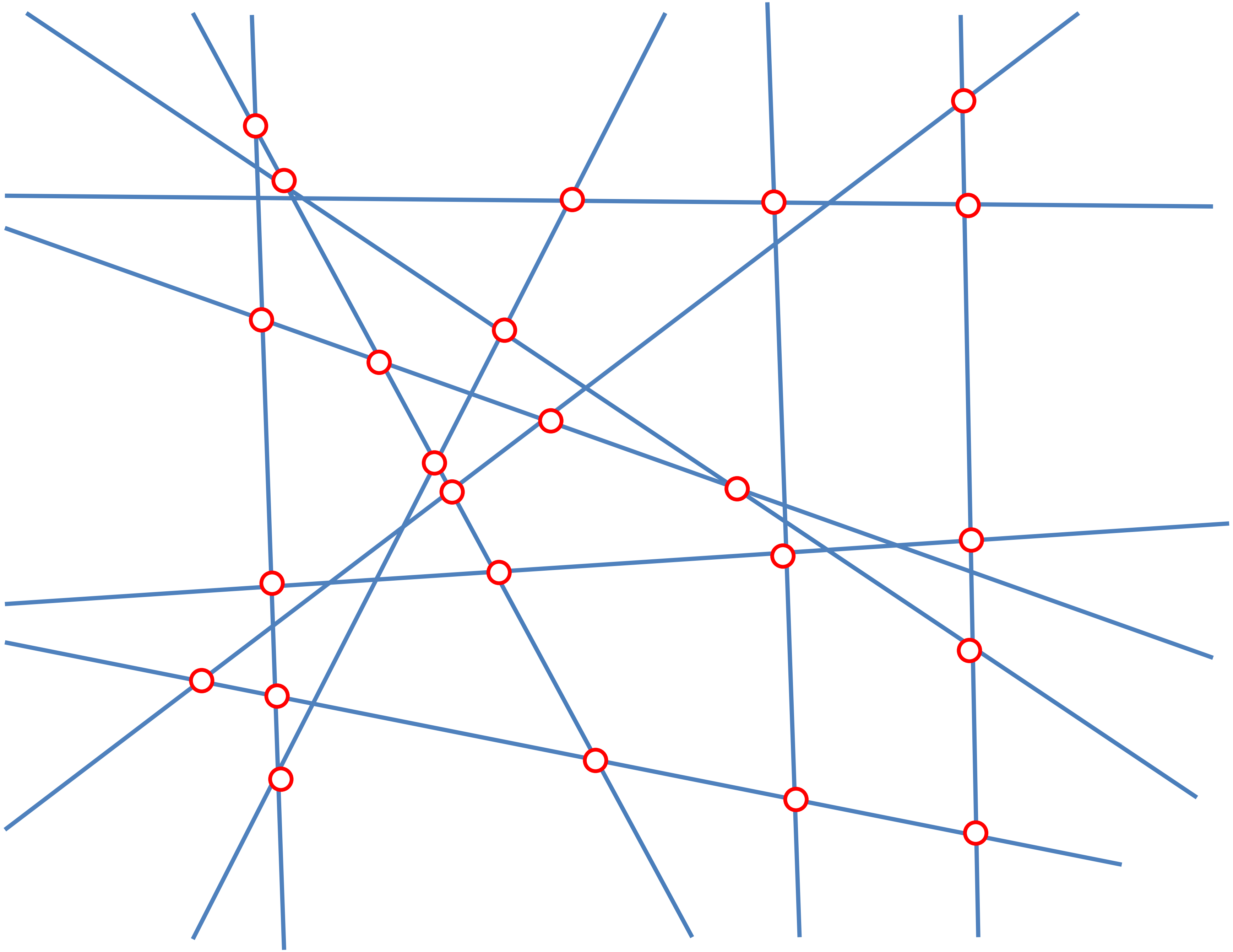

Input: Some lines

A set **T** of intersection points

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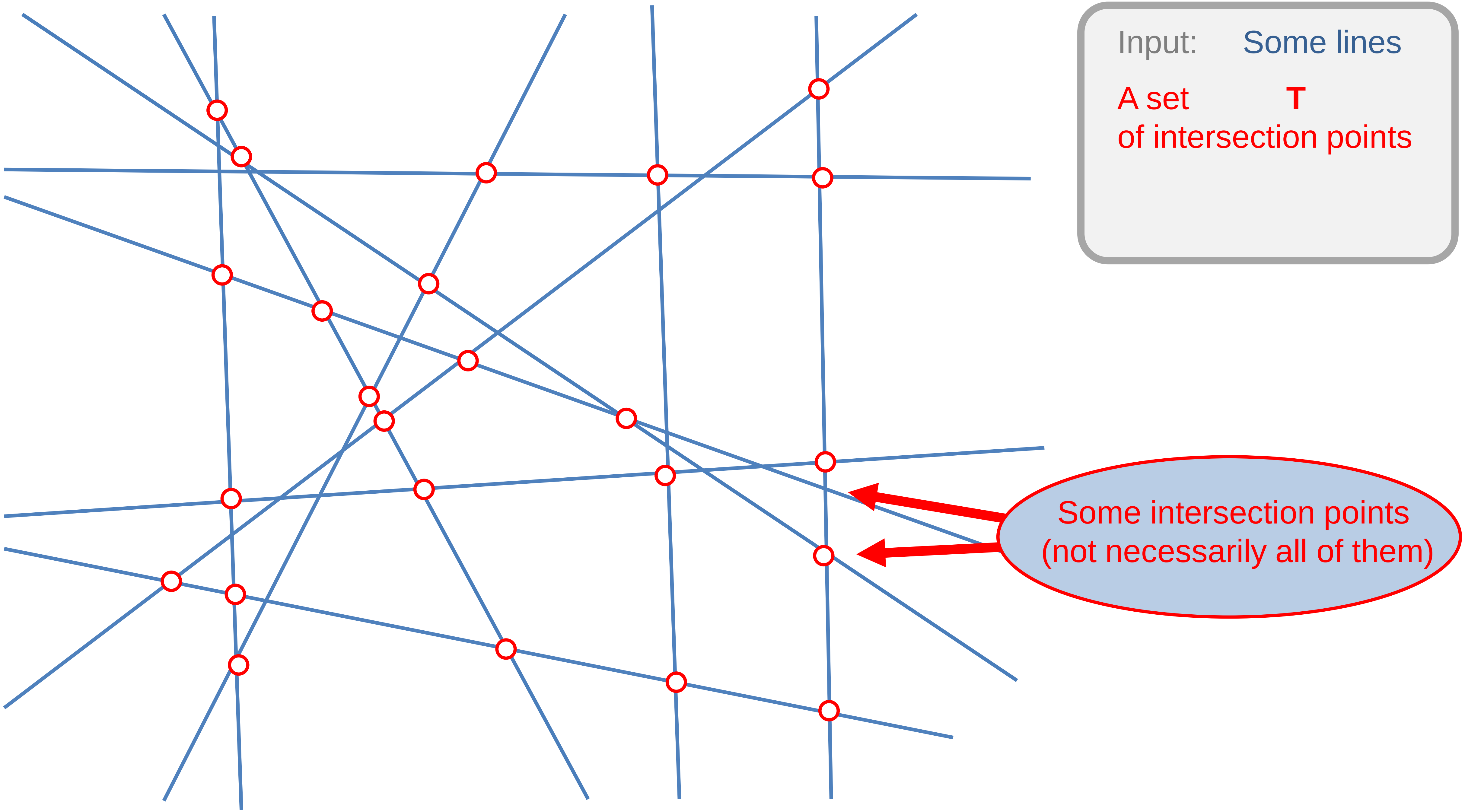
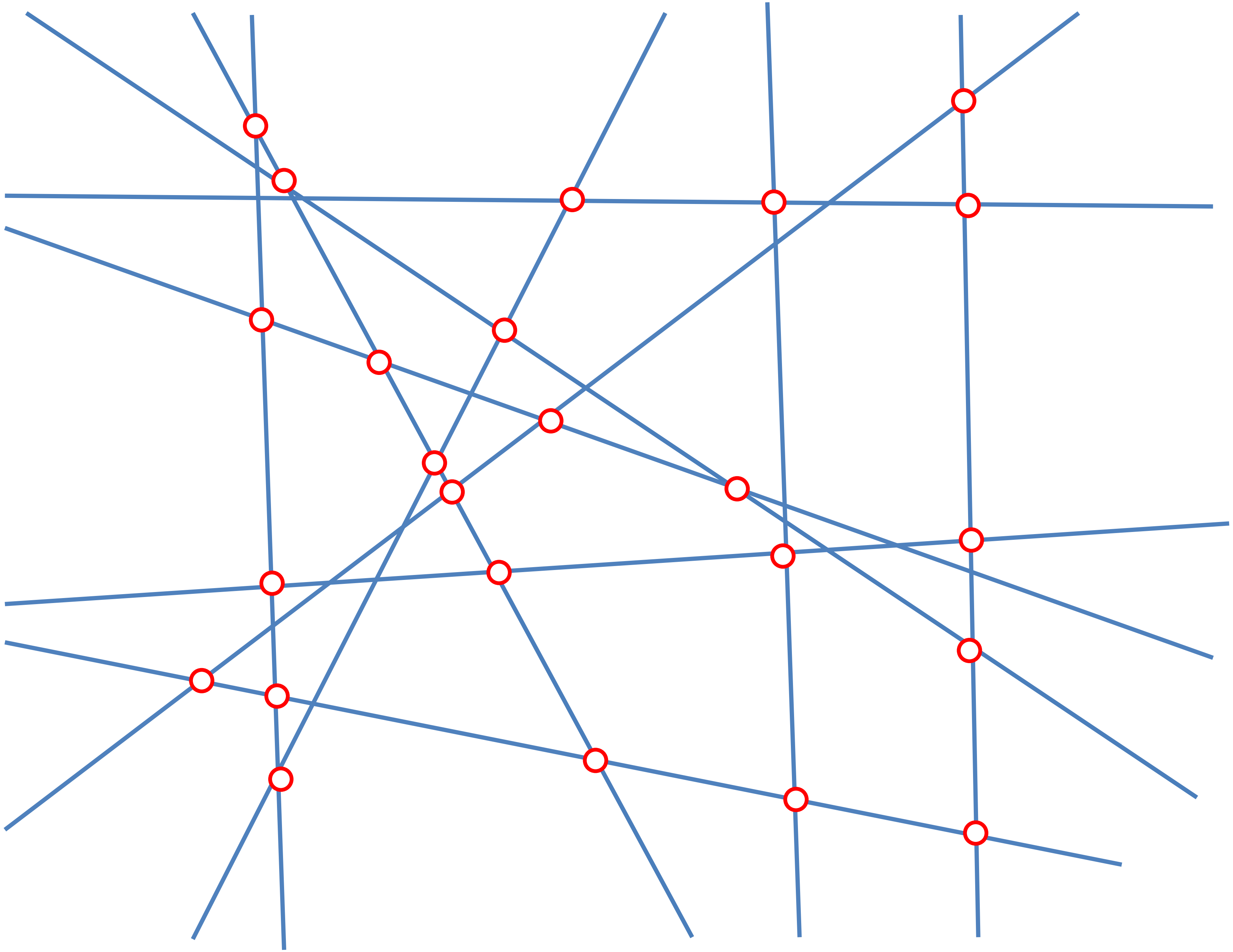


Some intersection points
(not necessarily all of them)



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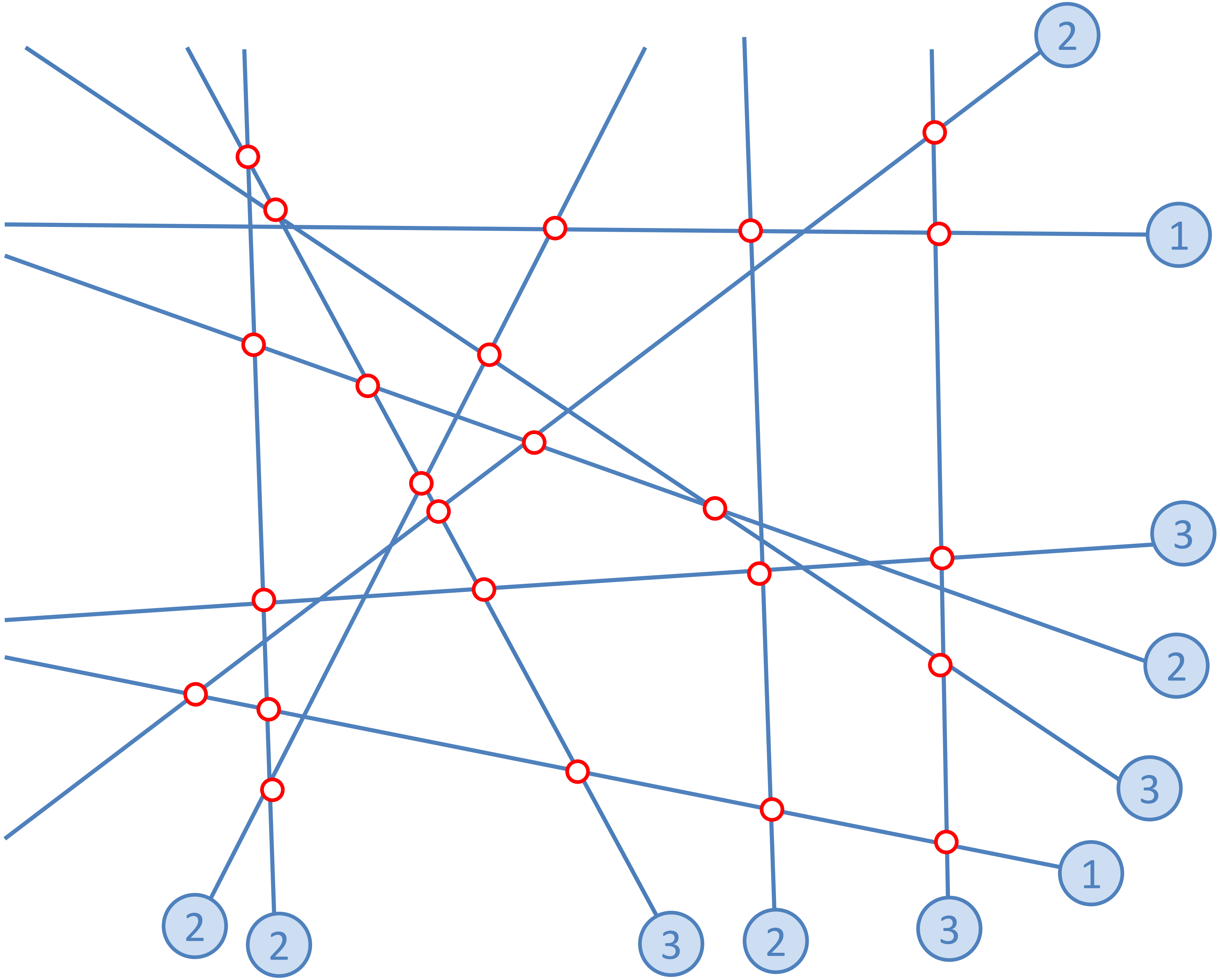

Input: Some lines

A set **T**
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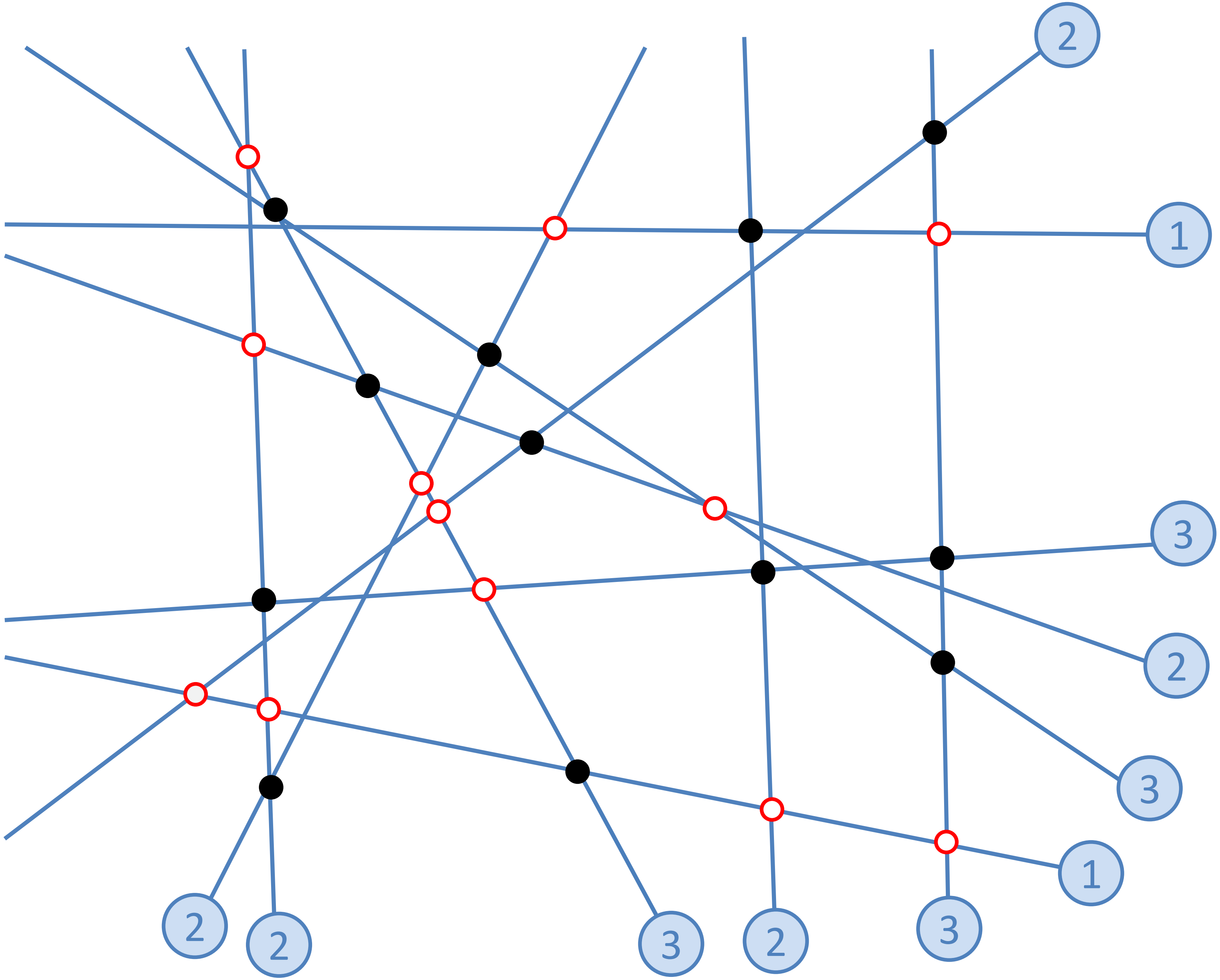
A prescribed number
of points per line

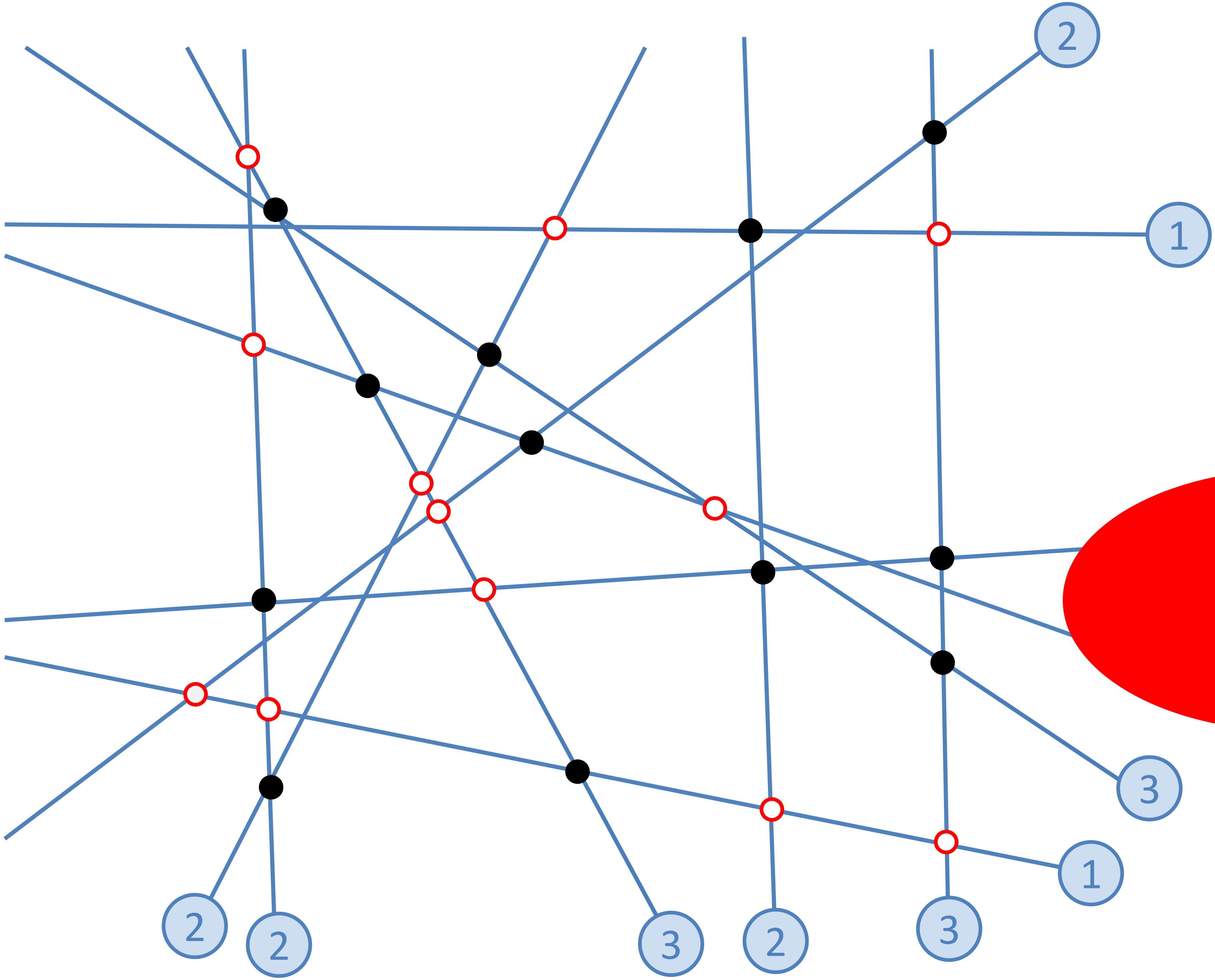


Input: Some lines

A set **T** of intersection points

A prescribed number of points per line

Output: Subset of **T** with prescribed number of points on each line



Input: Some lines

A set **T**
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NP-hard

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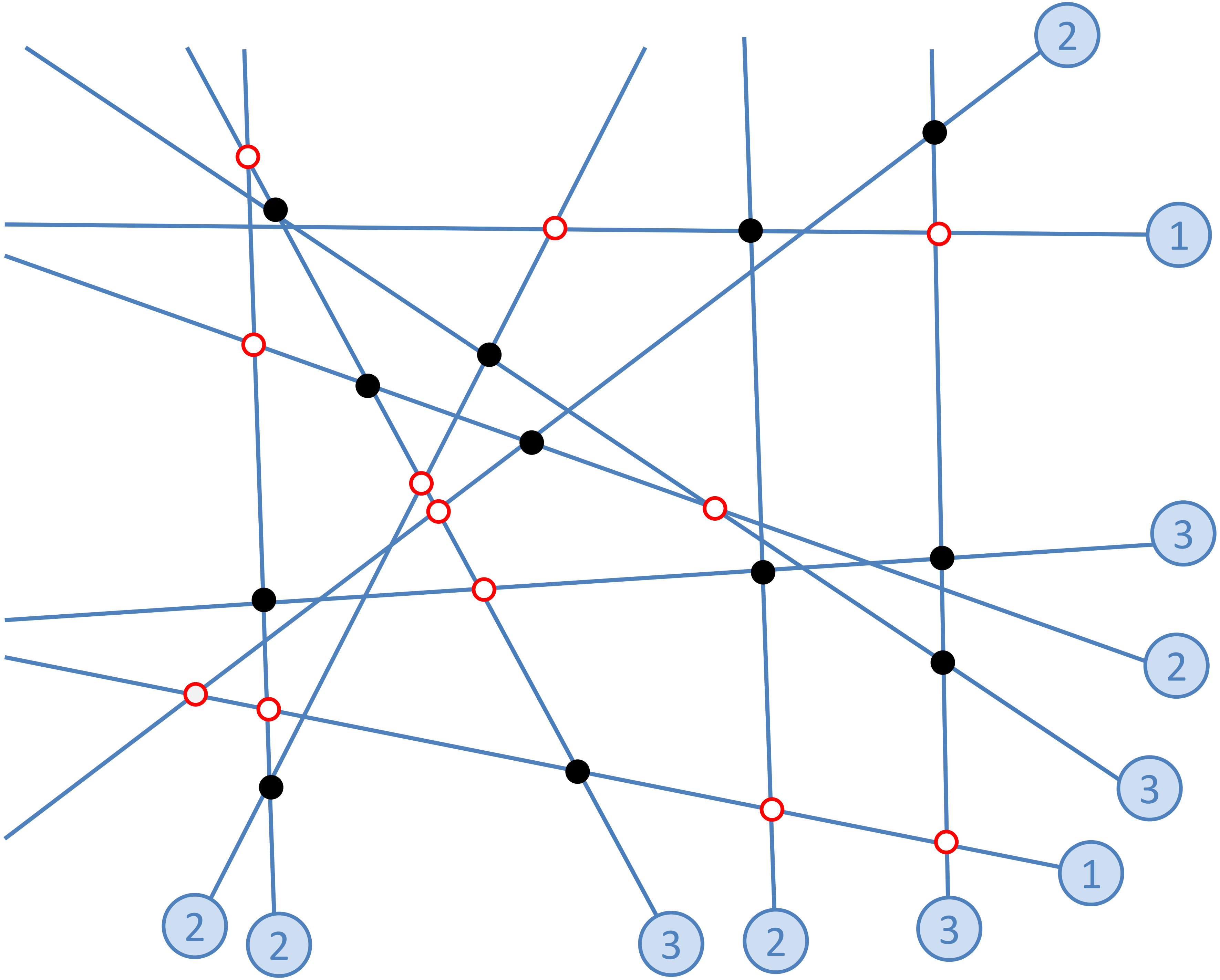

Input: Some lines

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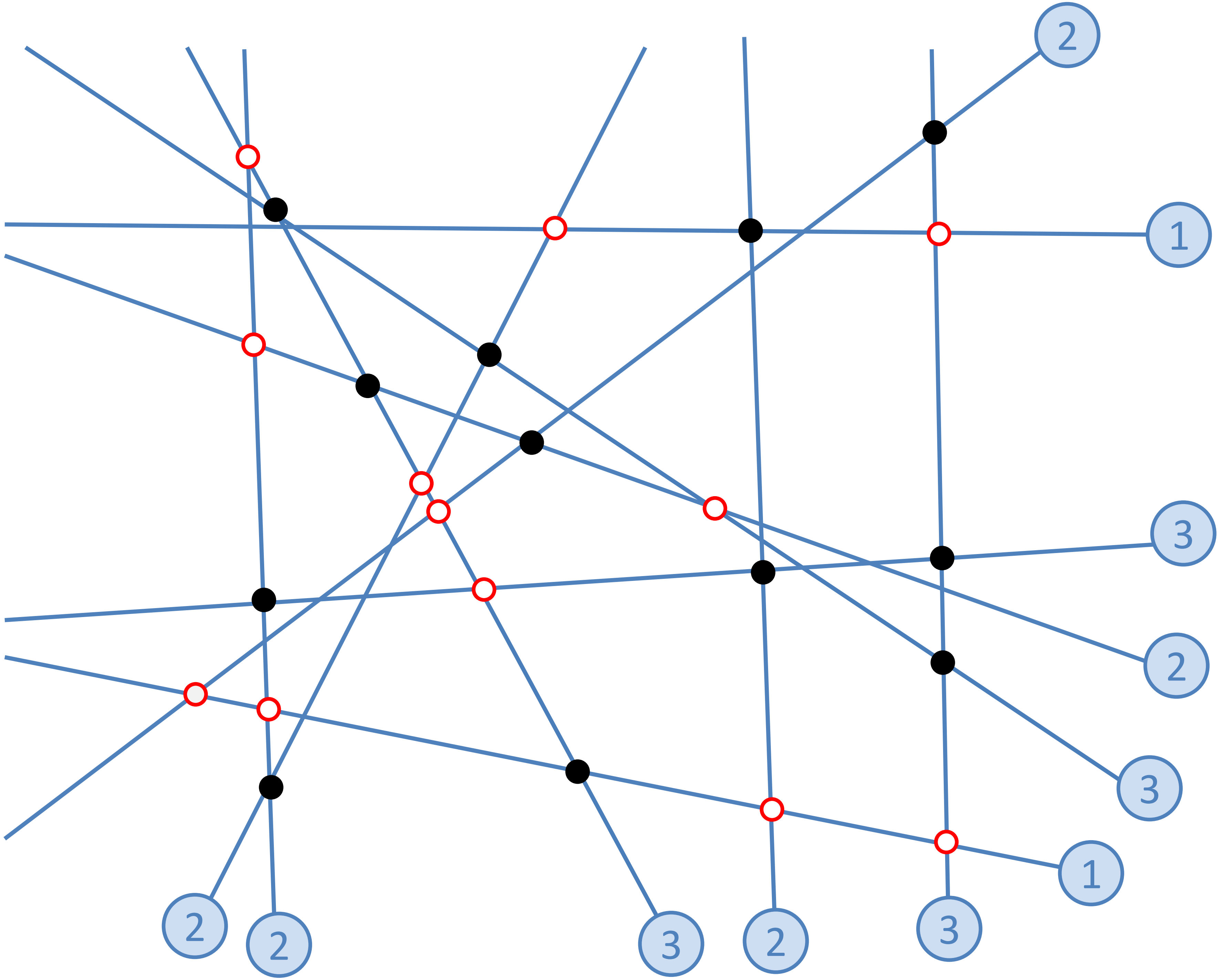
Computation

Output: Subset of **T** with prescribed number of points on each line



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A prescribed number of points per line

Algebraic rewriting:

Solve $A_T x = B$ with unknown vector x in $\{0,1\}^n$

Output: Subset of **T** with prescribed number of points on each line

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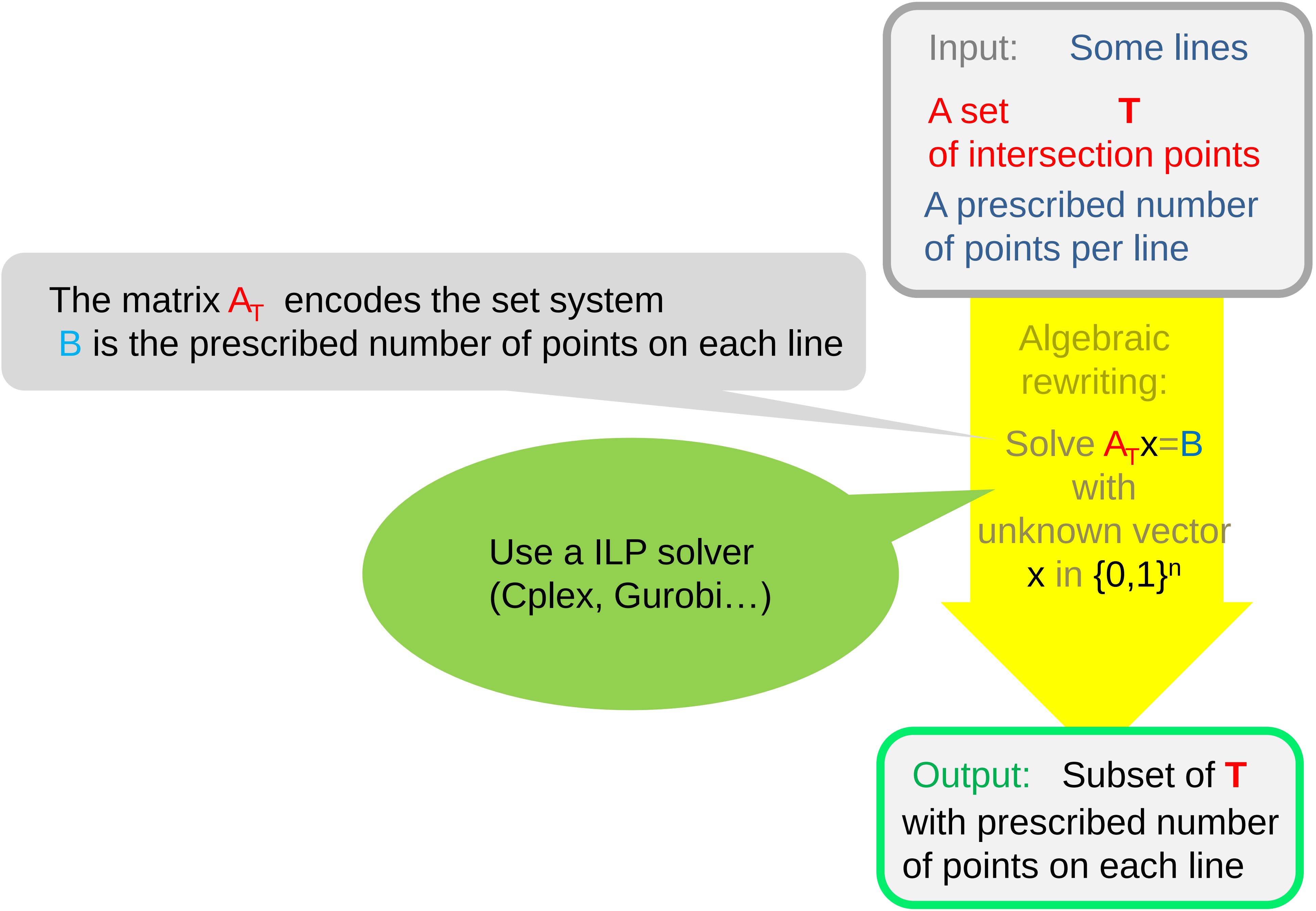
Algebraic
rewriting:

Solve $A_T x = B$
with
unknown vector
x in $\{0,1\}^n$

The matrix A_T encodes the set system
B is the prescribed number of points on each line

Output: Subset of **T**
with prescribed number
of points on each line



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Use a ILP solver
(Cplex, Gurobi…)

Output: Subset of **T**
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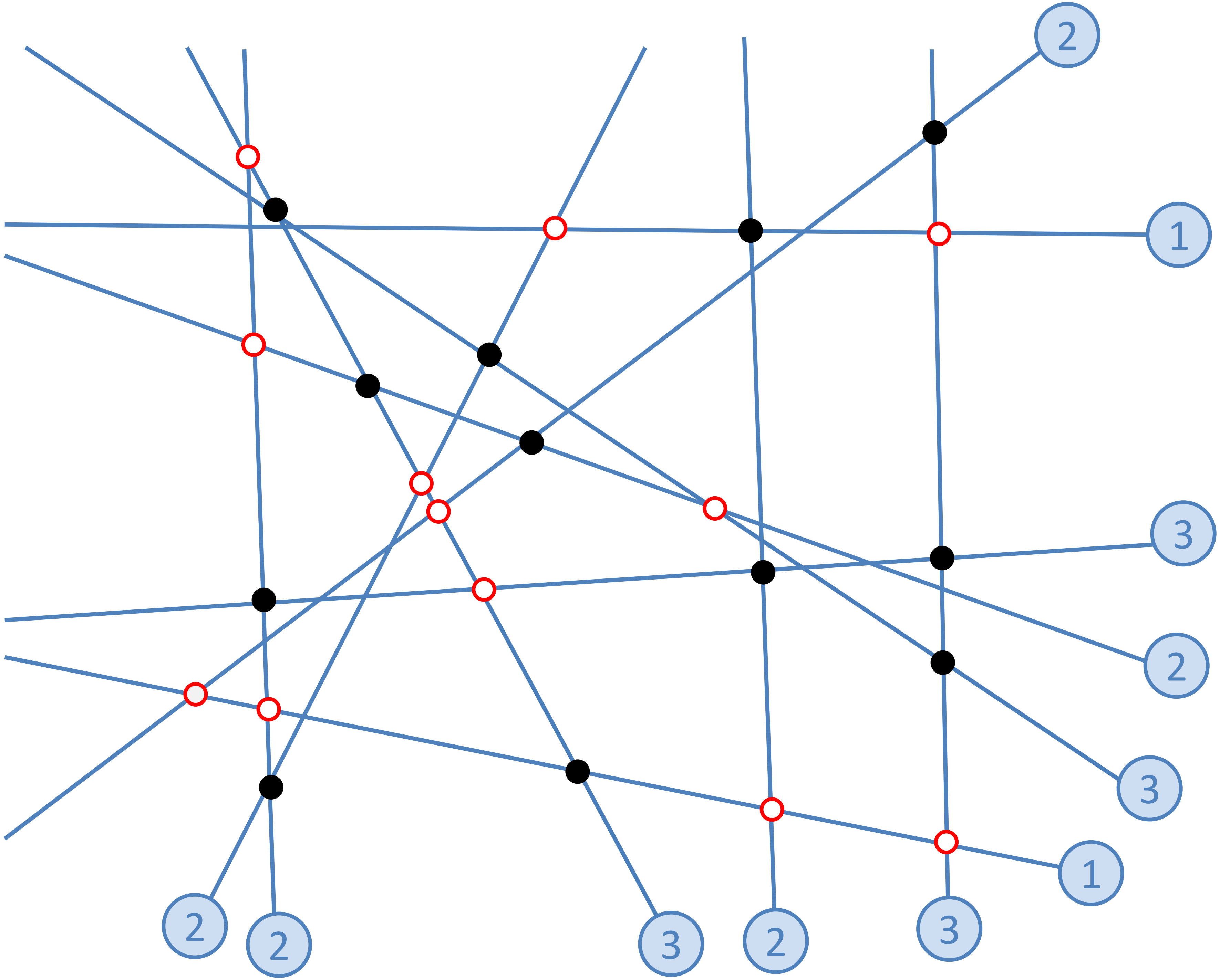
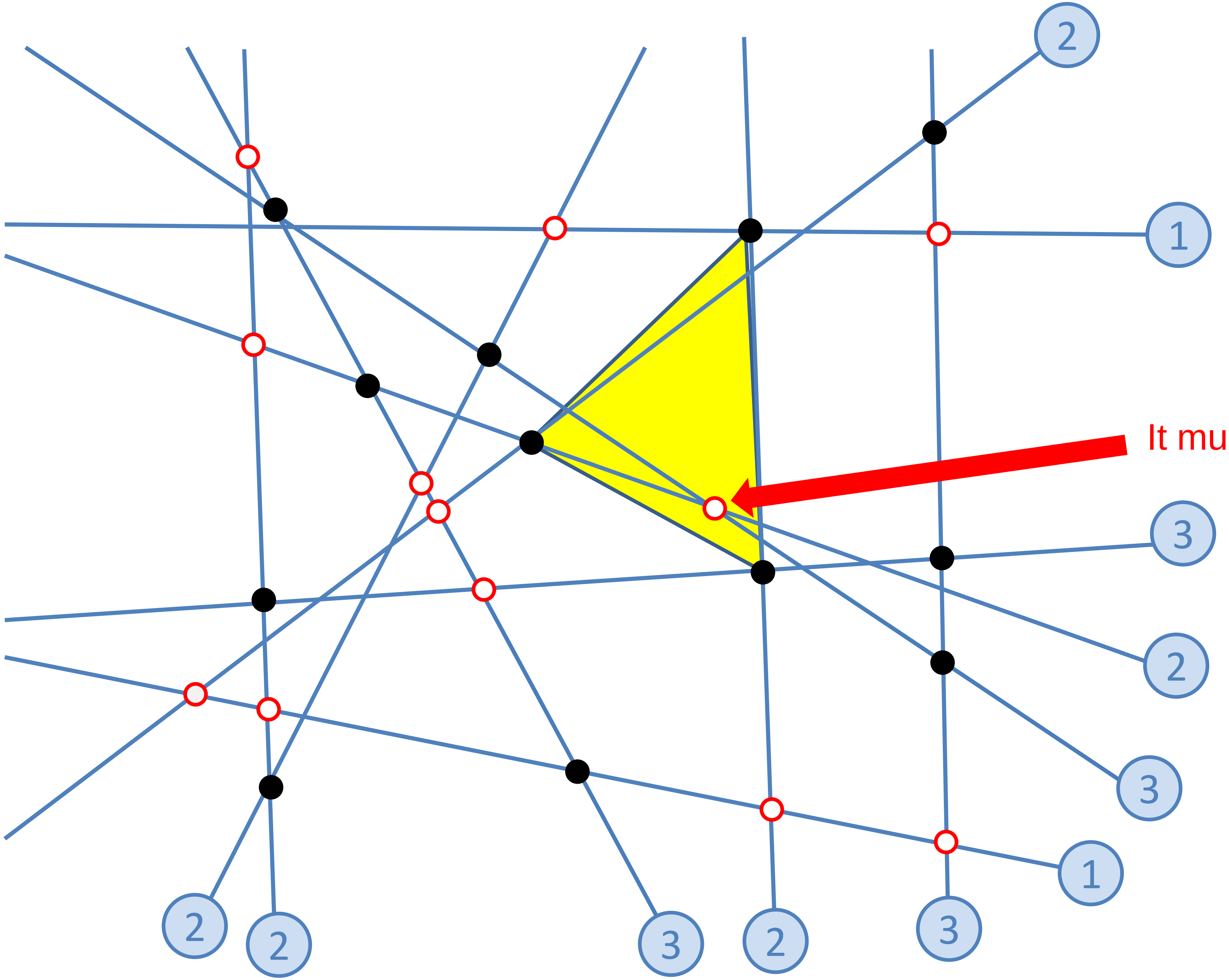


Add
geometric
or
topological constraints
(convexity…)



Add
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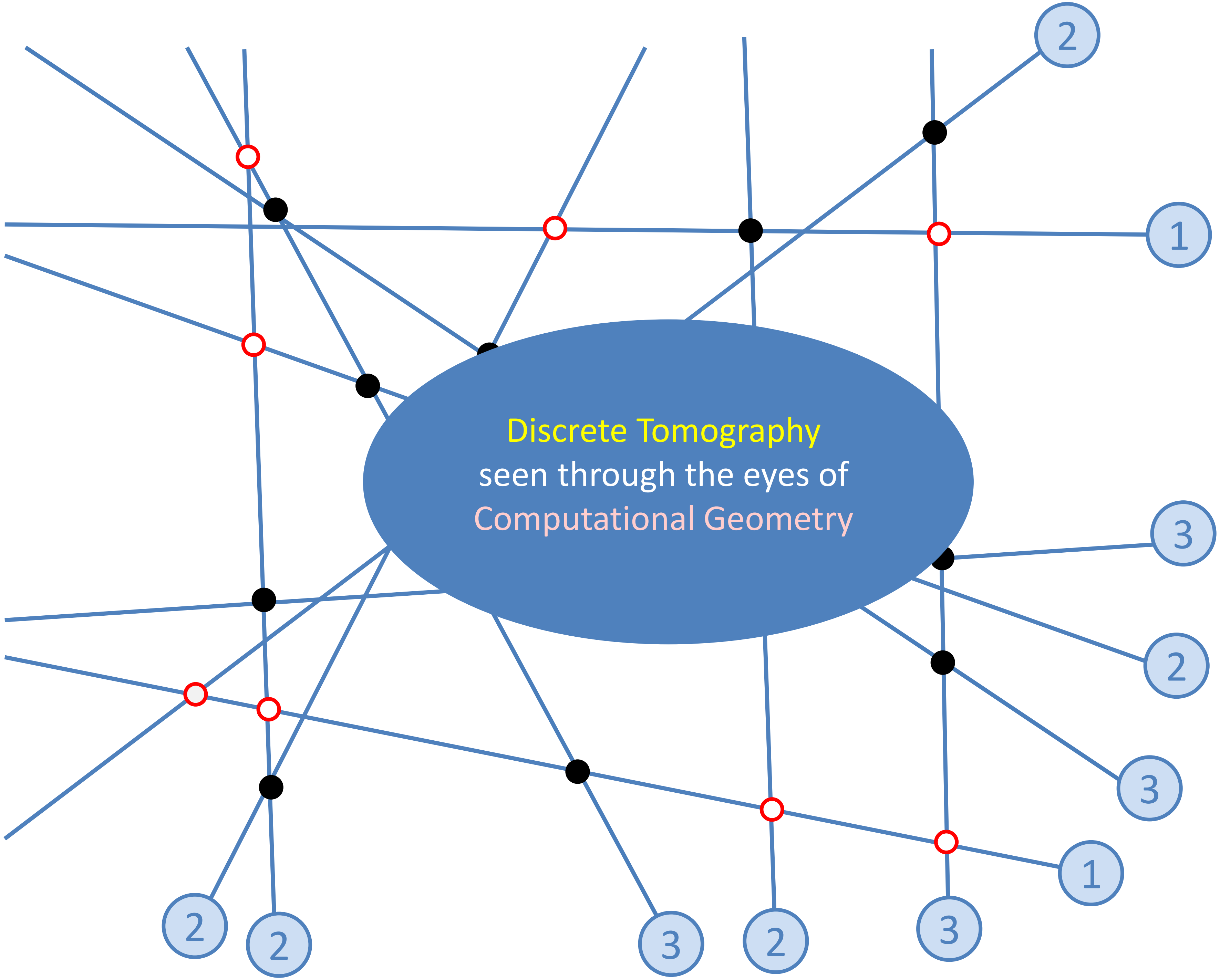
It must be in the solution



Discrete Tomography
seen through the eyes of
Computational Geometry

Input: Some lines

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Computation

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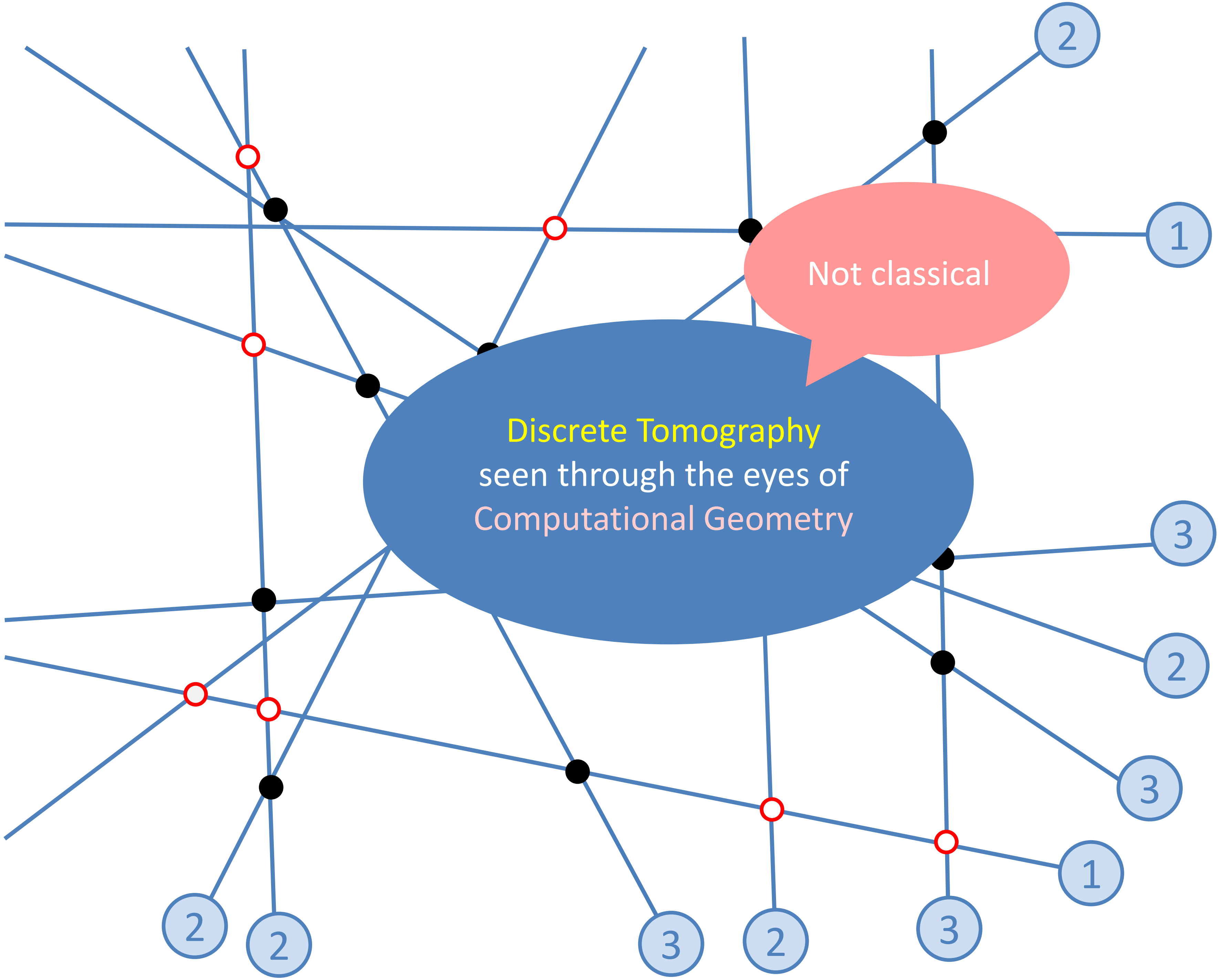

Discrete Tomography
seen through the eyes of
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Not classical

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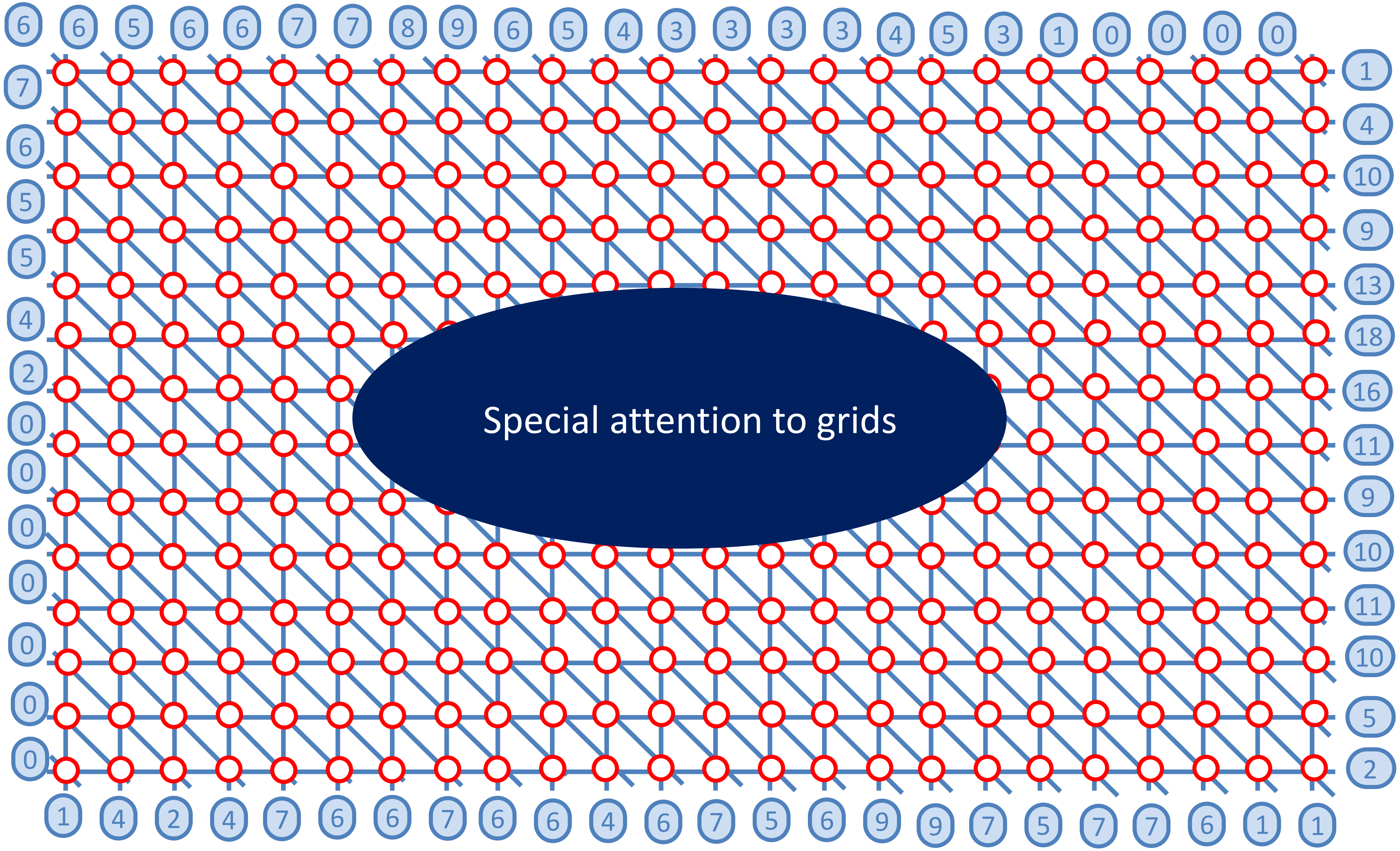

Special attention to grids

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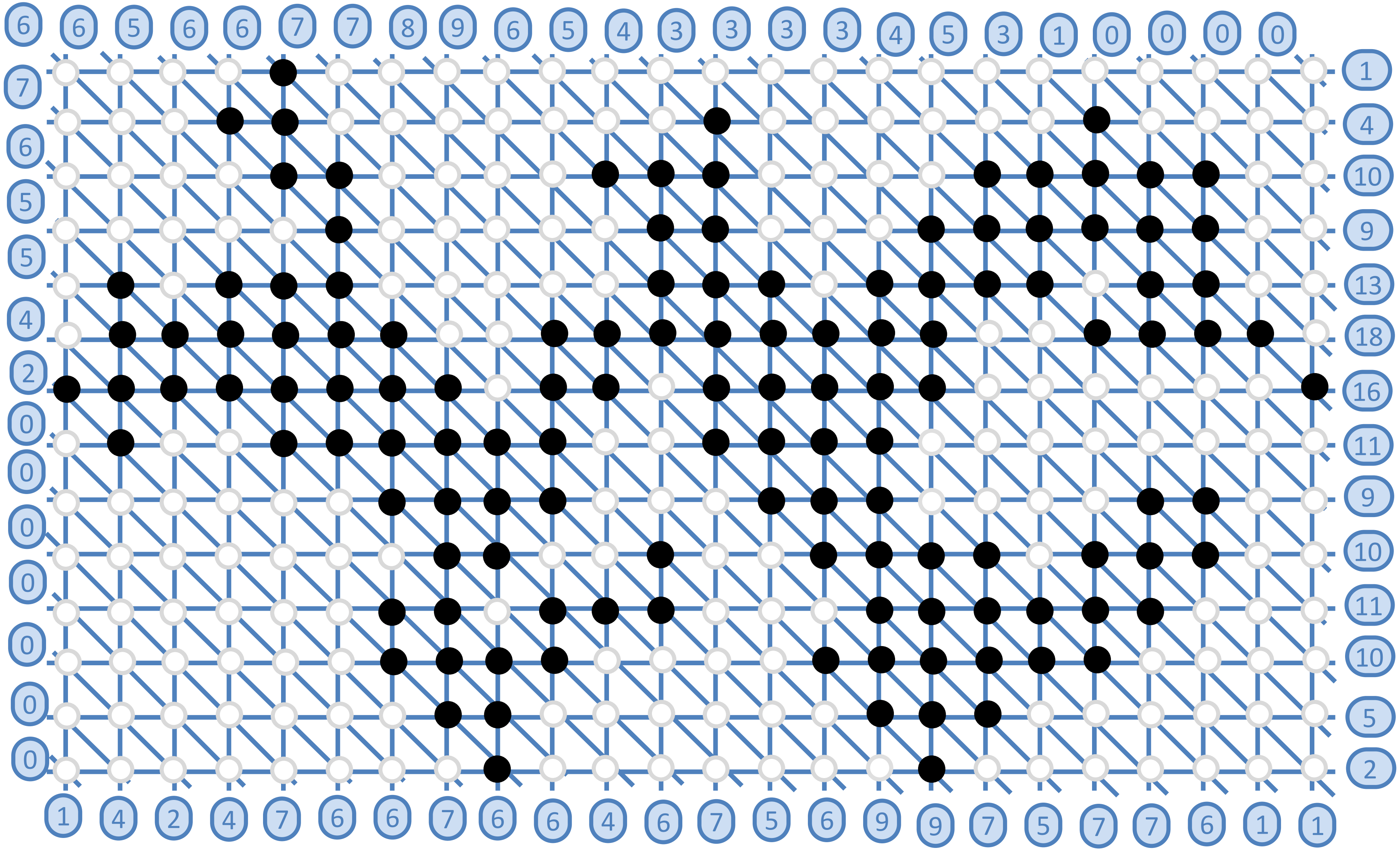

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Alberto Del Lungo
(1965-2003)

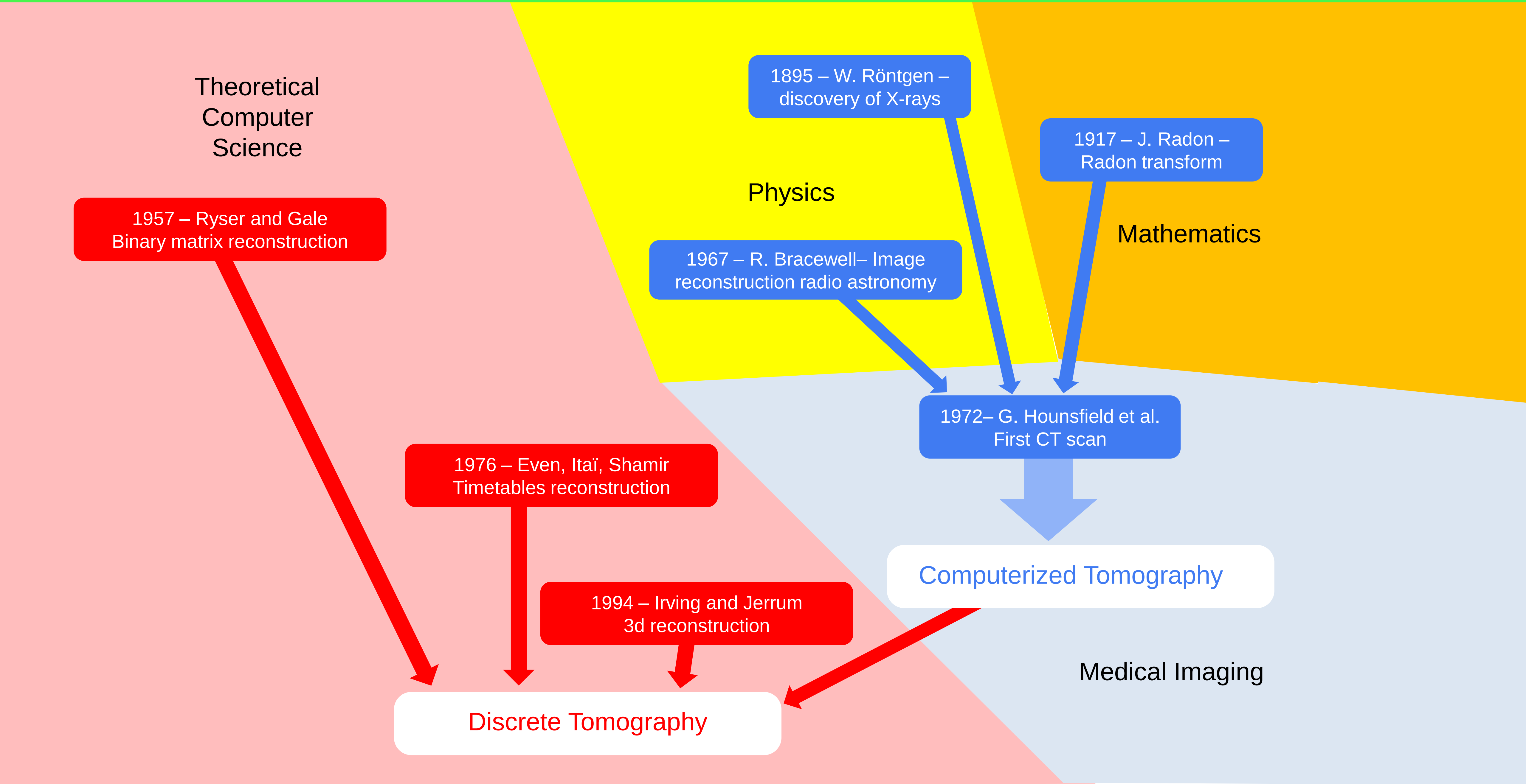
Attila Kuba
(1953-2006)

Alain Daurat
(1973-2010)

Maurice Nivat
(1937-2017)

① The Origins of Discrete Tomography

Multidisciplinary area



Theoretical Computer Science

1957 – Ryser and Gale
Binary matrix reconstruction

1976 – Even, Itaï, Shamir
Timetables reconstruction

1994 – Irving and Jerrum
3d reconstruction

Discrete Tomography

Physics

1895 – W. Röntgen – discovery of X-rays

1967 – R. Bracewell– Image reconstruction radio astronomy

Mathematics

1917 – J. Radon – Radon transform

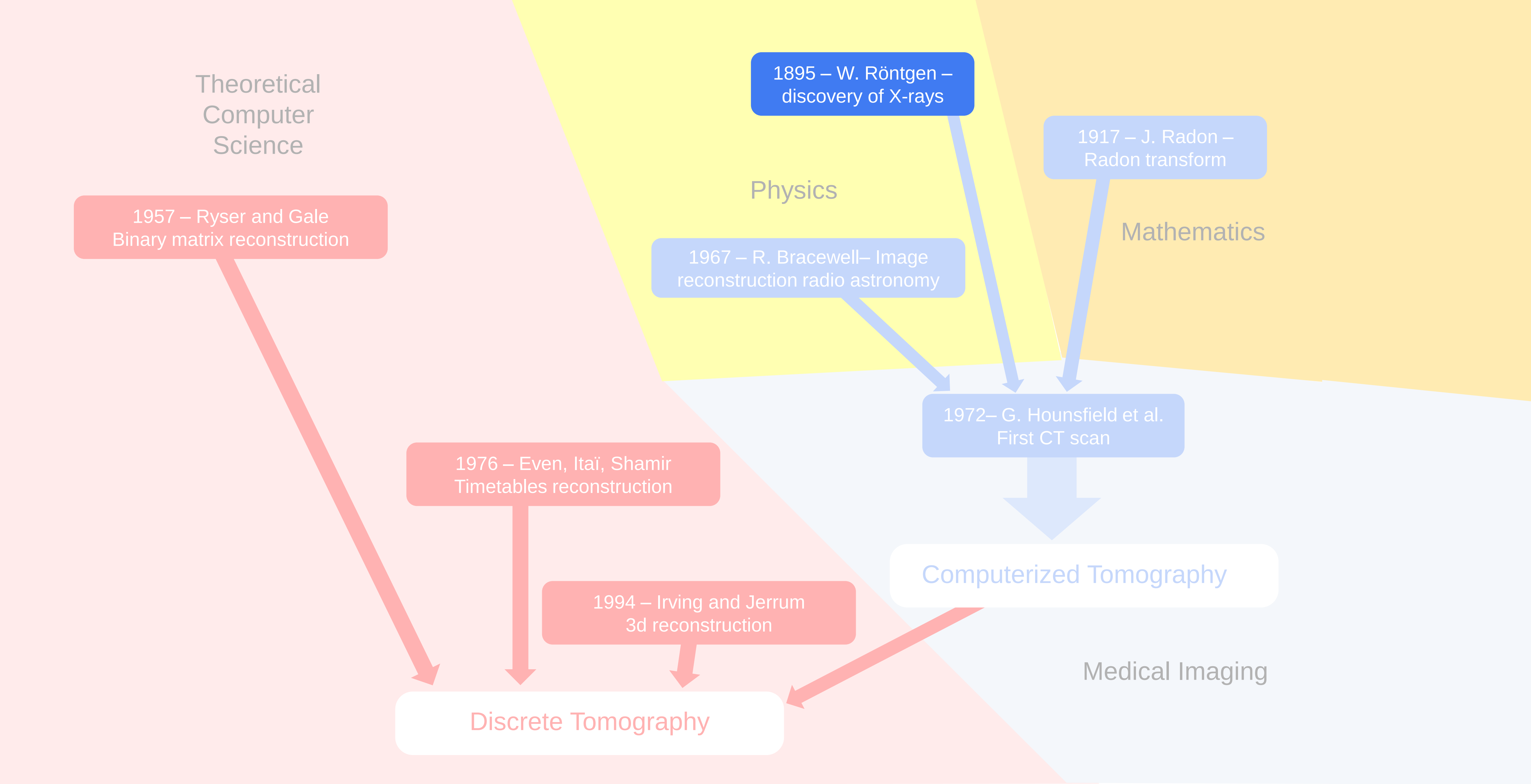
1972– G. Hounsfield et al.
First CT scan

Computerized Tomography

Medical Imaging

① The Origins of ~~Discrete~~ Tomography

Starting point



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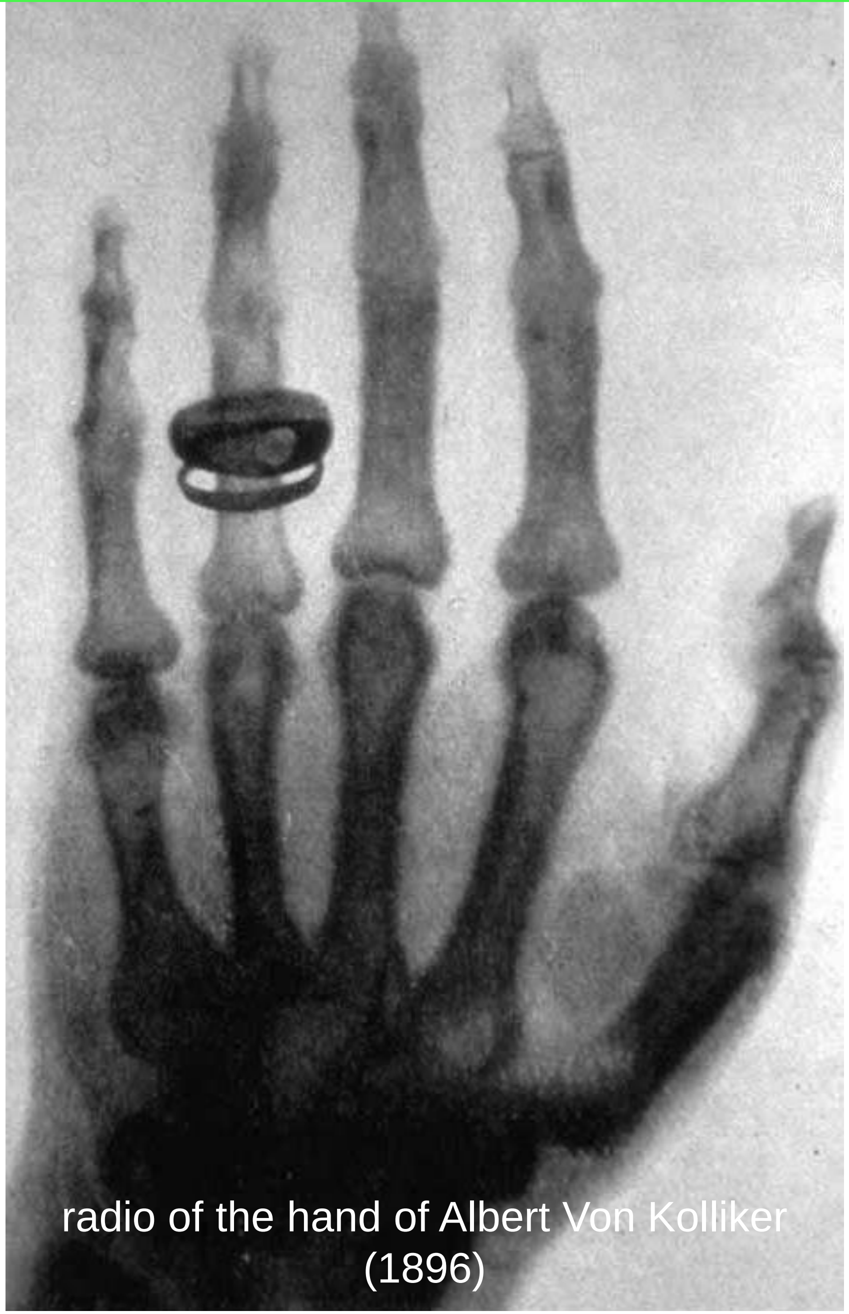


radio of the hand of Albert Von Kolliker (1896)

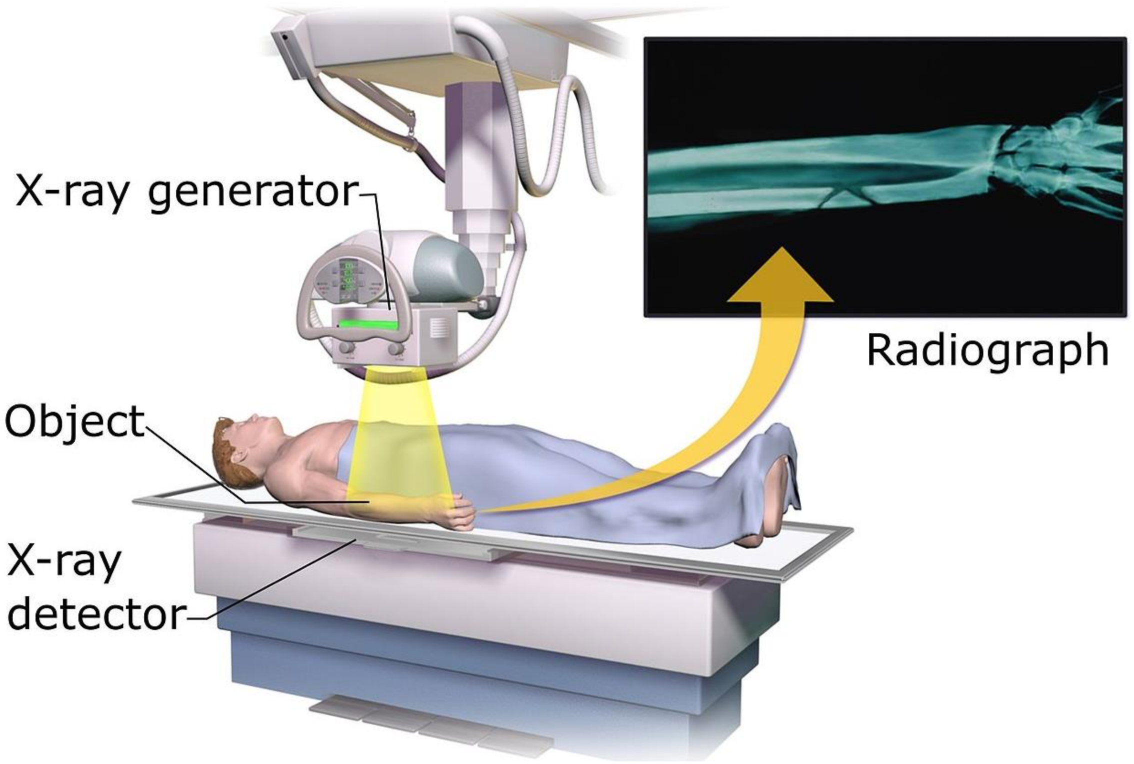
radio of the hand of his wife (1895)

William Röntgen

1895: William Röntgen discovers the X-rays and makes the first Röntgenograms

① The Origins of ~~Discrete~~ Tomography

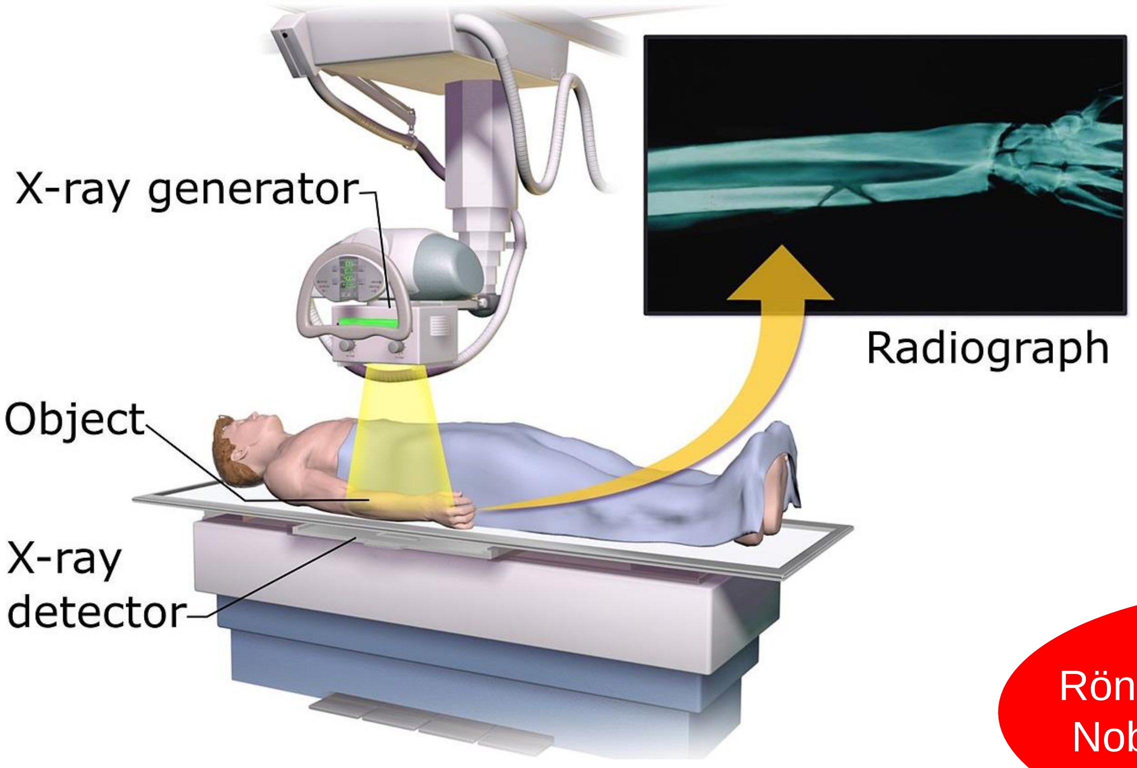
Medical Imaging / X-rays



William Röntgen

The discovery of X-rays and radiography marks the birth of Medical Imaging.

Nobel prize



Röntgen recieved the first Nobel prize of Medecine

William Röntgen

The discovery of X-rays and radiography marks the birth of Medical Imaging.

① The Origins of ~~Discrete~~ Tomography

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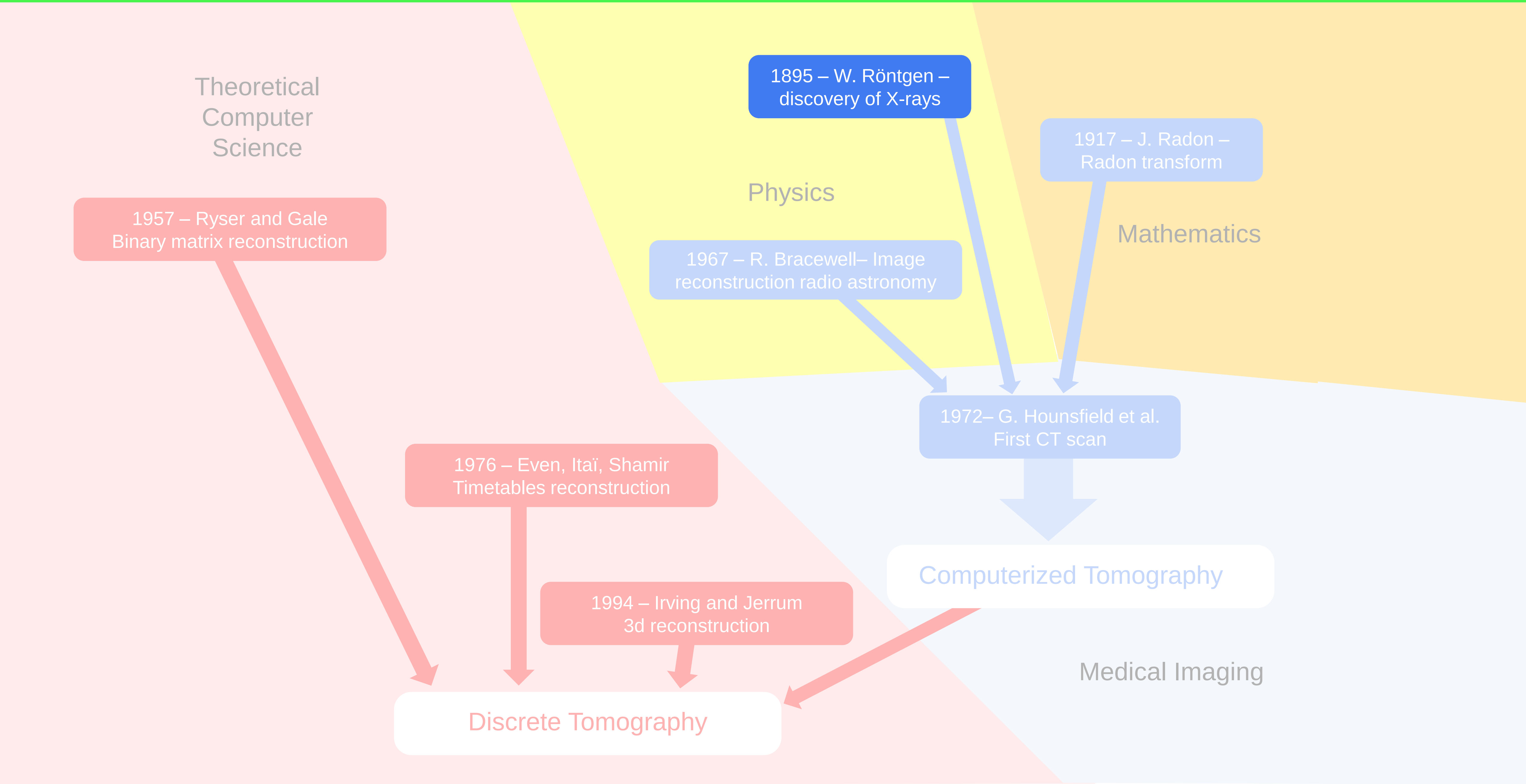

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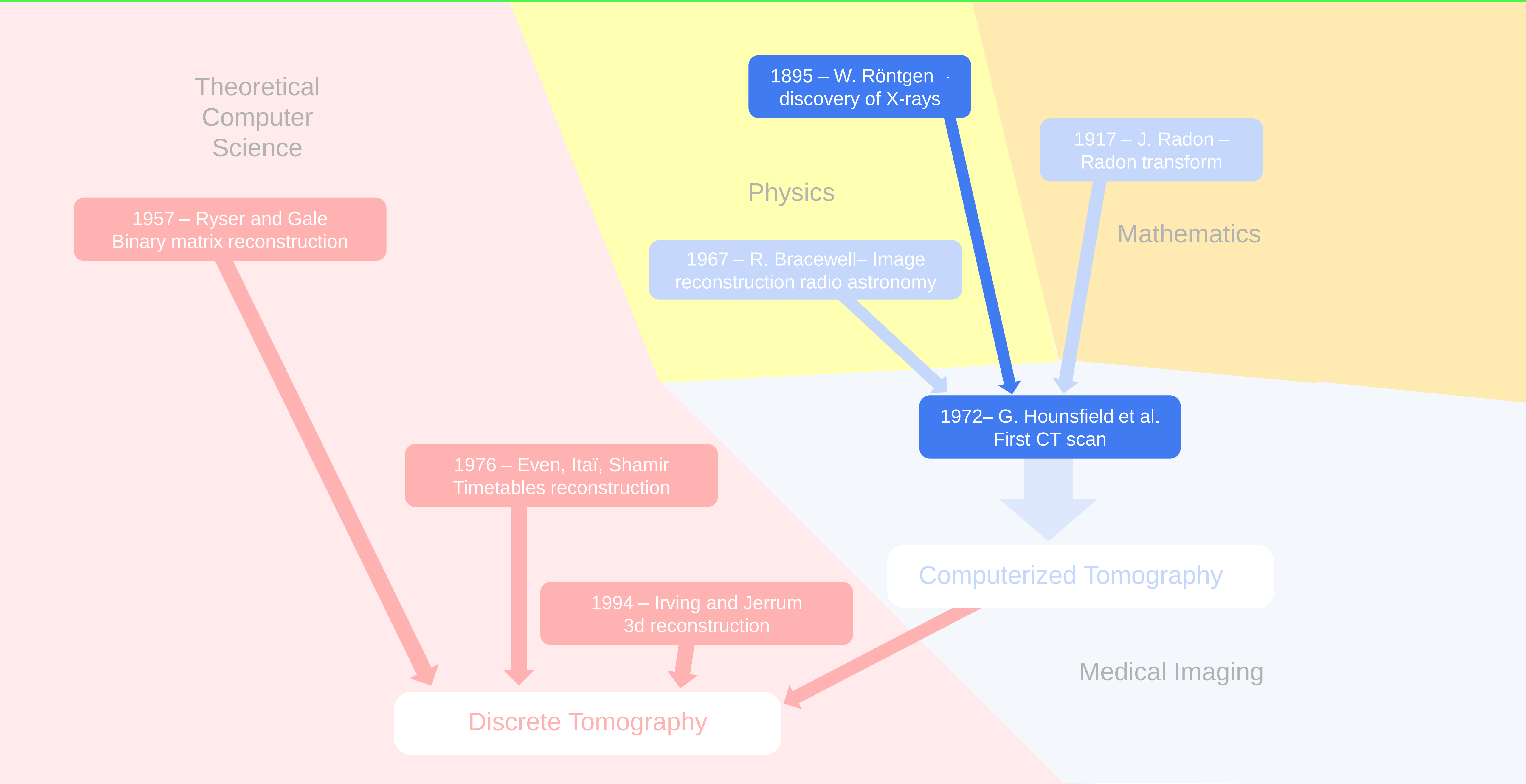
① The Origins of ~~Discrete~~ Tomography

First scanners



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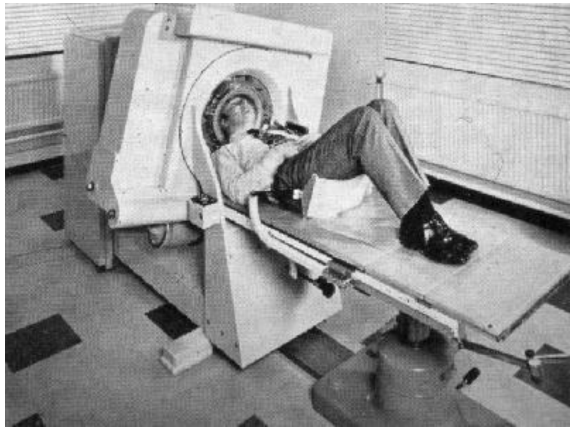
1979 Nobel Prize of Medecine

A.M. Cormack

G.H. Hounsfield



The EMI CT-scanner
(the first scanner)

The first CT scanner has been developped by Allan McLeod Cormack and Godfrey Hounsfield in 1971.

① The Origins of ~~Discrete~~ Tomography

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A.M. Cormack

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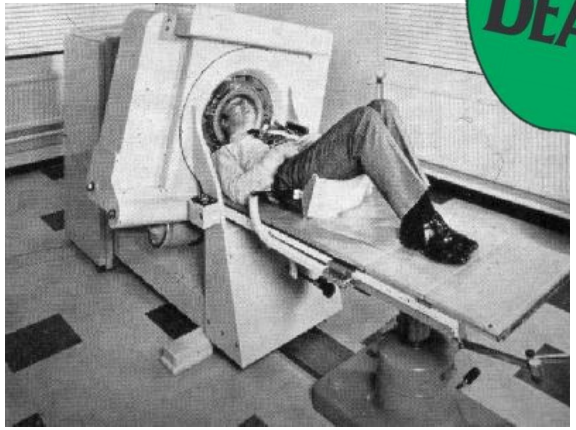

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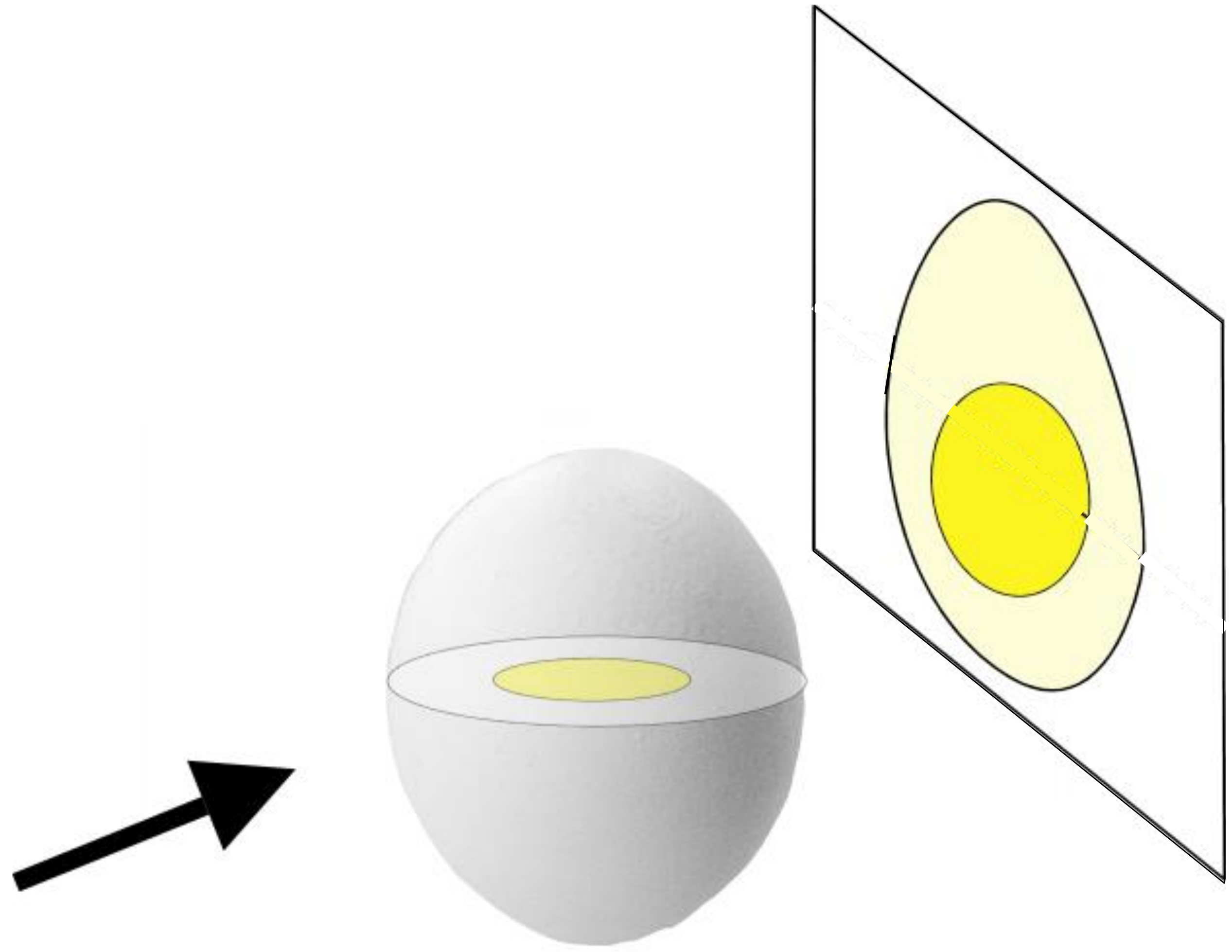
No Omelette

How can we see the interior of an egg without breaking it ?

37

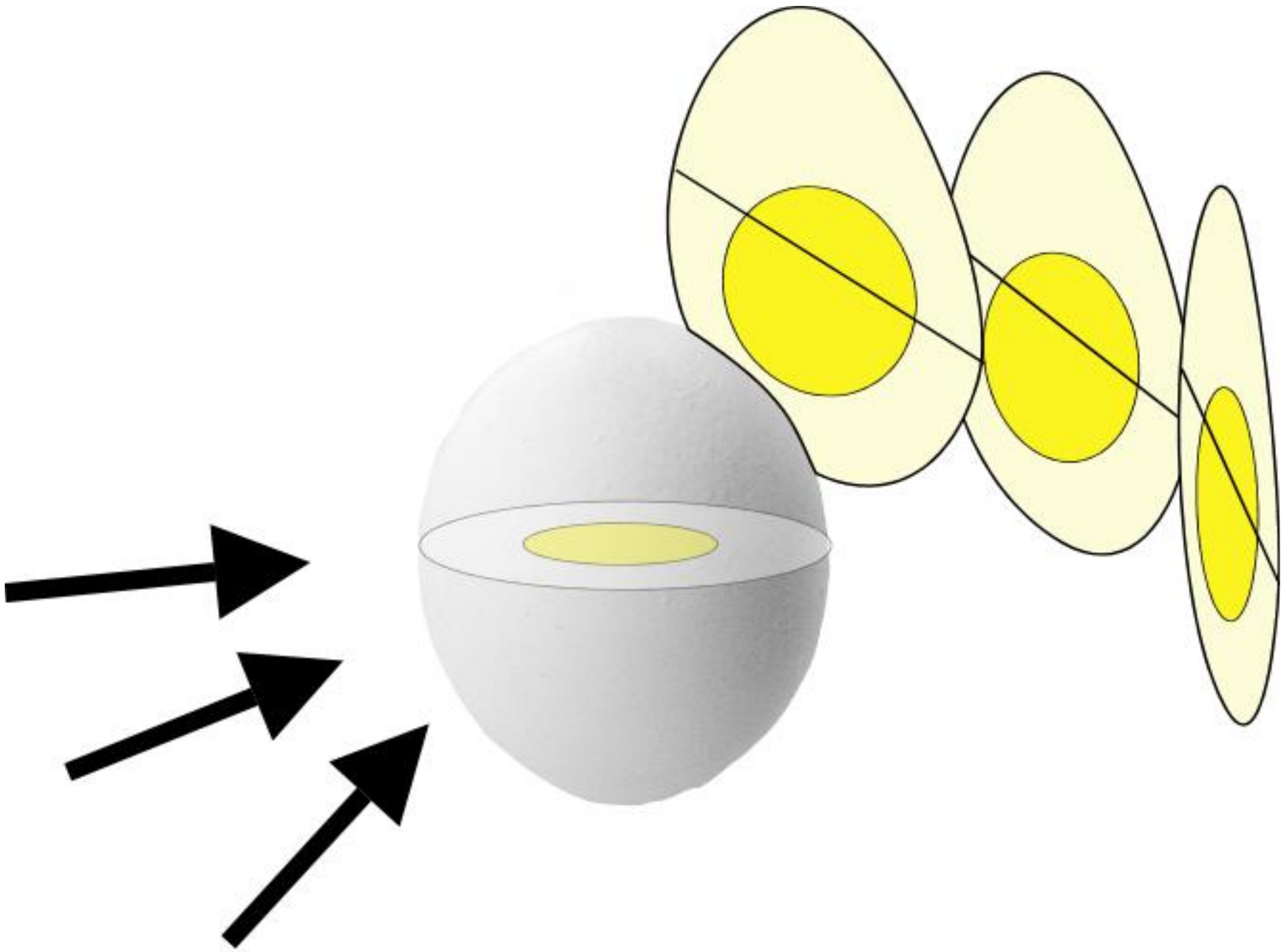
No Omelette



With one X-ray

① The Origins of ~~Discrete~~ Tomography

Multiple X-rays



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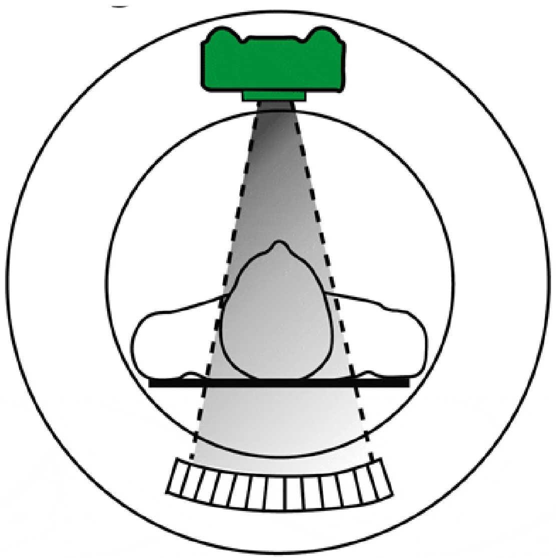
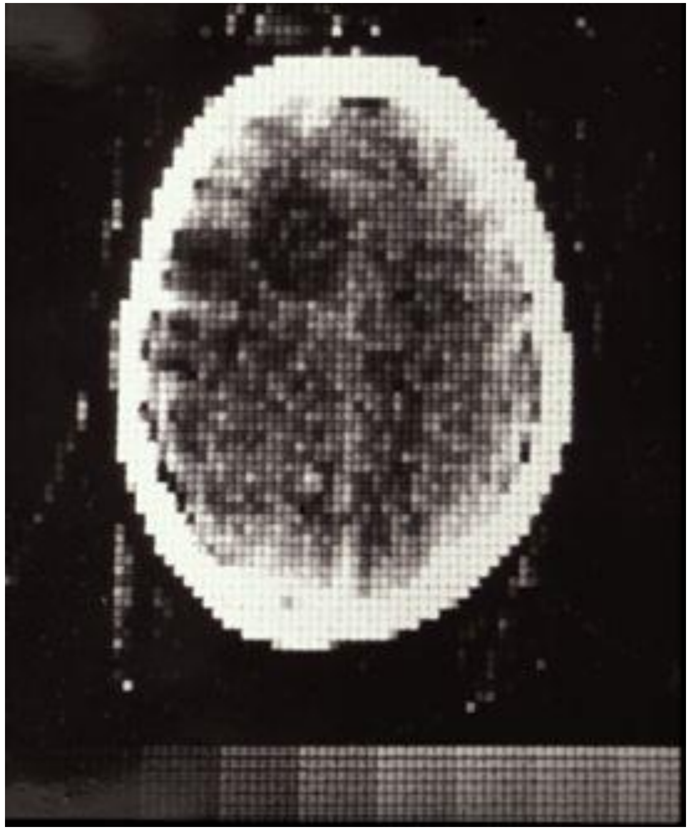
① The Origins of ~~Discrete~~ Tomography

Inverse problem



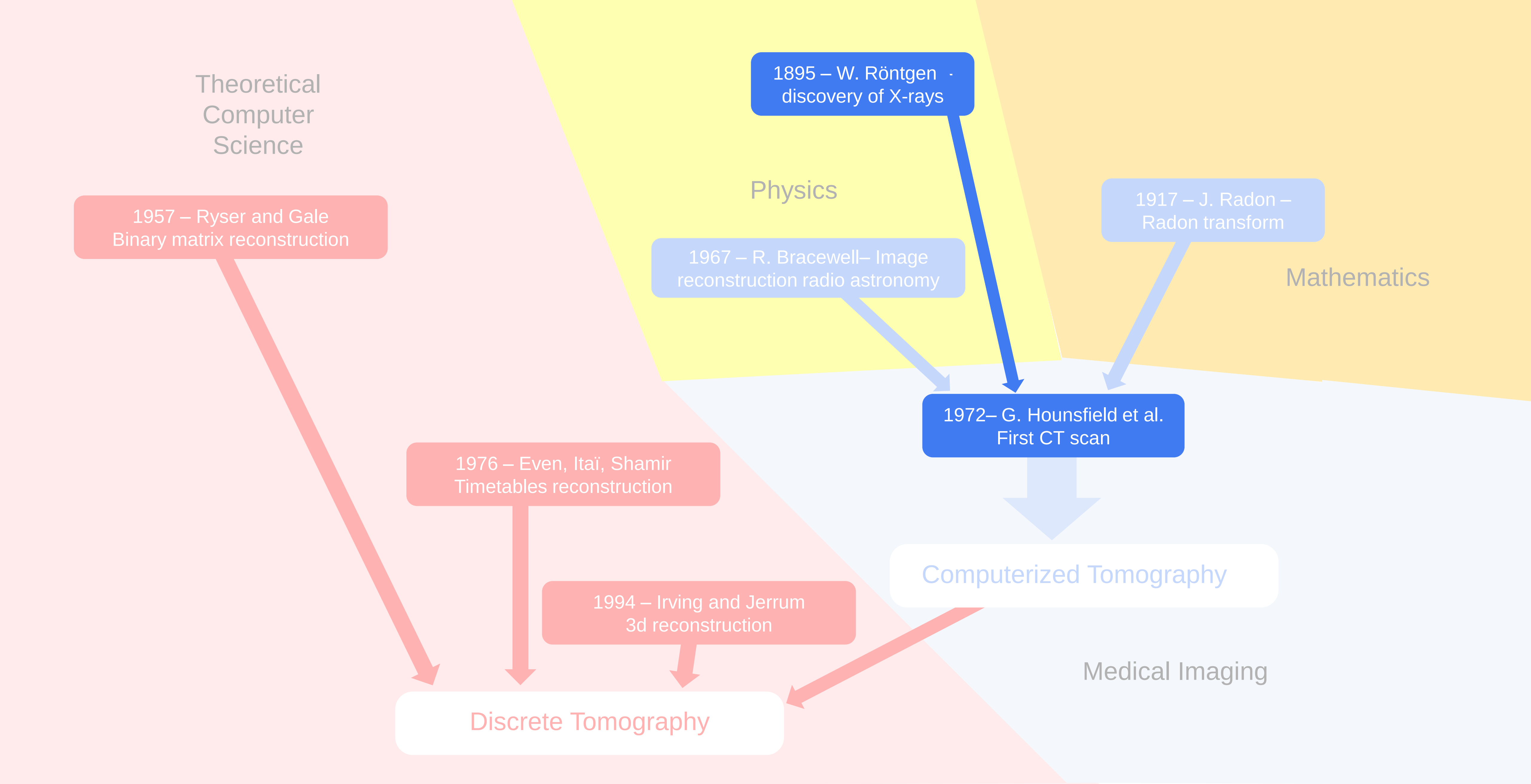
Reconstruction problem:
Compute the image from the X-rays



The first clinical scan
with an EMI scanner
Atkinson Morley hospital (1971)

The Origins of ~~Discrete~~ Tomography

First scanners



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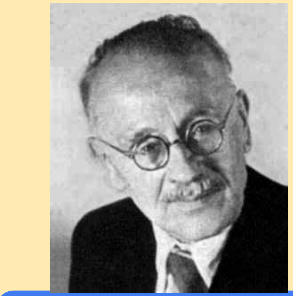
Johannes Radon



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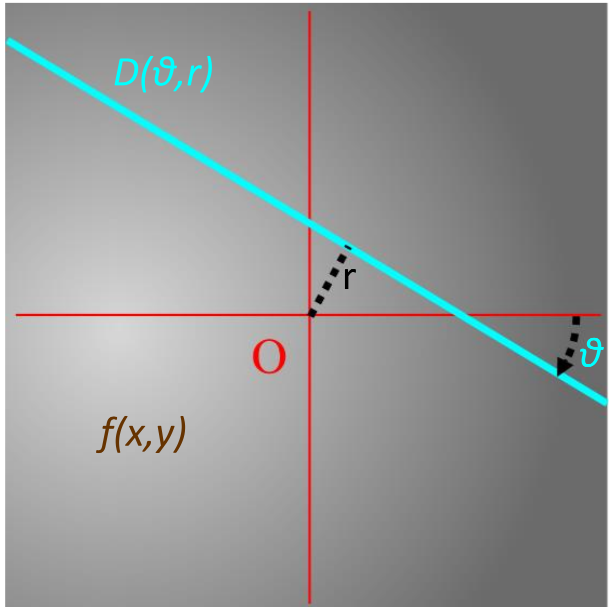
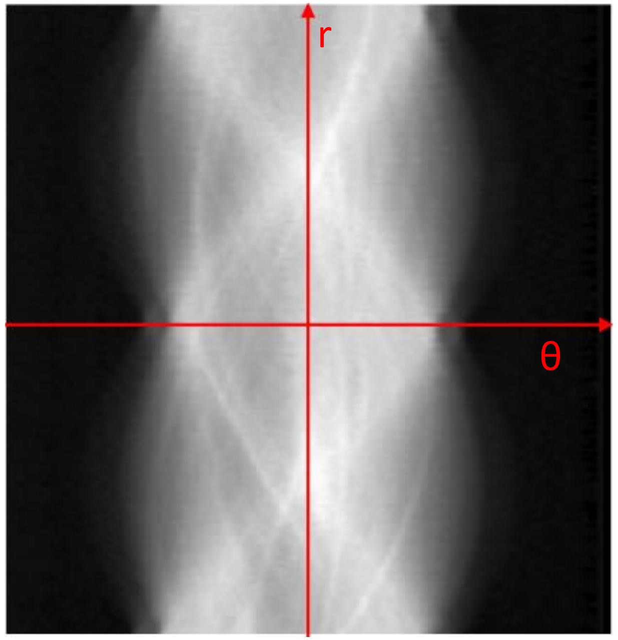
Radon transform



f(x,y) is represented by grey levels

Radon transform $\mathfrak{R}$

$$\mathfrak{R}\, f\,(\theta,r) = {}_{D_r}\!\int f(x,y)\,dl$$

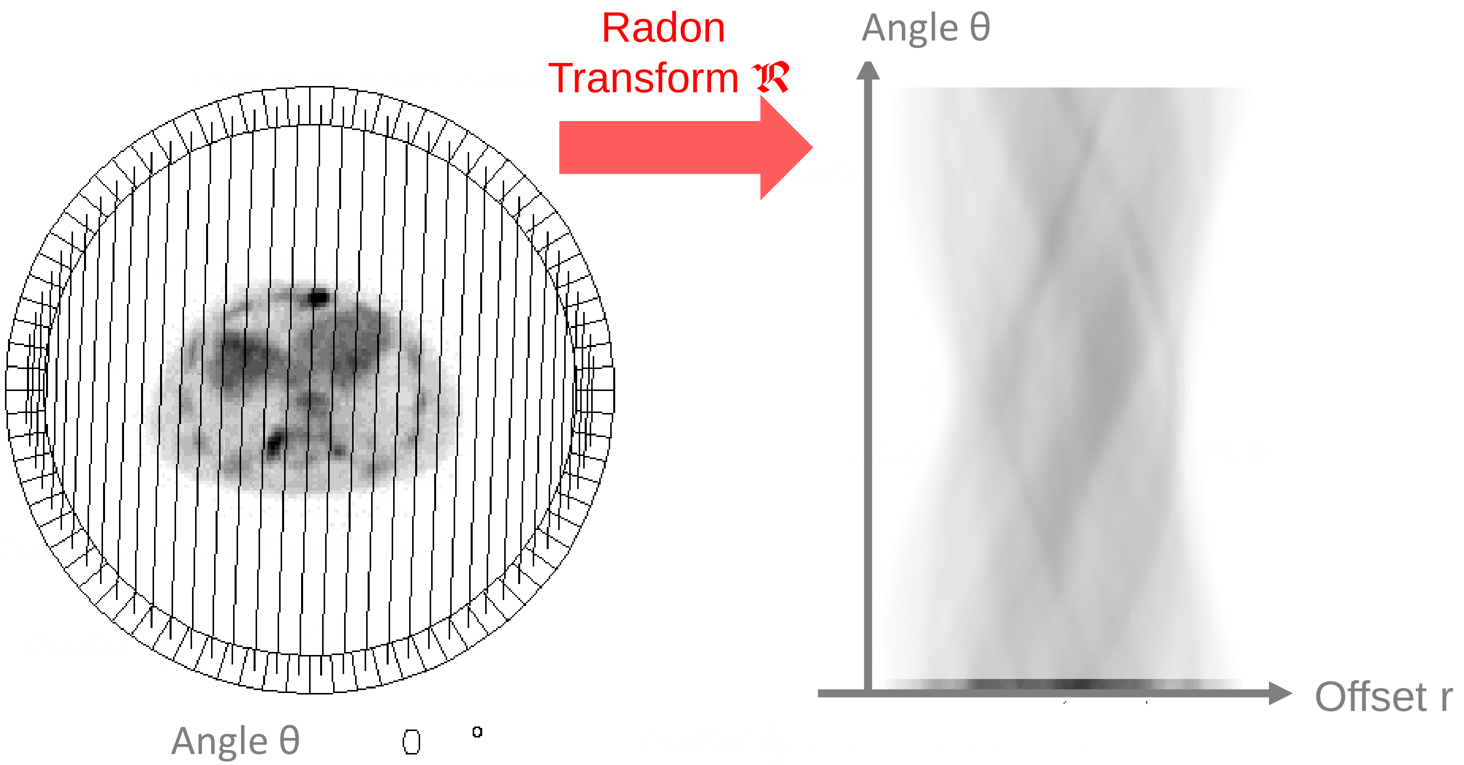


Radon transform of *f(x,y)* : $\mathfrak{R}$ *f (ϑ,r)*

$\mathfrak{R}$ *f (ϑ,r)* is represented by its grey level

Johann Radon (1887-1956) introduced in 1917
the mathematical transform now called the *Radon transform*.

① The Origins of ~~Discrete~~ Tomography



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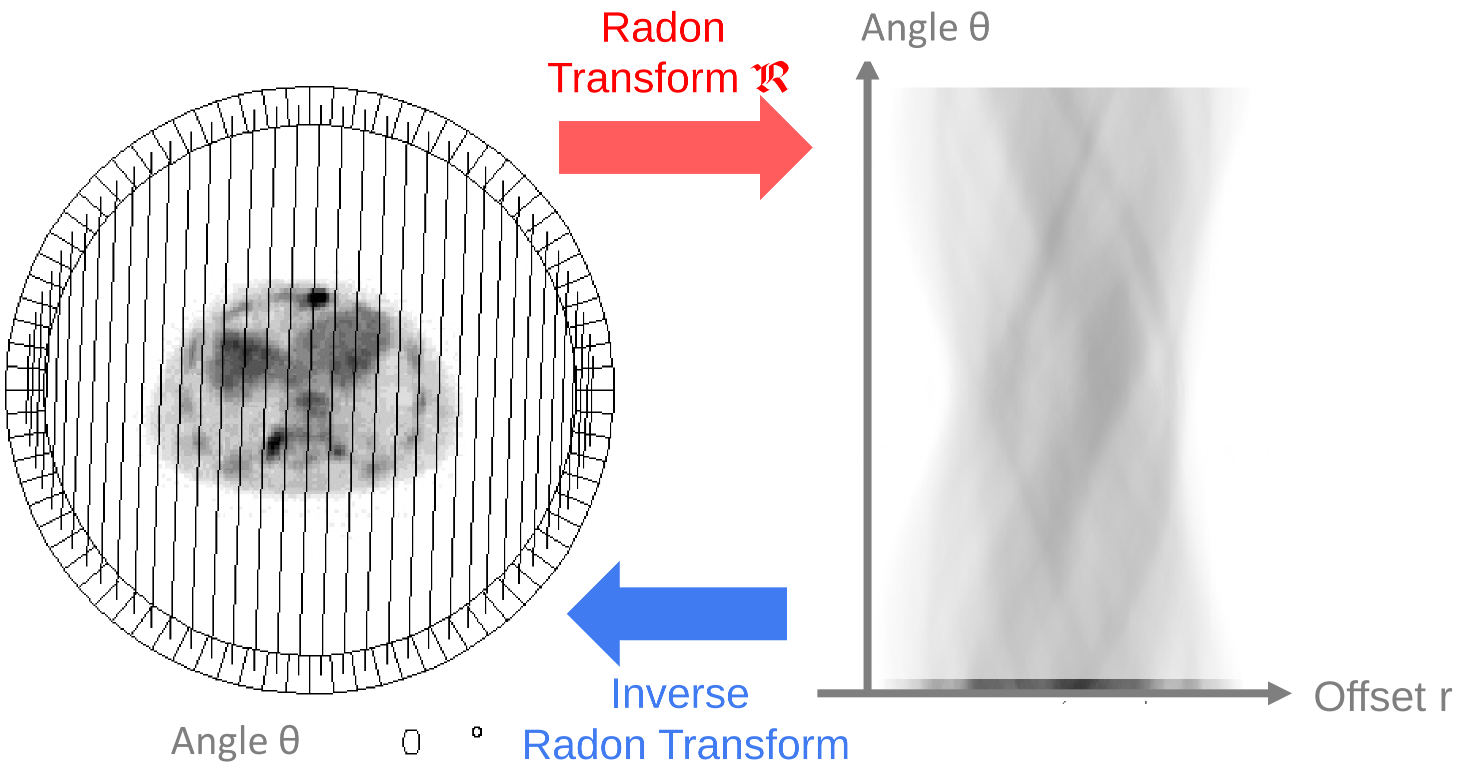
Radon transform



Radon
Transform ℜ

Angle θ

Offset r

Inverse
Radon Transform

Angle θ 0 °

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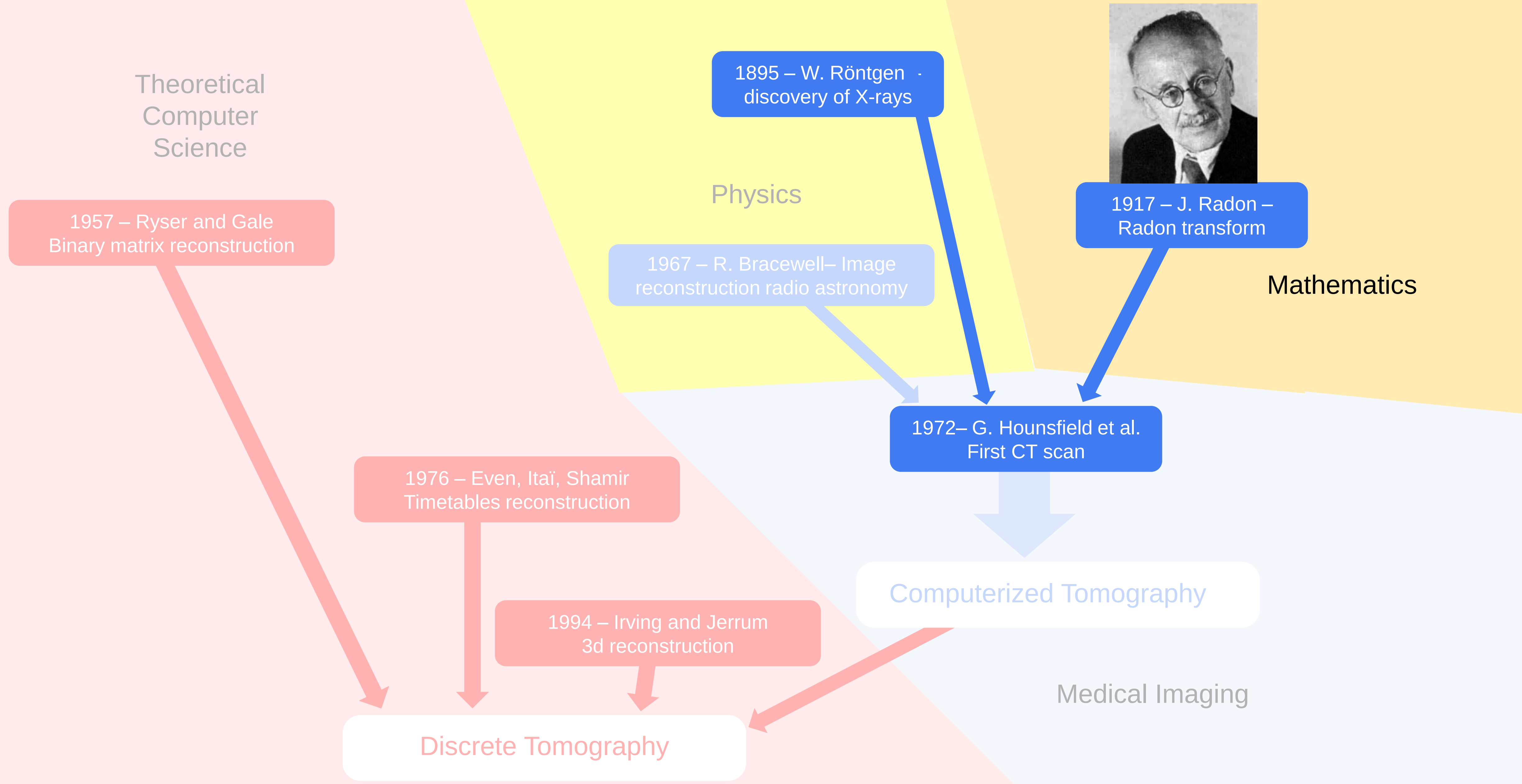
① The Origins of ~~Discrete~~ Tomography

Johannes Radon



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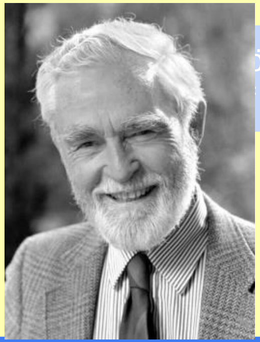

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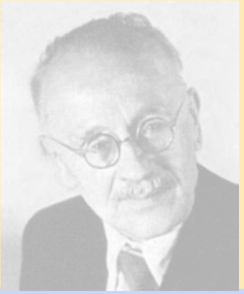

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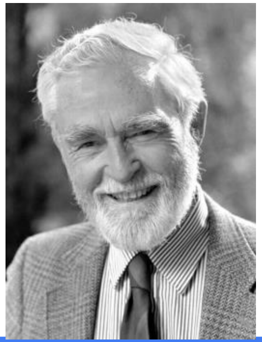

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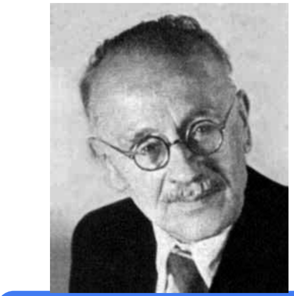
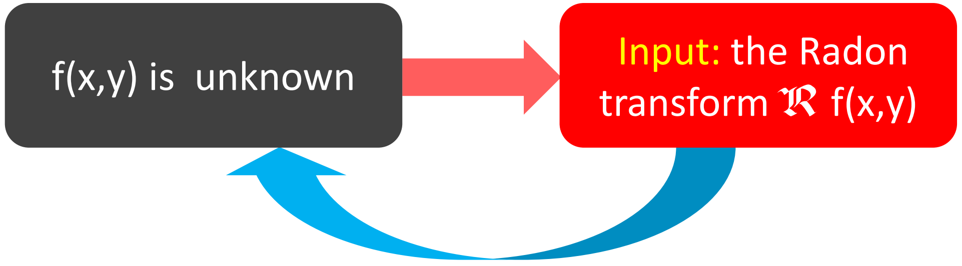


1917 – J. Radon – Radon transform



f(x,y) is unknown

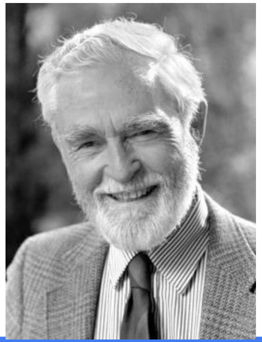
Input: the Radon transform $\mathfrak{R}$ f(x,y)

Inverse problem

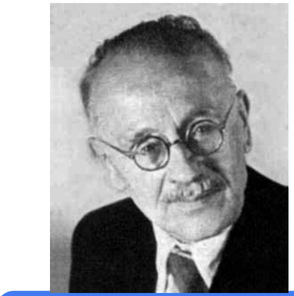
①

The Origins of ~~Discrete~~ Tomography

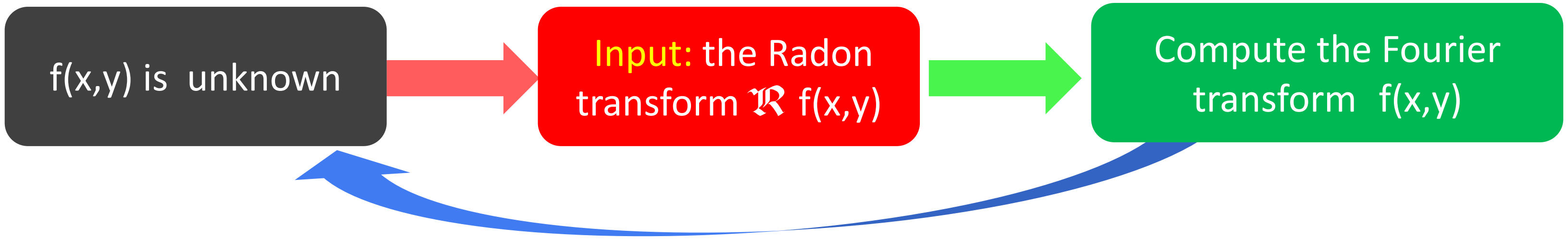
Fourier slice theorems



1967 – R. Bracewell– Image reconstruction radio astronomy



1917 – J. Radon – Radon transform



f(x,y) is unknown

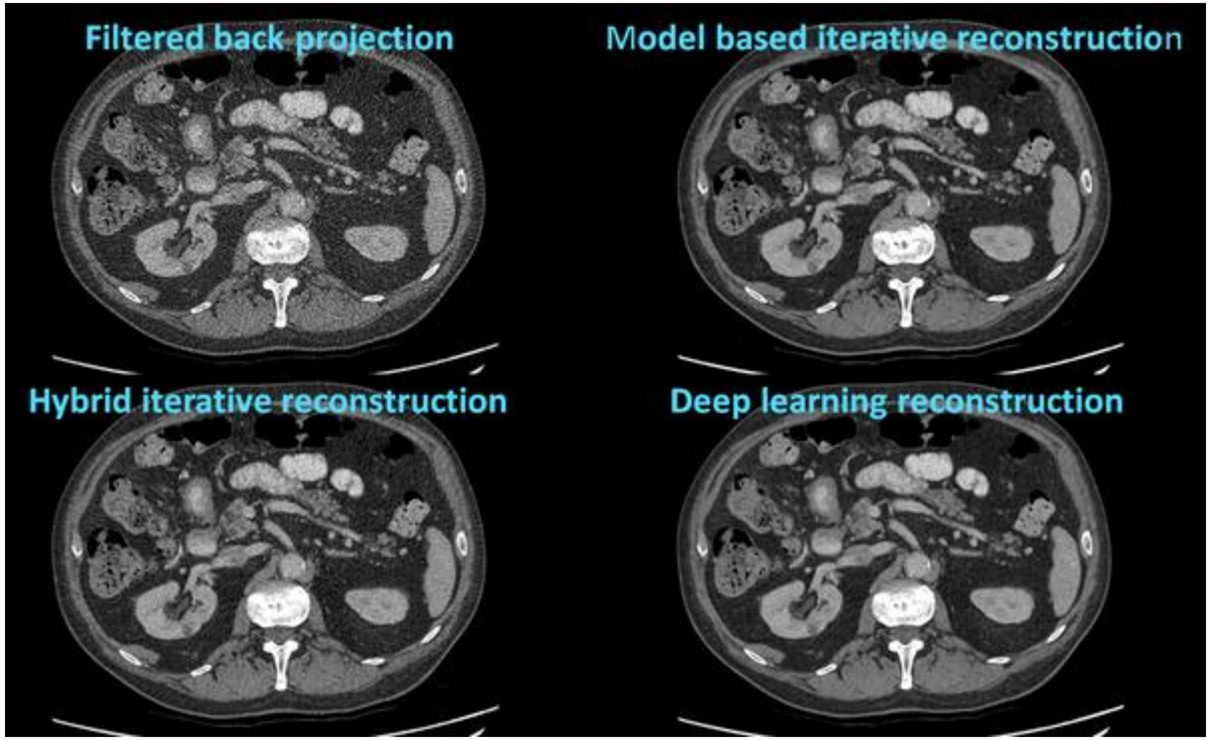
Input: the Radon transform 𝕽 f(x,y)

Compute the Fourier transform f(x,y)

Compute the inverse Fourier transform of 𝕱 f(x,y)

FBP, ART, OSEM…

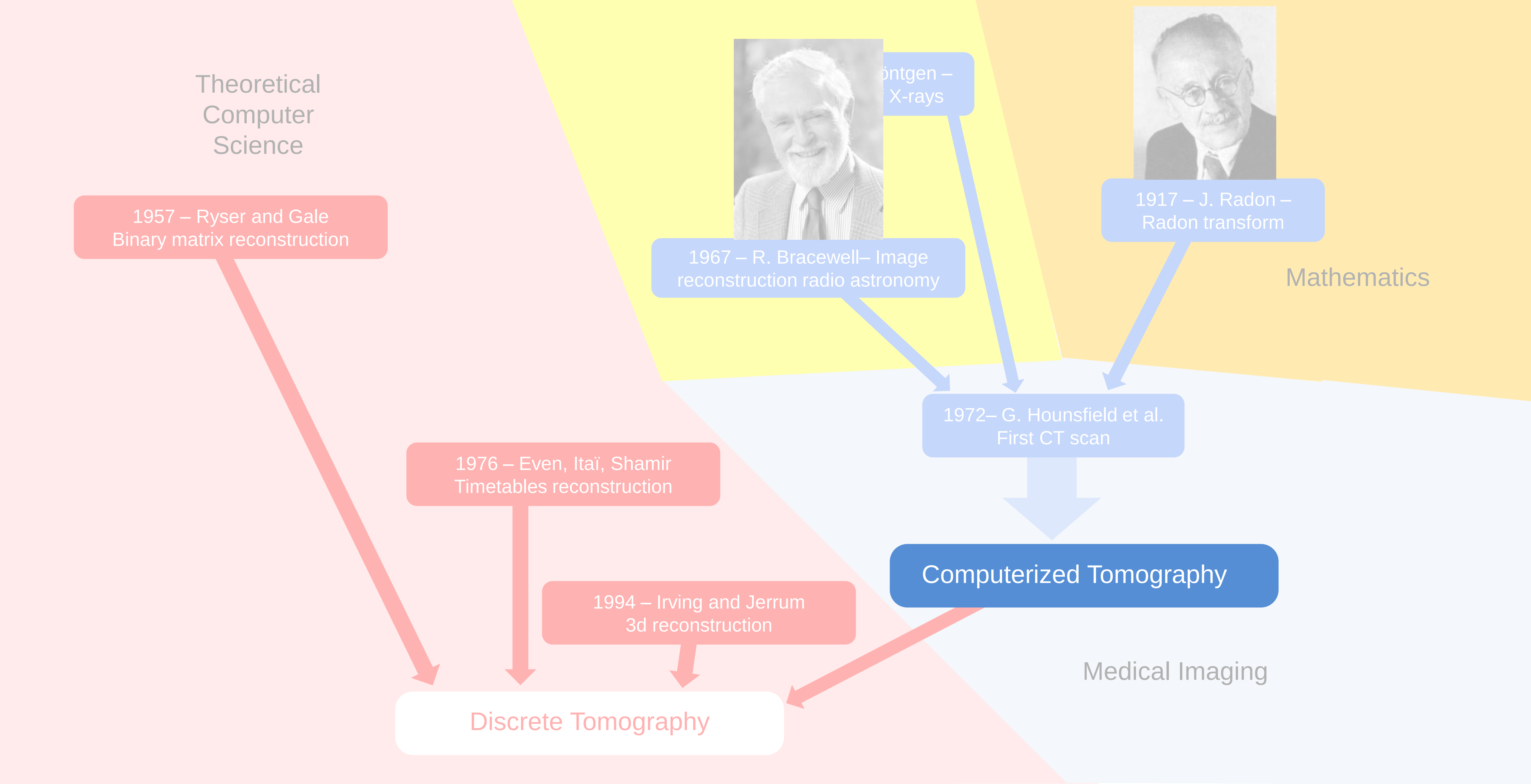
Many practical algorithms
(Filtered Back Projection, Algebraic Reconstruction Tecniques… and more recent ones)



L. Oostveen, K. Boedeker, M. Brink, M. Prokop, F. de Lange and I. Sechopoulos.
"*Physical evaluation of an ultra-high-resolution CT scanner*", 2020.

①

The Origins of ~~Discrete~~ Tomography



Theoretical Computer Science

1957 – Ryser and Gale
Binary matrix reconstruction

1976 – Even, Itaï, Shamir
Timetables reconstruction

1994 – Irving and Jerrum
3d reconstruction

Discrete Tomography

öntgen –
X-rays

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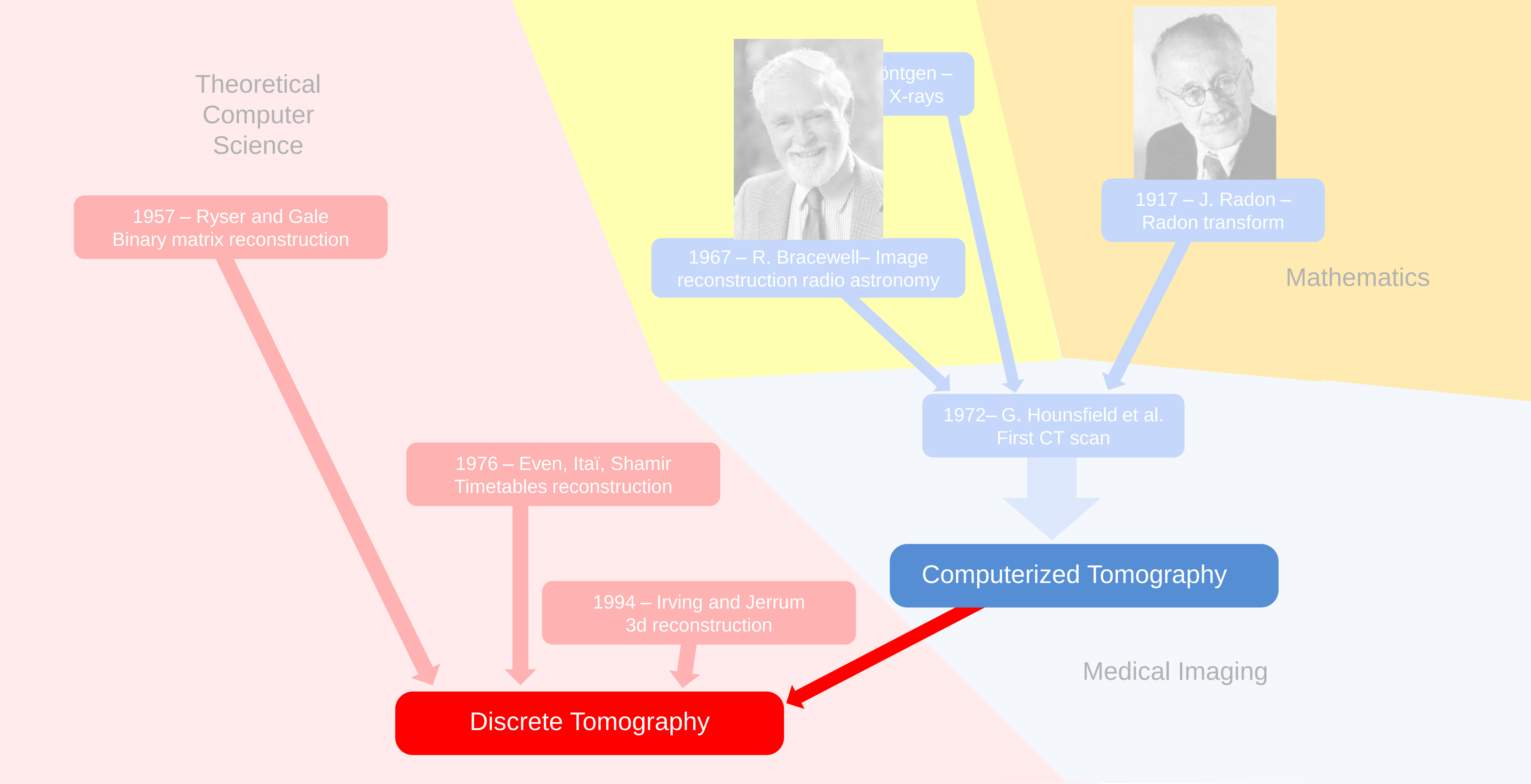
Mathematics

1972– G. Hounsfield et al.
First CT scan

Computerized Tomography

Medical Imaging

① The Origins of Discrete Tomography



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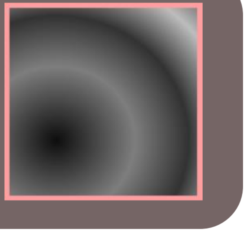
Computerized Tomography

Medical Imaging

Computerized Tomography deals with the reconstruction
of a continuous function
on a continuous domain

$f : [0 , 1]^2 \rightarrow [0 , 1]$



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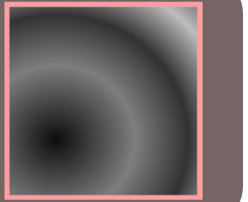
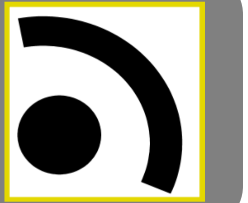


Geometric Tomography deals with the reconstruction
of a binary function
on a continuous domain

$f : [0 , 1]^2 \rightarrow \{0 , 1\}$

namely a subset of $[0 , 1]^2$



The Origins of Discrete Tomography

Computerized Tomography deals with the reconstruction
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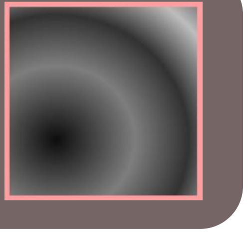

Geometric Tomography deals with the reconstruction
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on a continuous domain

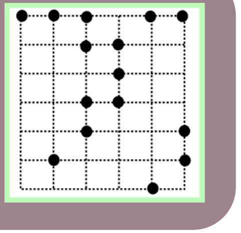
$f : [0 , 1]^2 \rightarrow \{0 , 1\}$

namely a subset of $[0 , 1]^2$

Discrete Tomography deals with the reconstruction
of a binary function
on a discrete domain

$f : Lattice \rightarrow \{0 , 1\}$

namely a lattice set



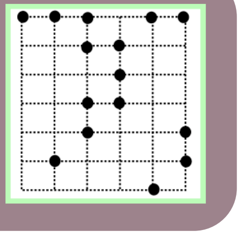
① The Origins of Discrete Tomography

1990s

Discrete Tomography deals with the reconstruction
of a binary function
on a discrete domain

$f : \mathit{Lattice} \to \{0 , 1\}$

namely a lattice set



① The Origins of Discrete Tomography

A simple question

Peter Schwander, physicist
at AT&T Bell labs
(in the 90s)

Question →

Larry Shepp, CT expert,
AT&T Bell labs
(in the 90s)

Discrete Tomography deals with the reconstruction
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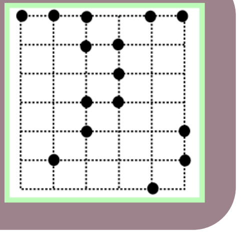
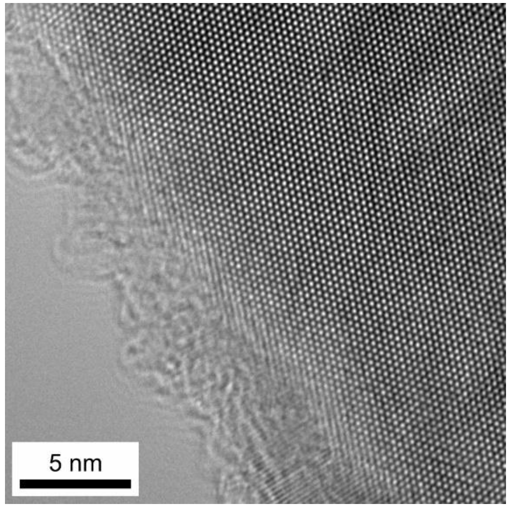


① The Origins of Discrete Tomography

Peter Schwander, physicist
at AT&T Bell labs
(in the 90s)

High Resolution Transmission
Electron Microscope (HRTEM)



HRTEM image

Discrete Tomography deals with the reconstruction
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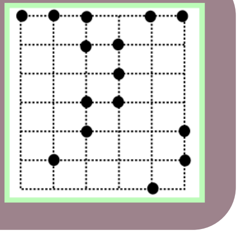
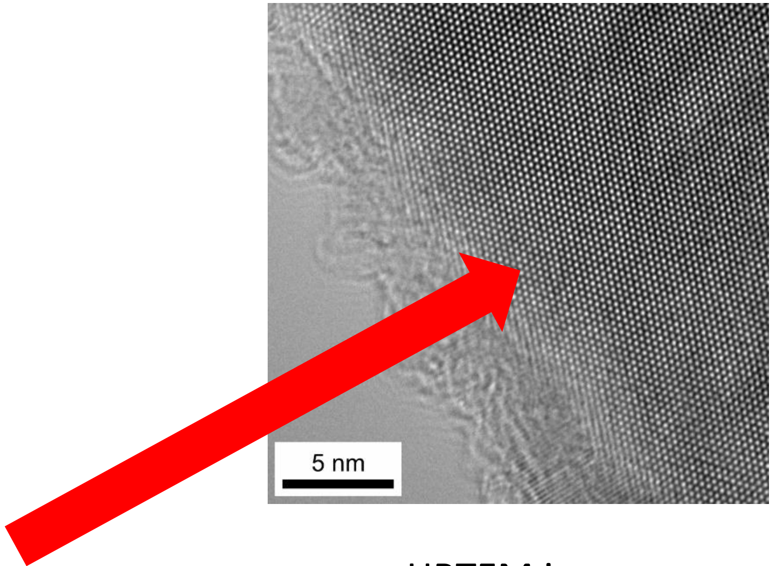
f : Lattice → {0 , 1}
namely a lattice set



① The Origins of Discrete Tomography

Counting atoms

Peter Schwander, physicist
at AT&T Bell labs
(in the 90s)



HRTEM image

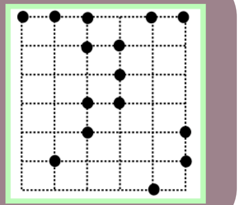
Data might provide the
number of atoms (with noise)
behind a point

Counting the number of atoms on a line is possible.

Discrete Tomography deals with the reconstruction
of a binary function
on a discrete domain

$f : Lattice \to \{0 , 1\}$

namely a lattice set



① The Origins of Discrete Tomography

Reconstructing the 3D structure of the atoms

Peter Schwander, physicist
at AT&T Bell labs
(in the 90s)

Can we recover
the 3D positions of atoms ?

Counting the number of atoms on a line is possible.

Discrete Tomography deals with the reconstruction
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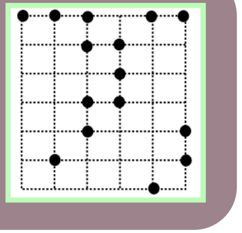

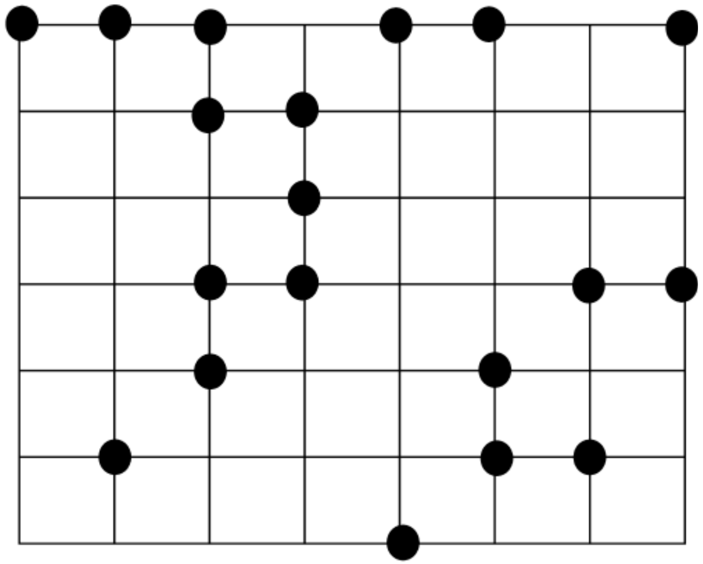

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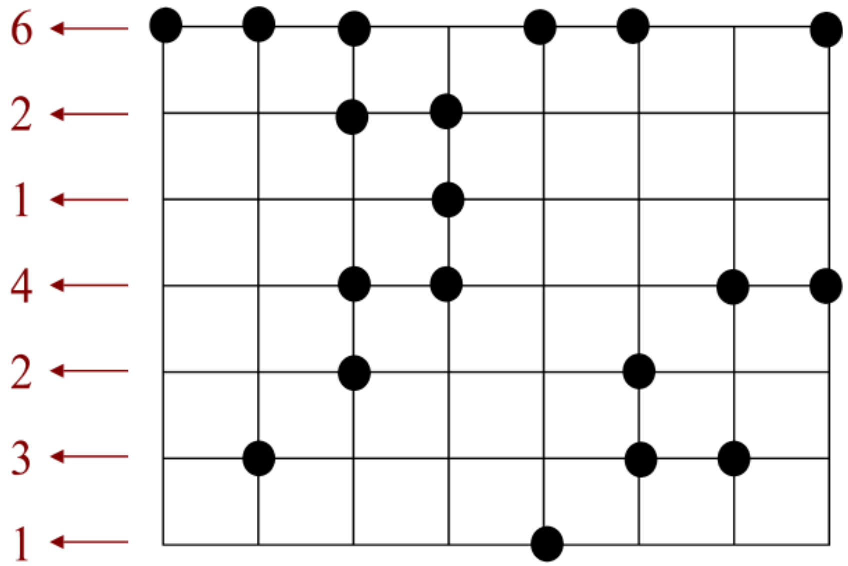
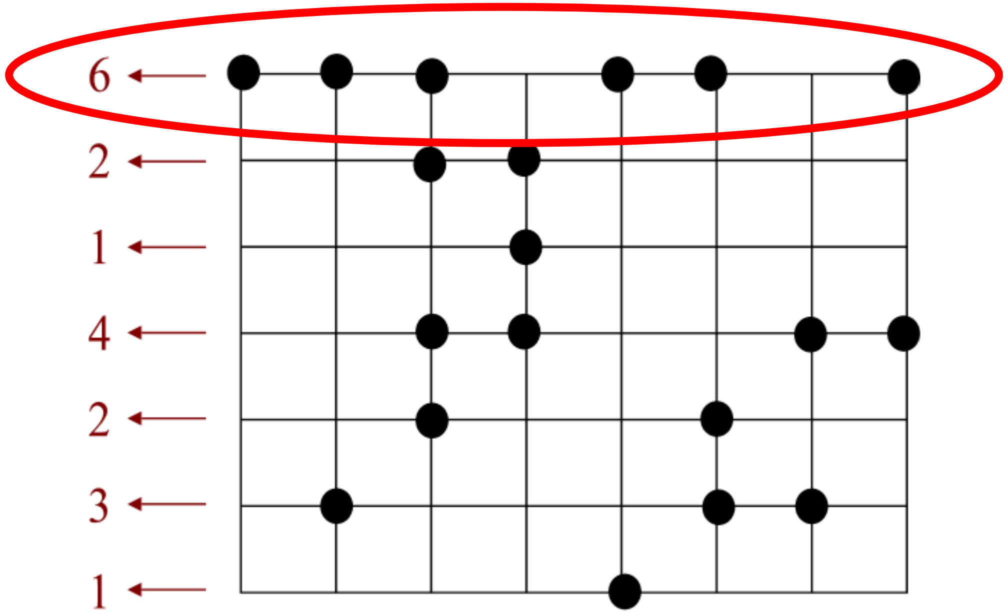

6
2
1
4
2
3
1


6
2
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4
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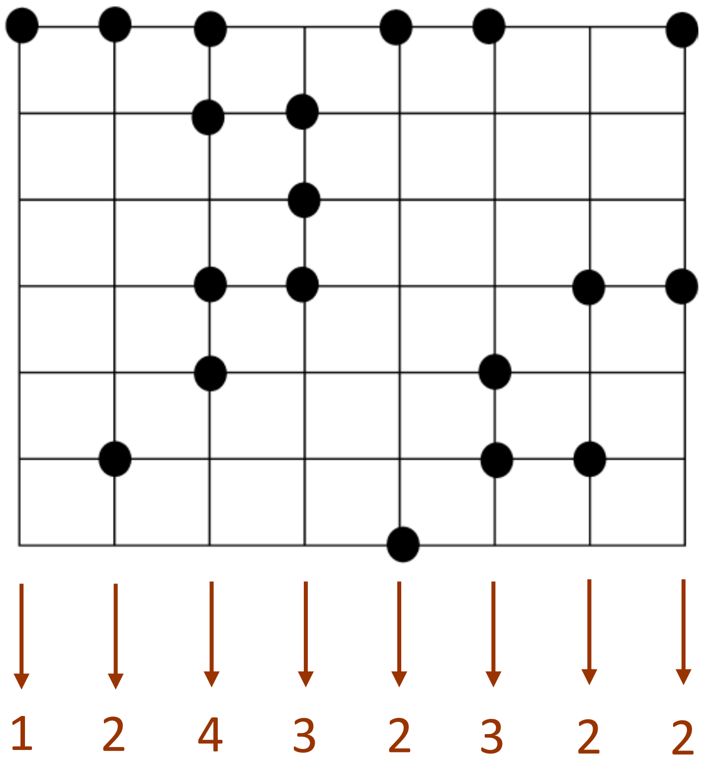
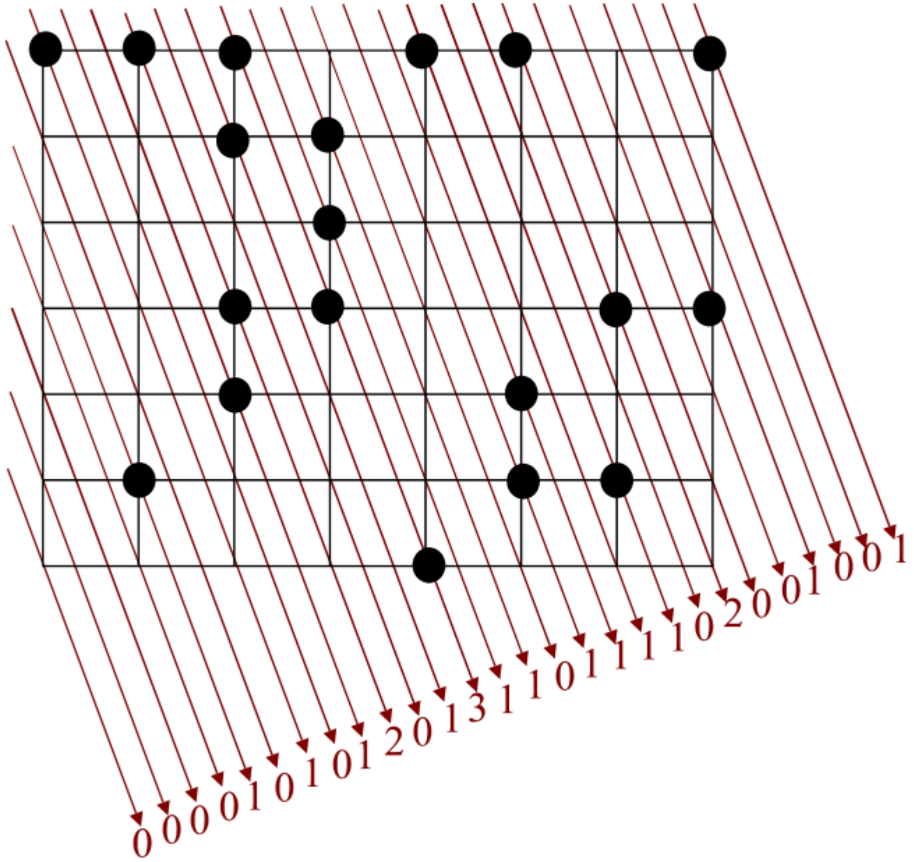
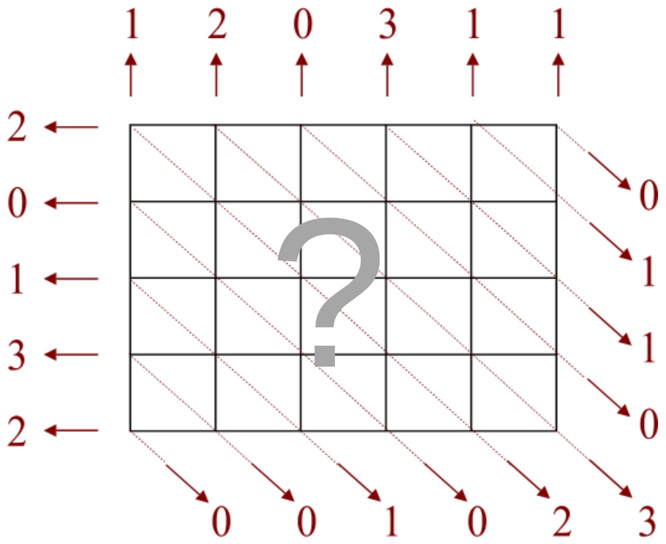
3
1


1
2
4
3
2
3
2
2


0 0 0 0 1 0 1 0 1 2 0 1 3 1 1 0 1 1 1 1 0 2 0 0 1 0 0 1



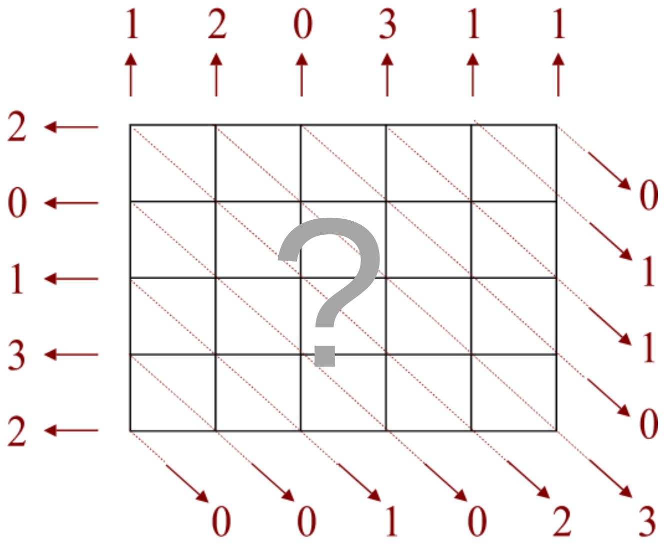
Combinatorial problem

① The Origins of Discrete Tomography

Can we solve this class of problems ?

Peter Schwander, physicist at AT&T Bell labs

(in the 90s)



Combinatorial problem

Larry Shepp, CT expert, AT&T Bell labs

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DIMACS

Center for Discrete mathematics and Theoretical Computer Science

Mini Symposium at DIMACS (Rutger University) in September 1994 with the title: Discrete Tomography

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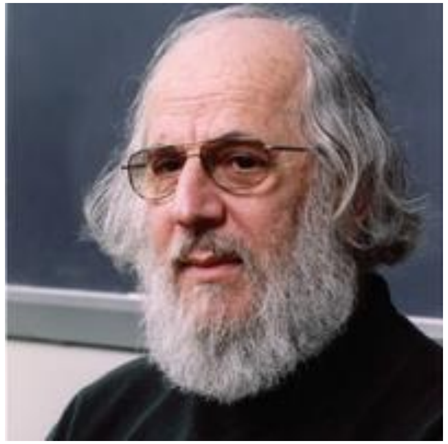
Research group in US and in Europ….

The Origins of Discrete Tomography

Maurice Nivat
(Université Paris Diderot)

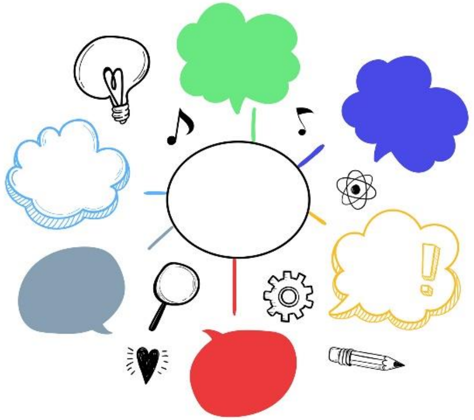
Peter Gritzmann
(Technical University of Munich)

Richard Gardner
(Wester Washingtom University)



Gabor Hermann
(City University of New York)

+ students, close researchers….

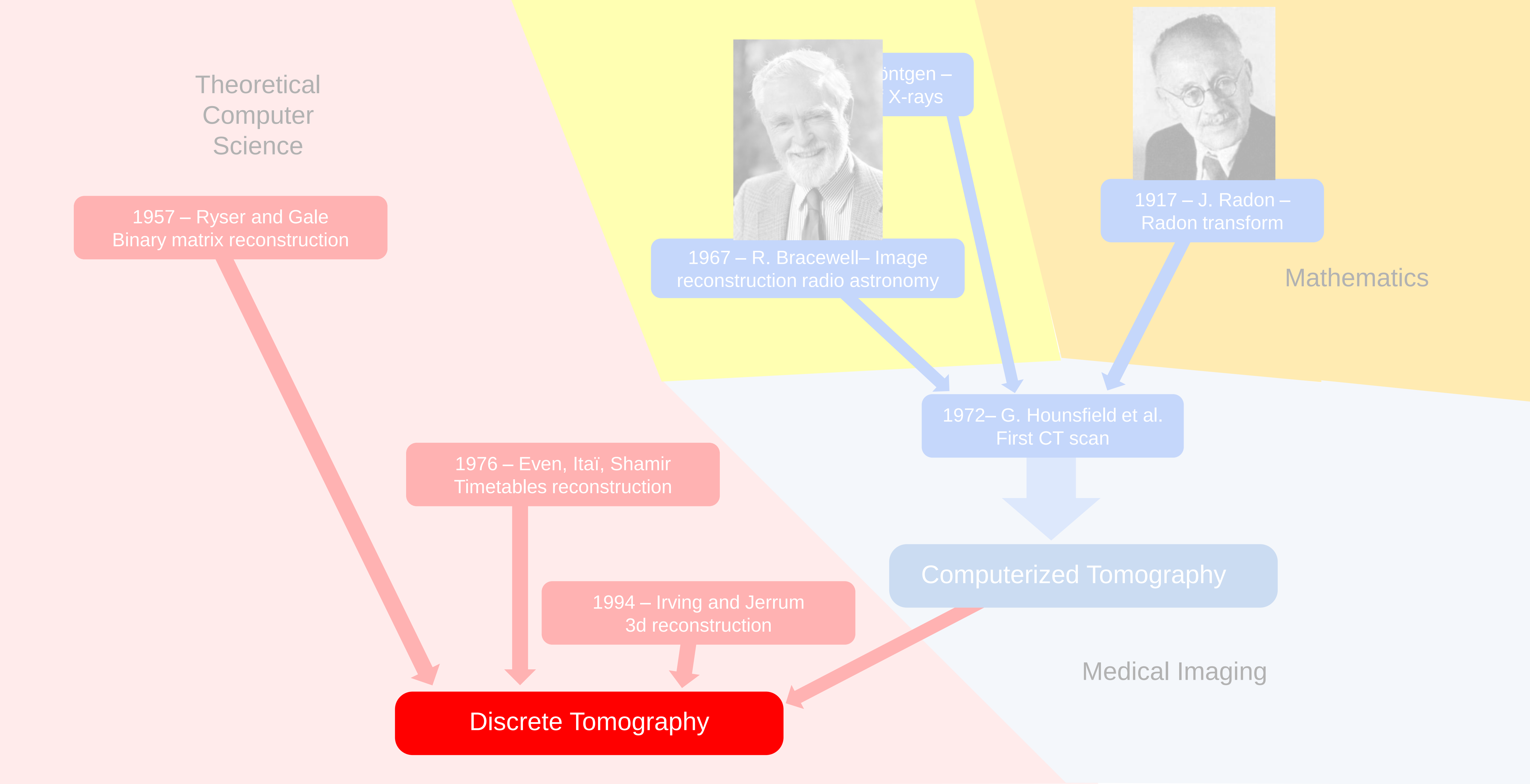


Research group in US and in Europ….

①

The Origins of Discrete Tomography

The other part of the story



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Binary matrix reconstruction

öntgen –
X-rays

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Mathematics

1972– G. Hounsfield et al.
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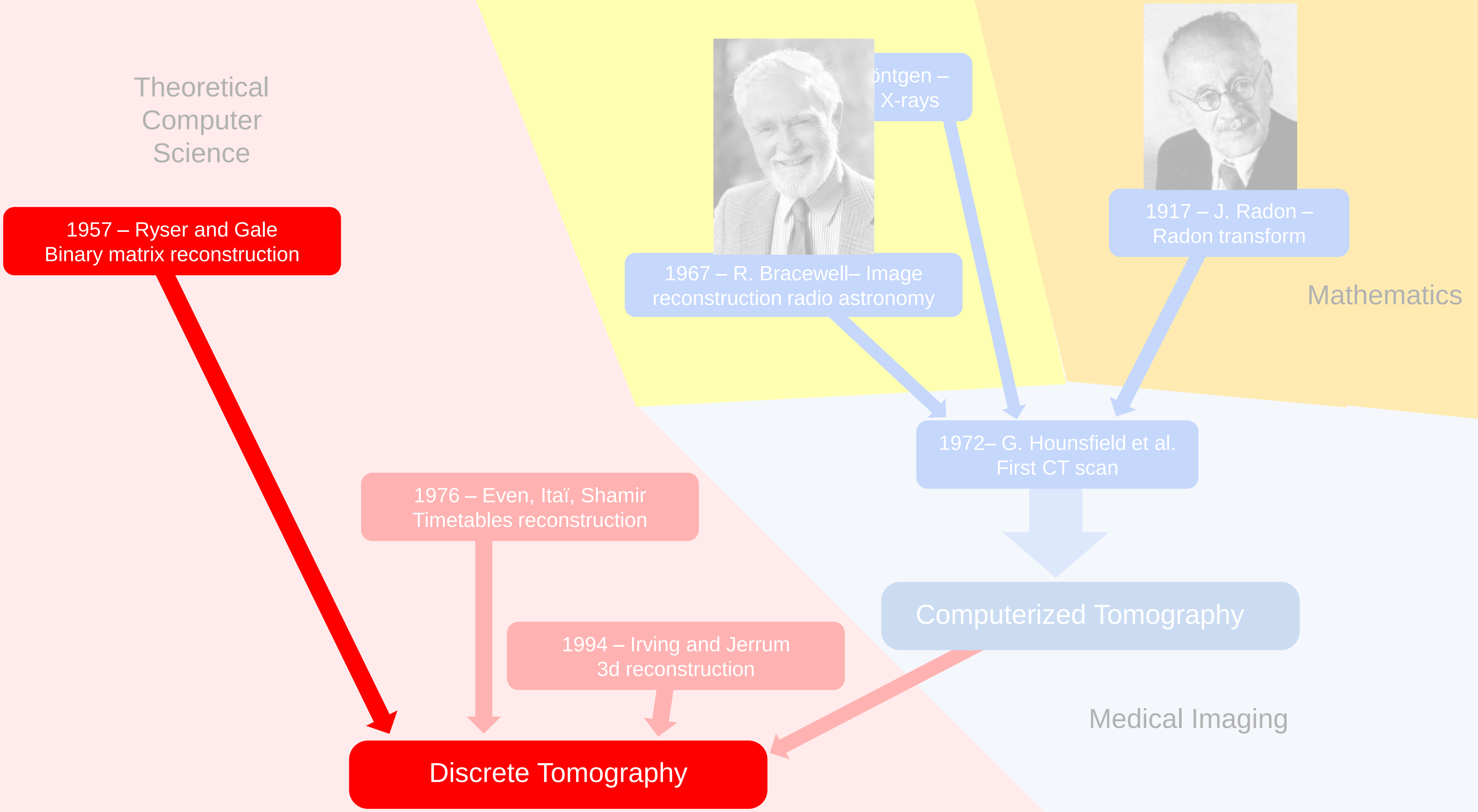
① The Origins of Discrete Tomography

The other part of the story



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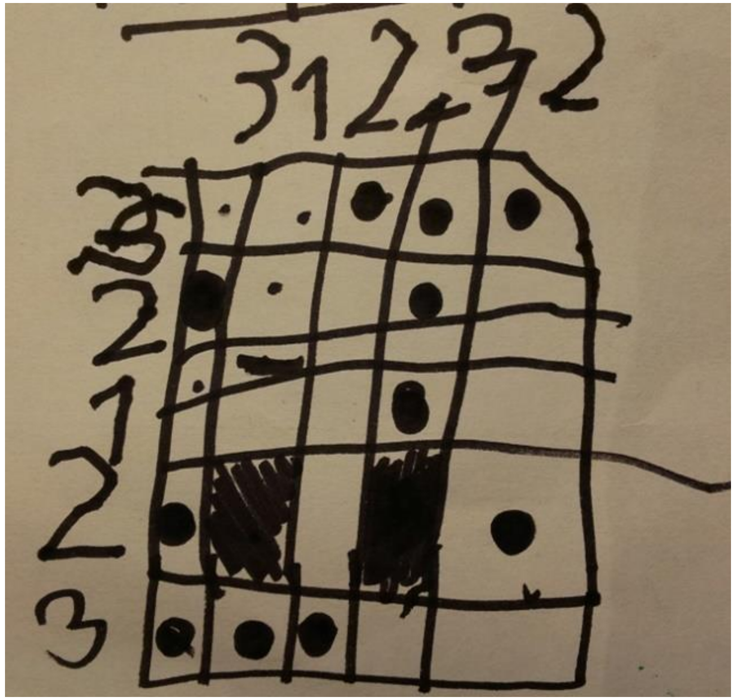


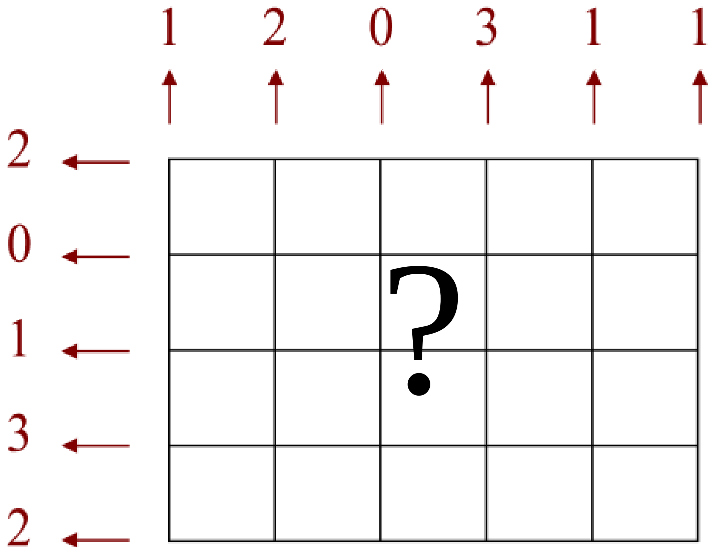
Jude (5 years old)

①

The Origins of Discrete Tomography

Two X-rays

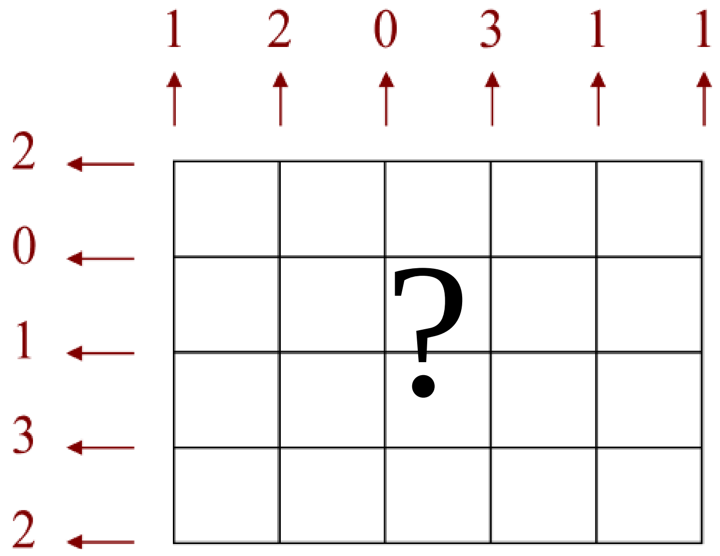
1957 – Ryser and Gale
Binary matrix reconstruction



An instance with 2 directions.

Polynomial time

1957 – Ryser and Gale
Binary matrix reconstruction

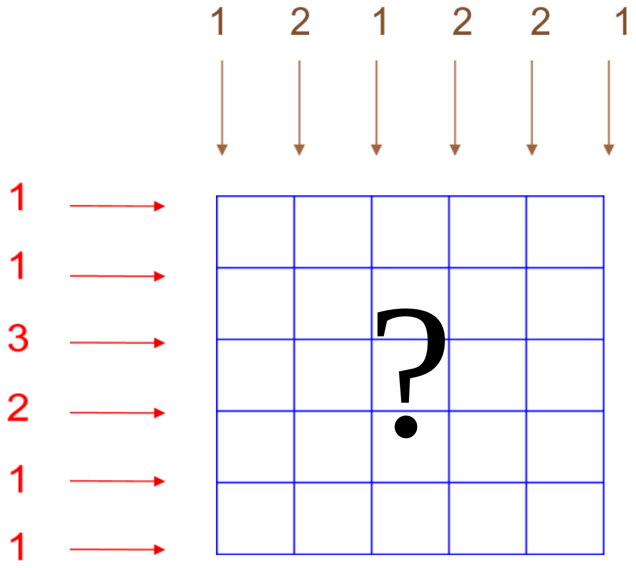


An instance with 2 directions.

Theorem (Gale & Ryser – 1957):
Discrete Tomography with 2 directions can be solved in polynomial time.

① The Origins of Discrete Tomography

Polynomial time



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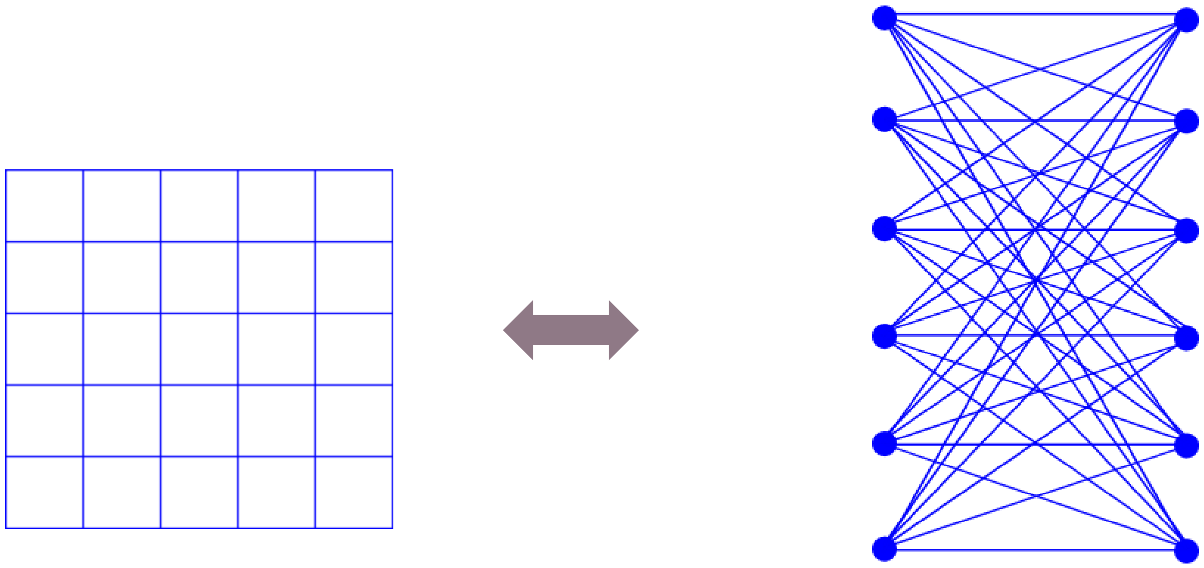

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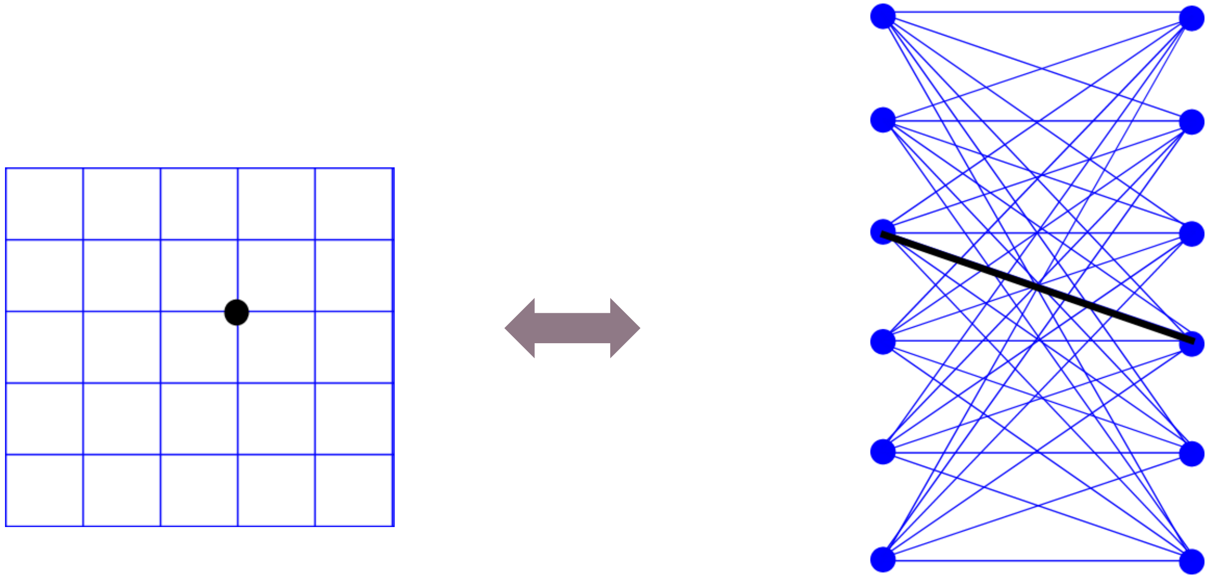
① The Origins of Discrete Tomography

Polynomial time



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The Origins of Discrete Tomography

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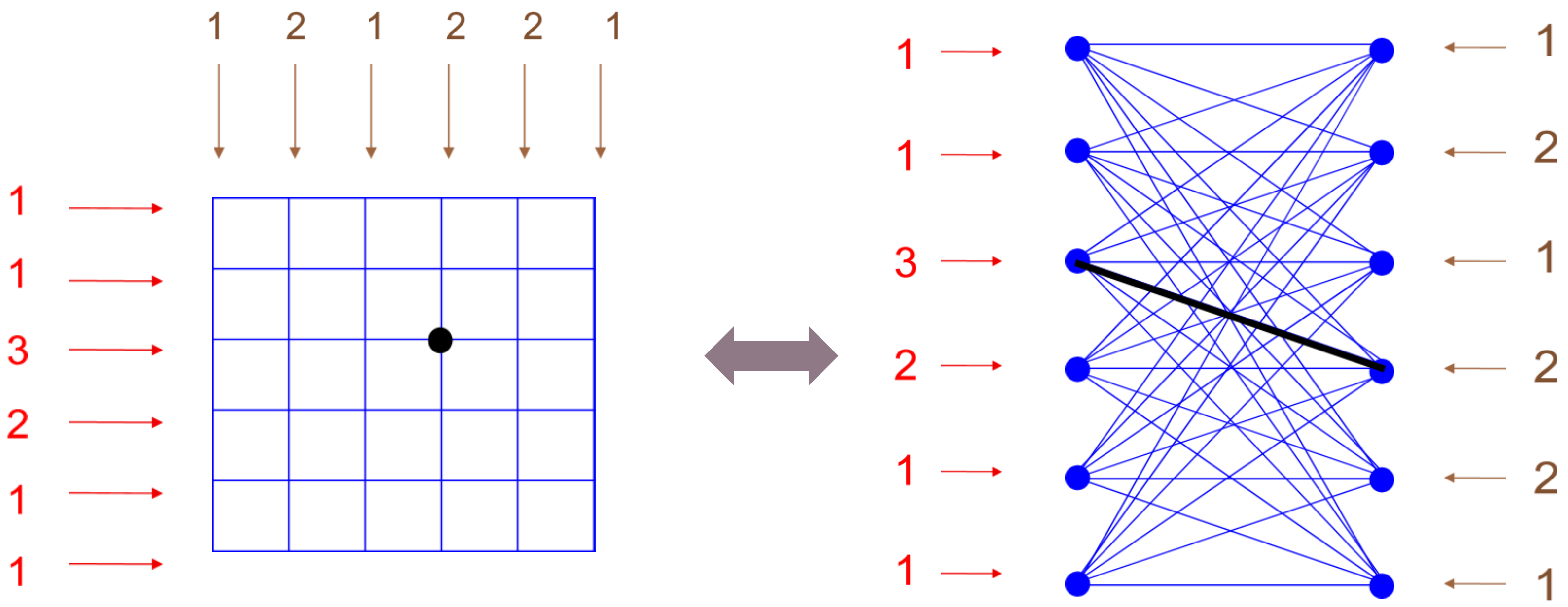

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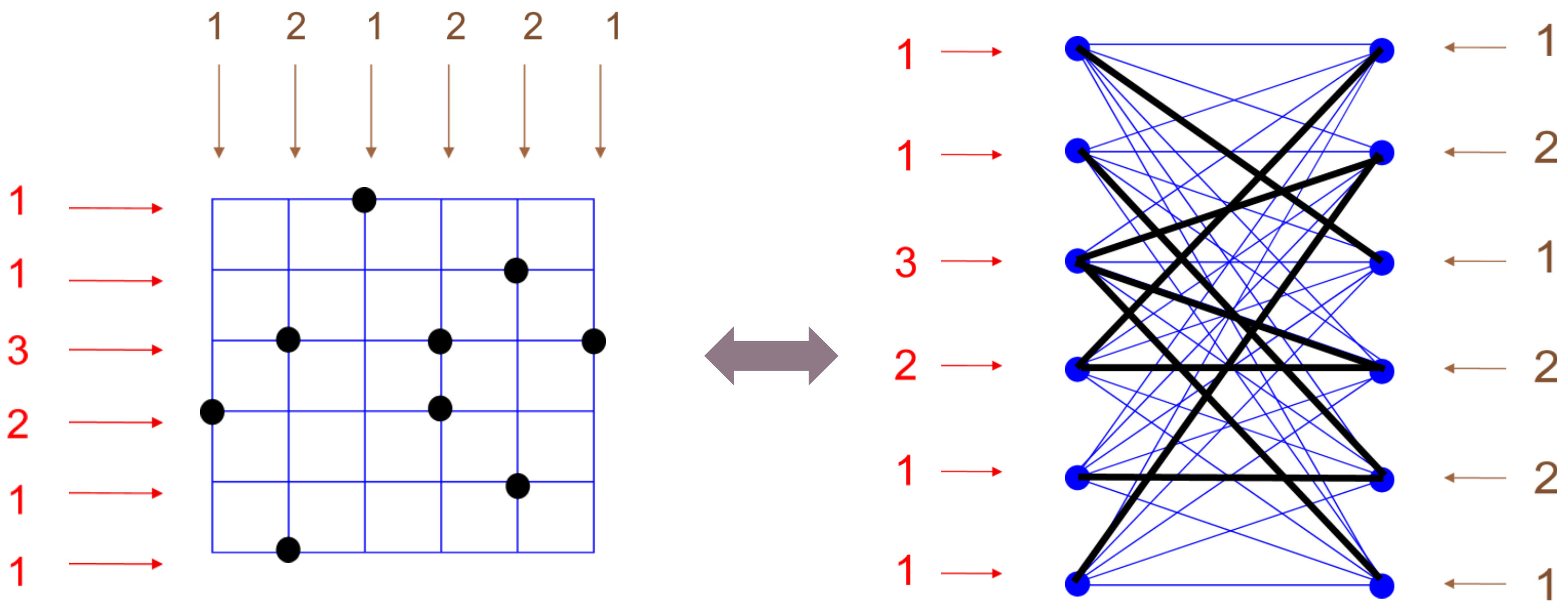


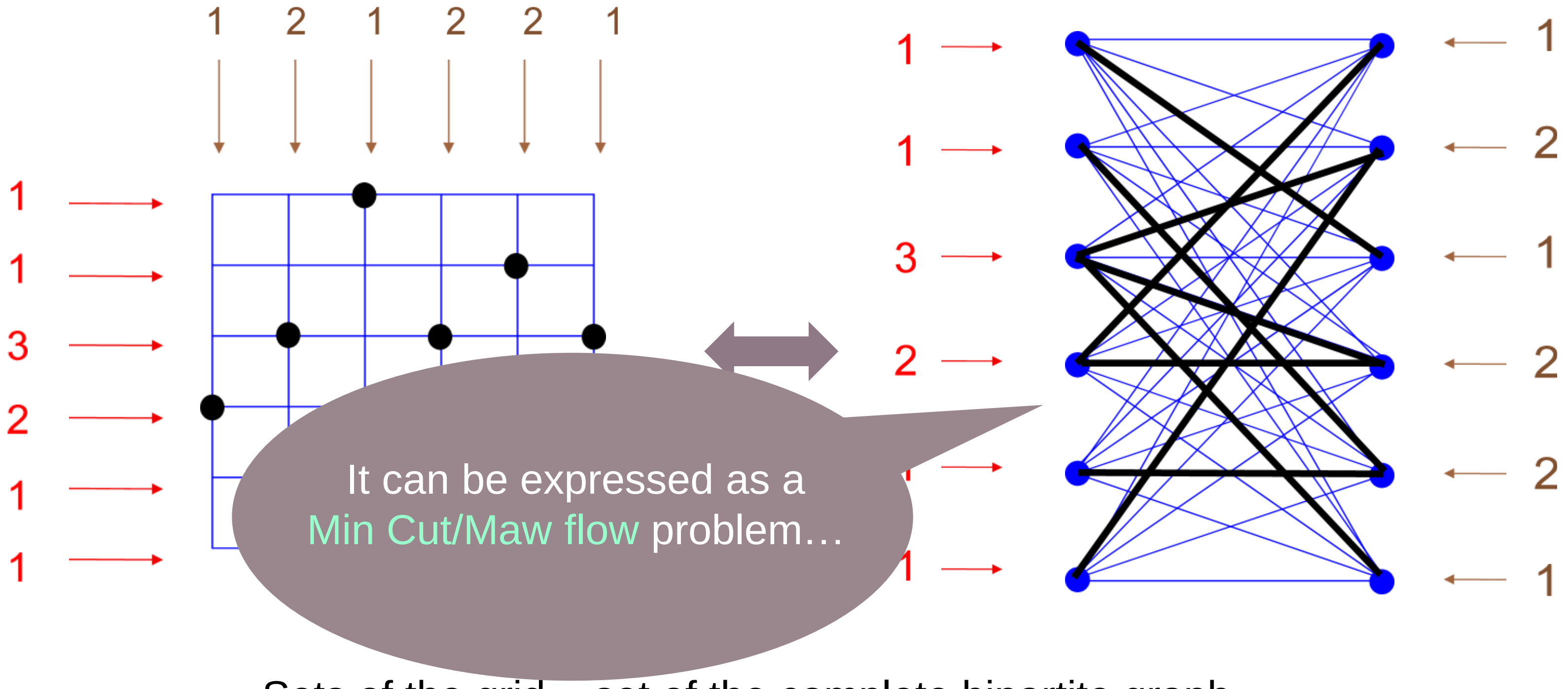
Sets of the grid = set of the complete bipartite graph.

Theorem (Gale & Ryser – 1957):

Discrete Tomography with 2 directions can be solved in polynomial time.

① The Origins of Discrete Tomography

Polynomial time



It can be expressed as a Min Cut/Maw flow problem…

Sets of the grid = set of the complete bipartite graph.

Theorem (Gale & Ryser – 1957):

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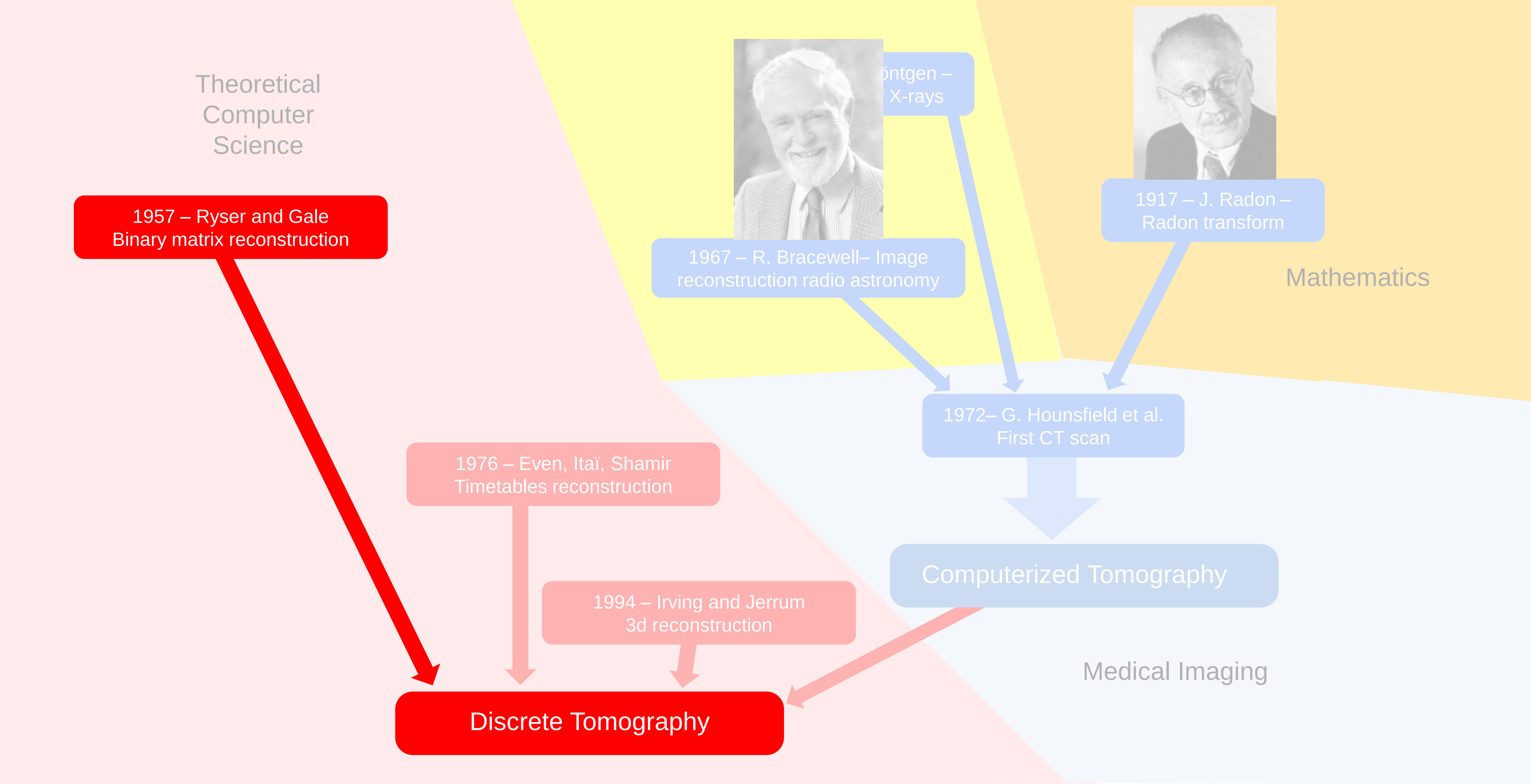
①

The Origins of Discrete Tomography

The other part of the story



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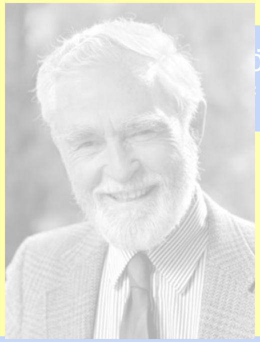

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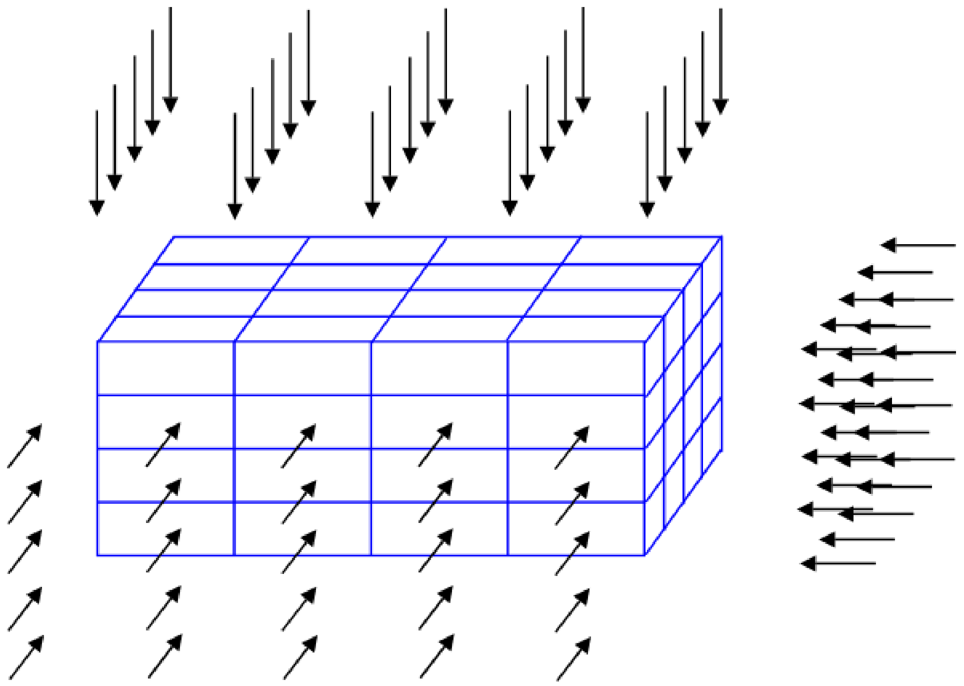
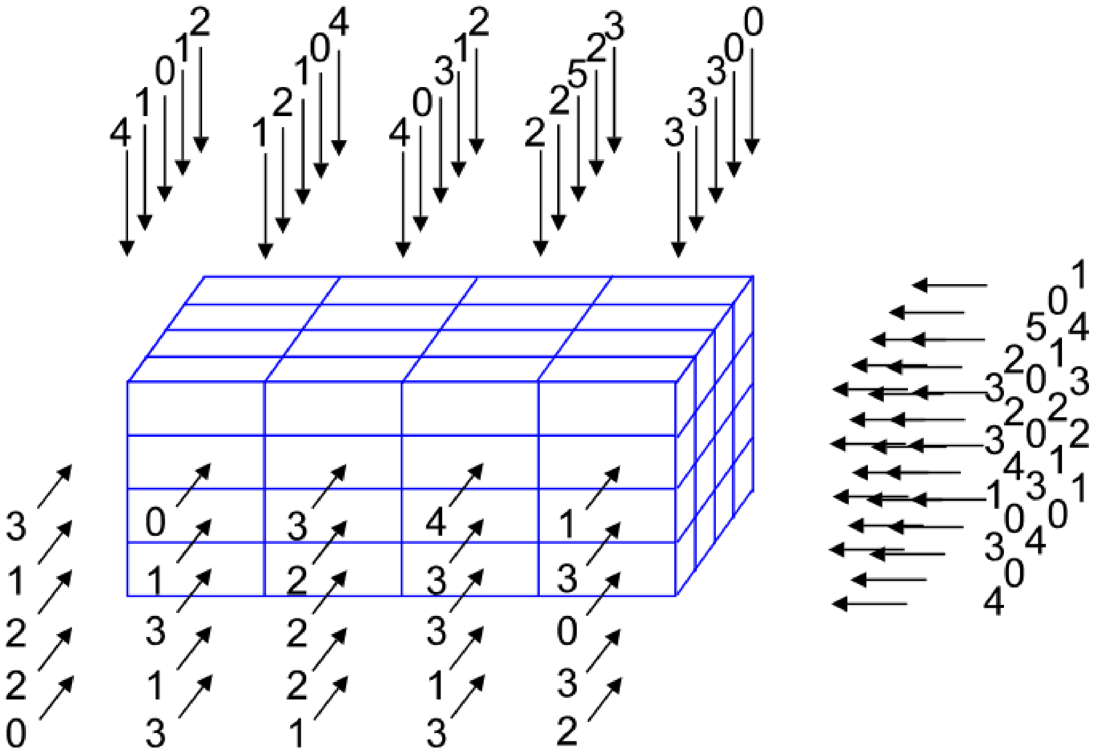
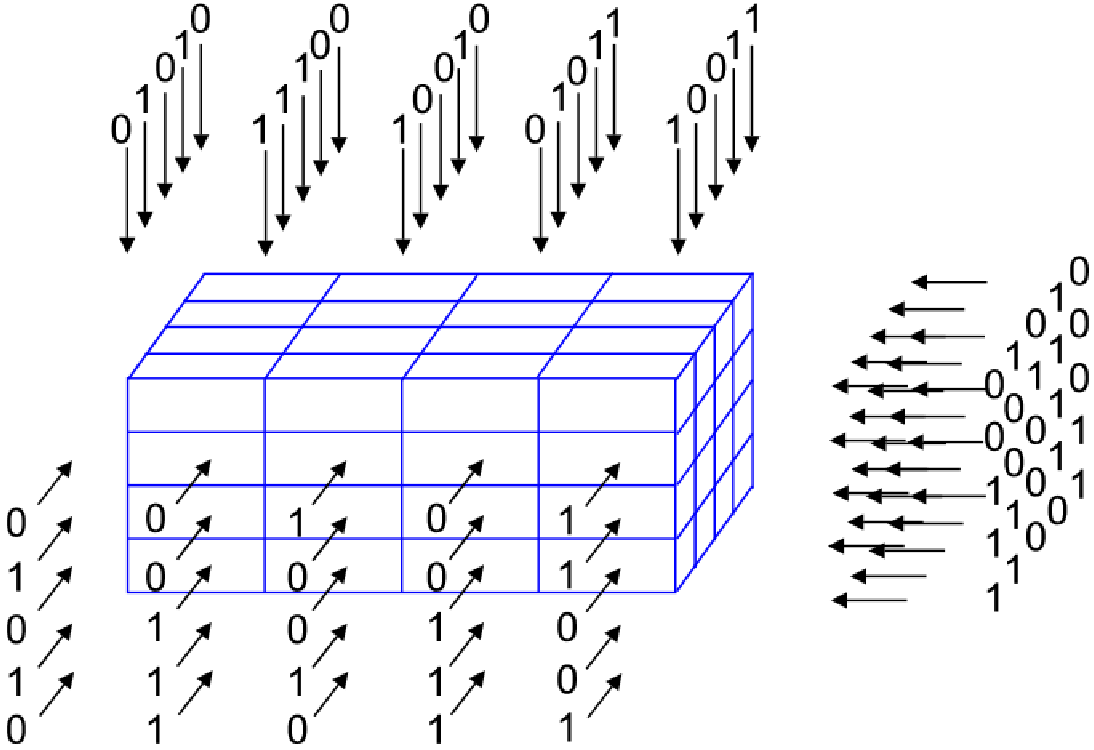


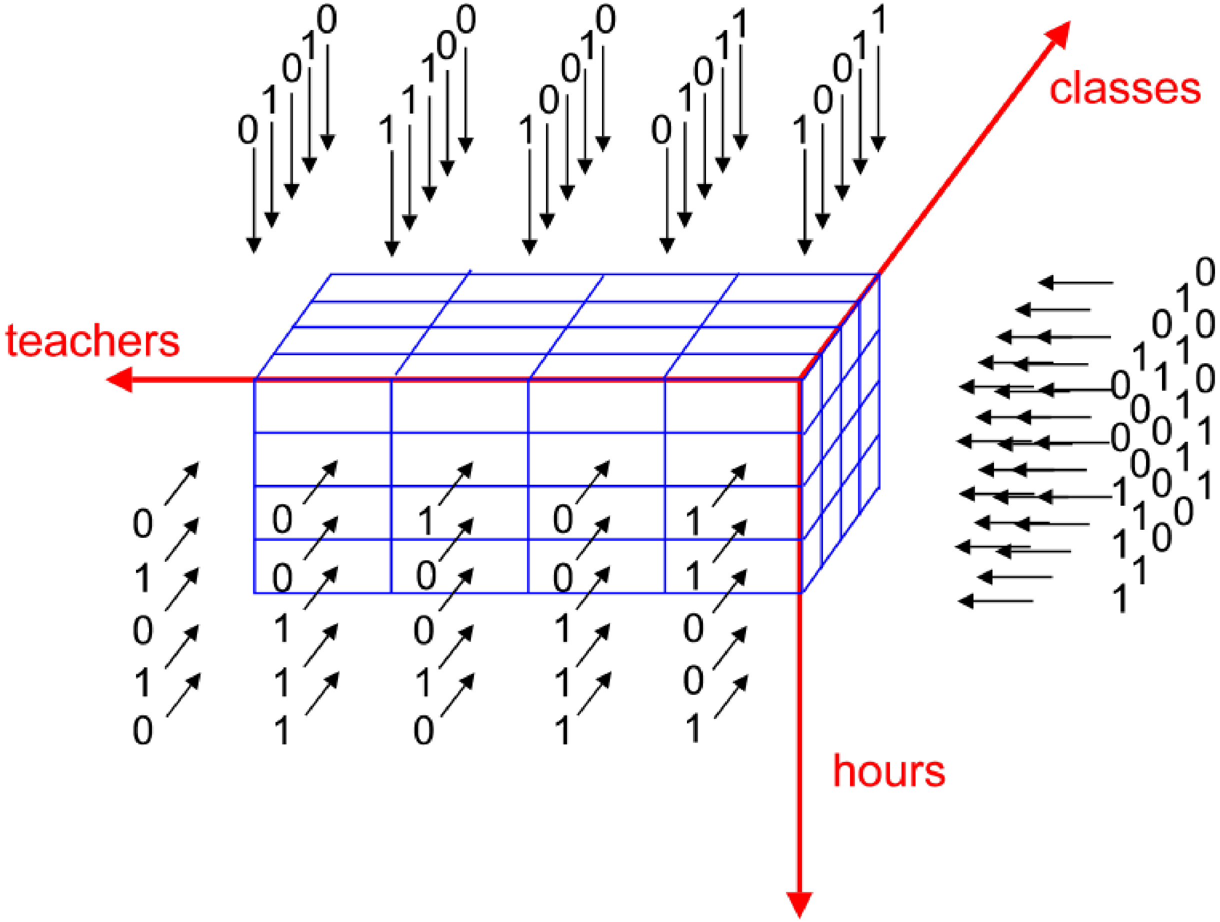
A 3D grid.



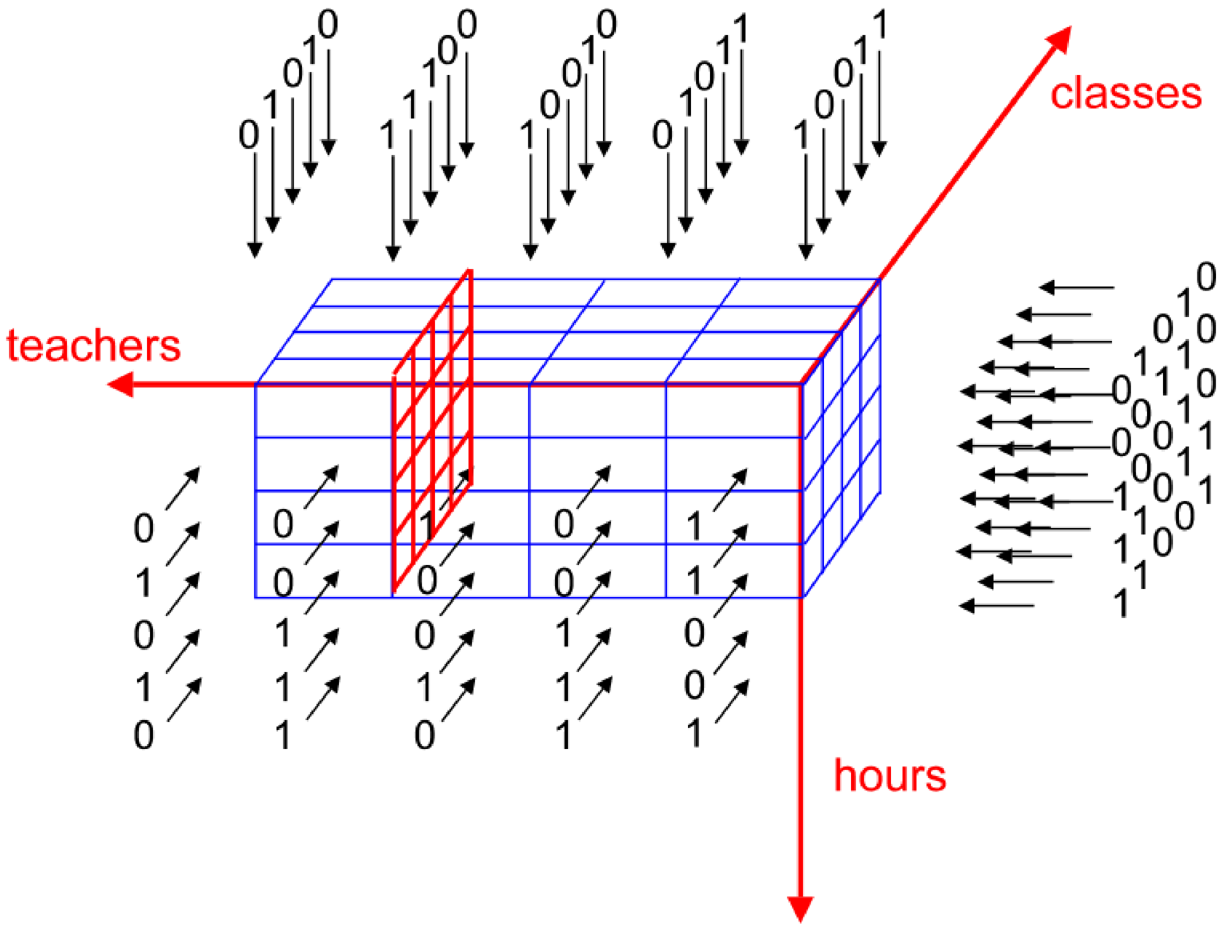
A 3D instance.



A 3D binary instance.



A 3D-instance = A timetable instance



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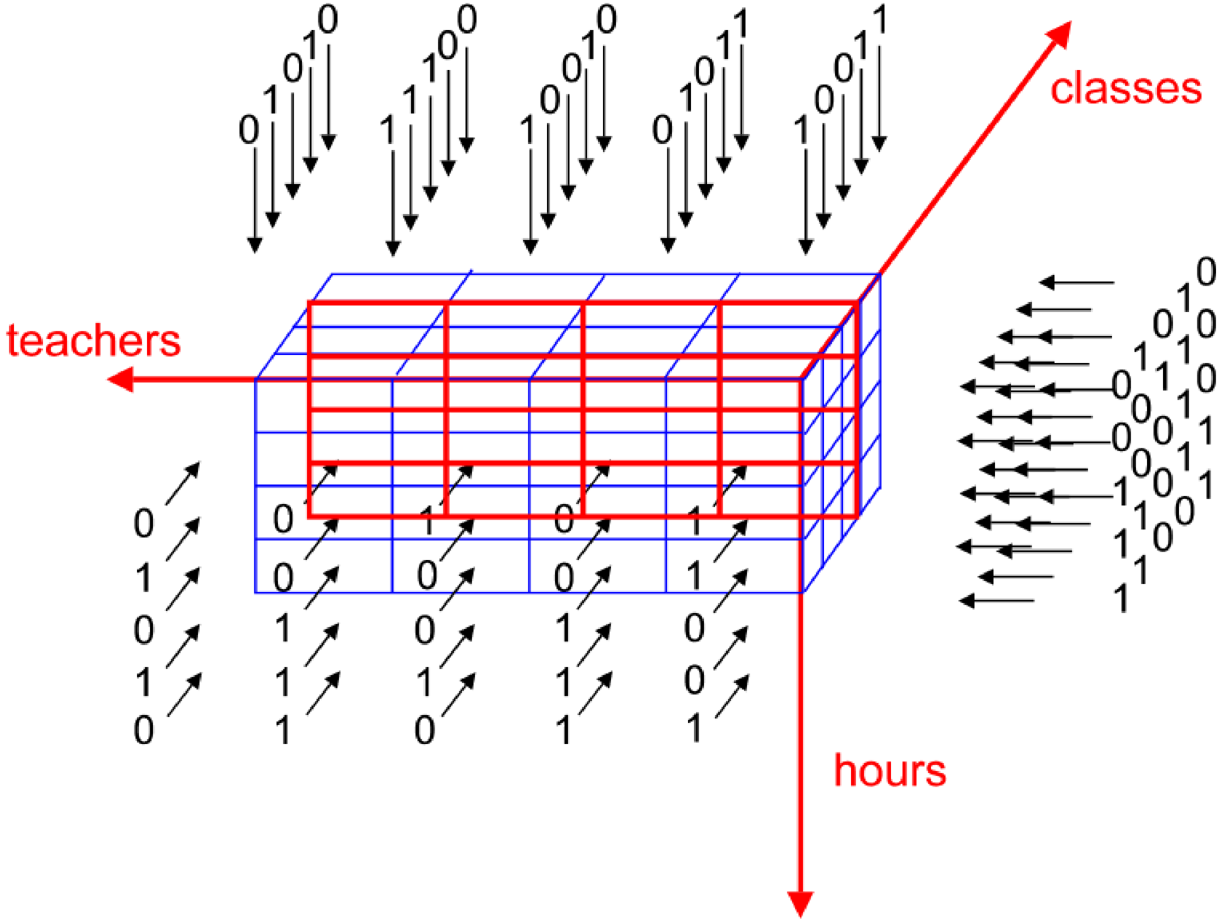
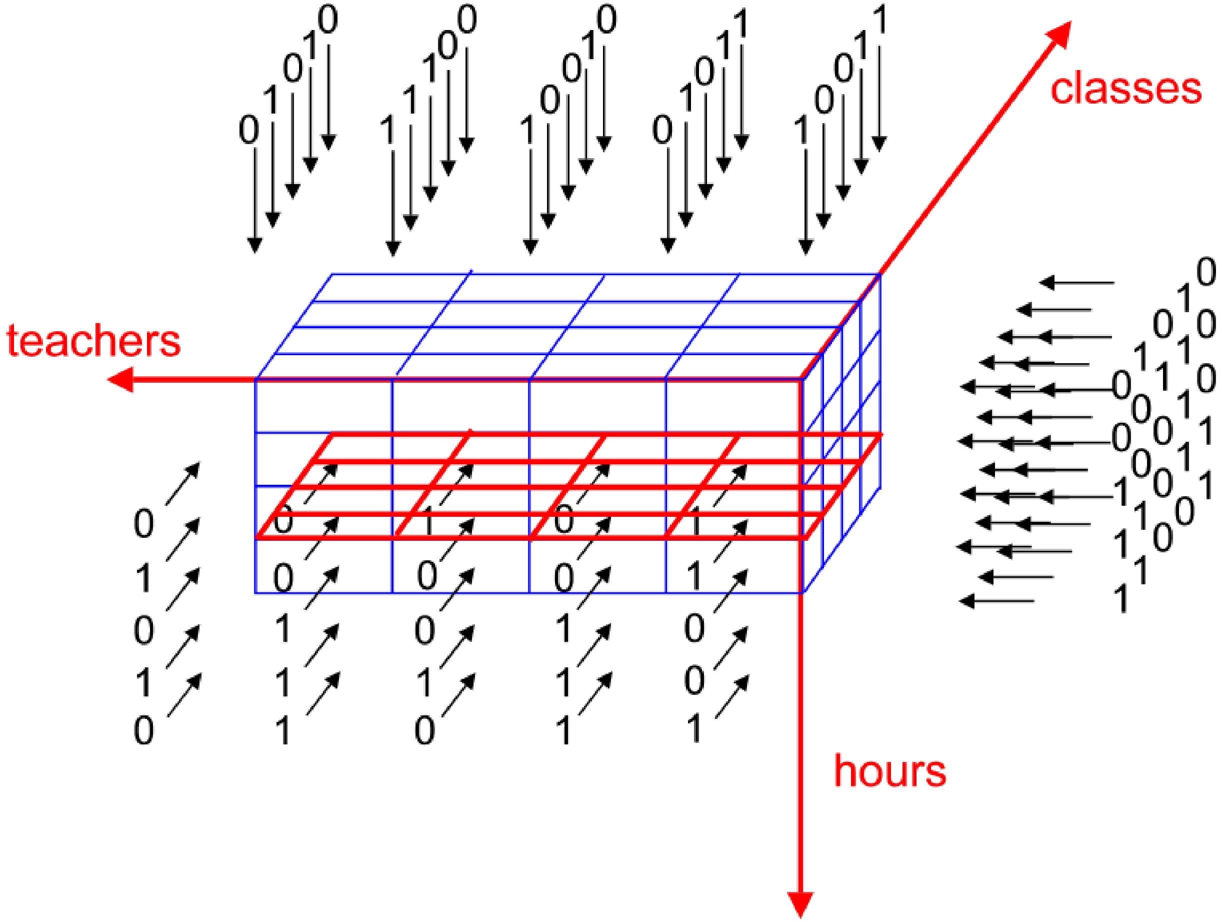


A 3D-instance = A timetable instance

①

The Origins of Discrete Tomography

Timetables



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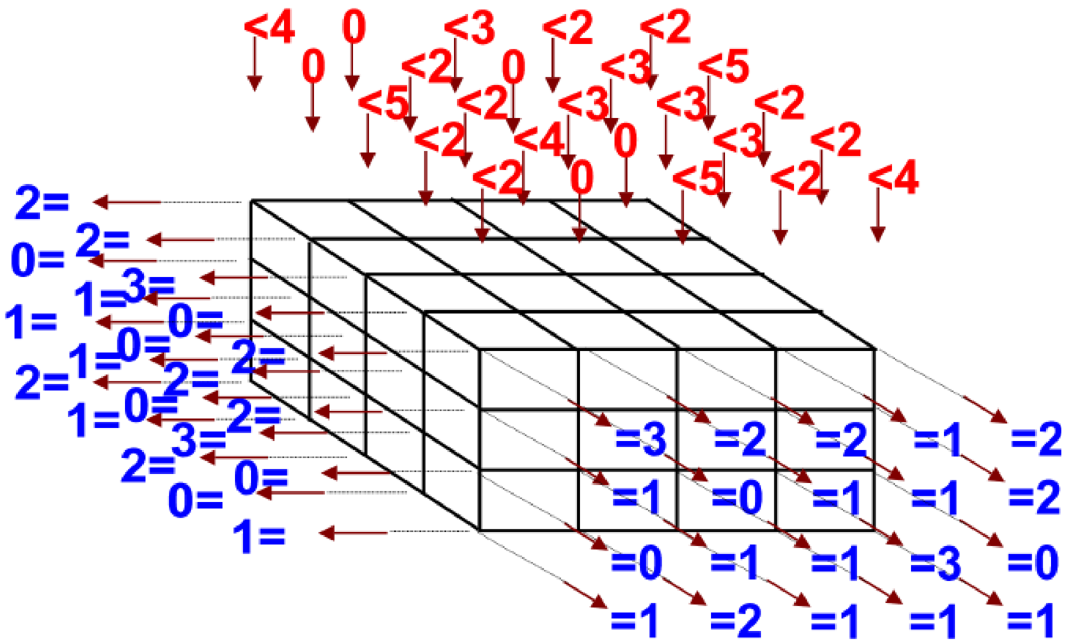
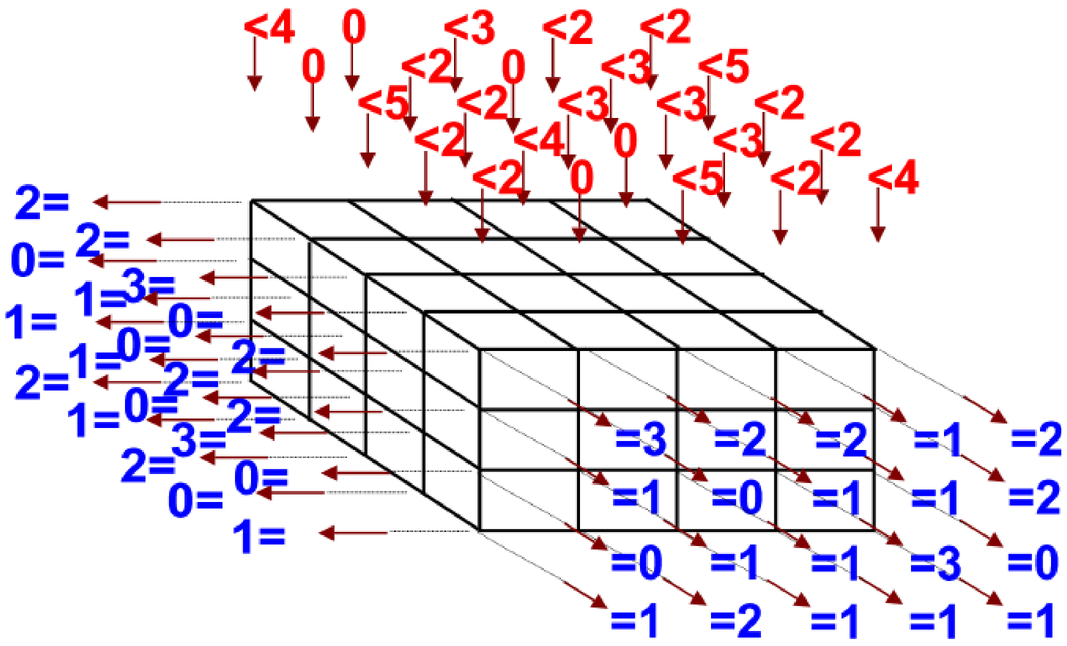
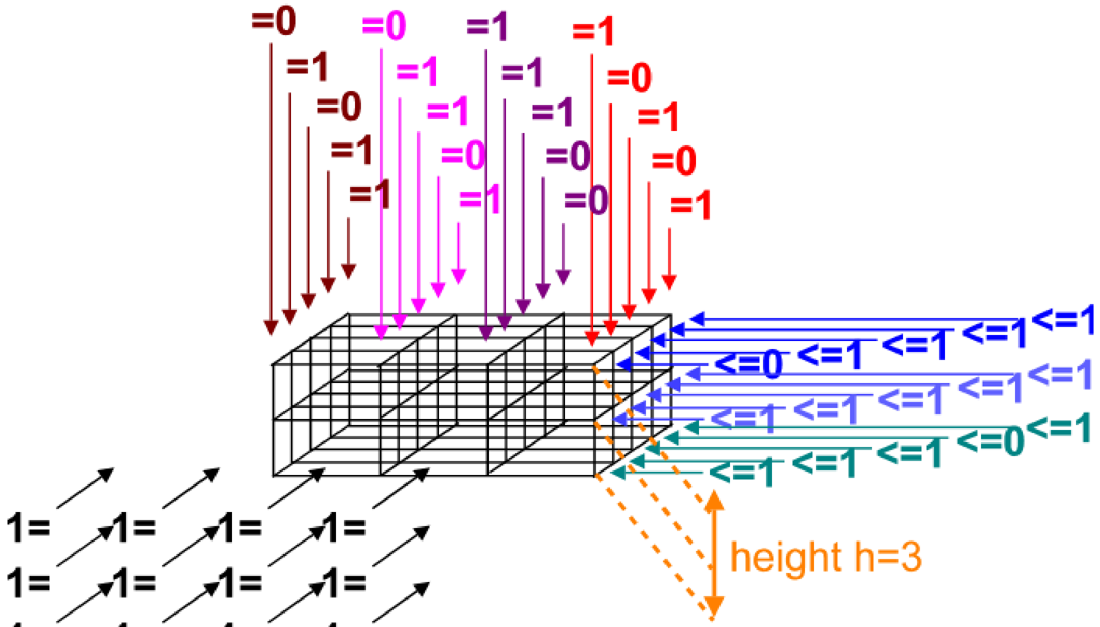


A timetable instance.



A timetable instance.

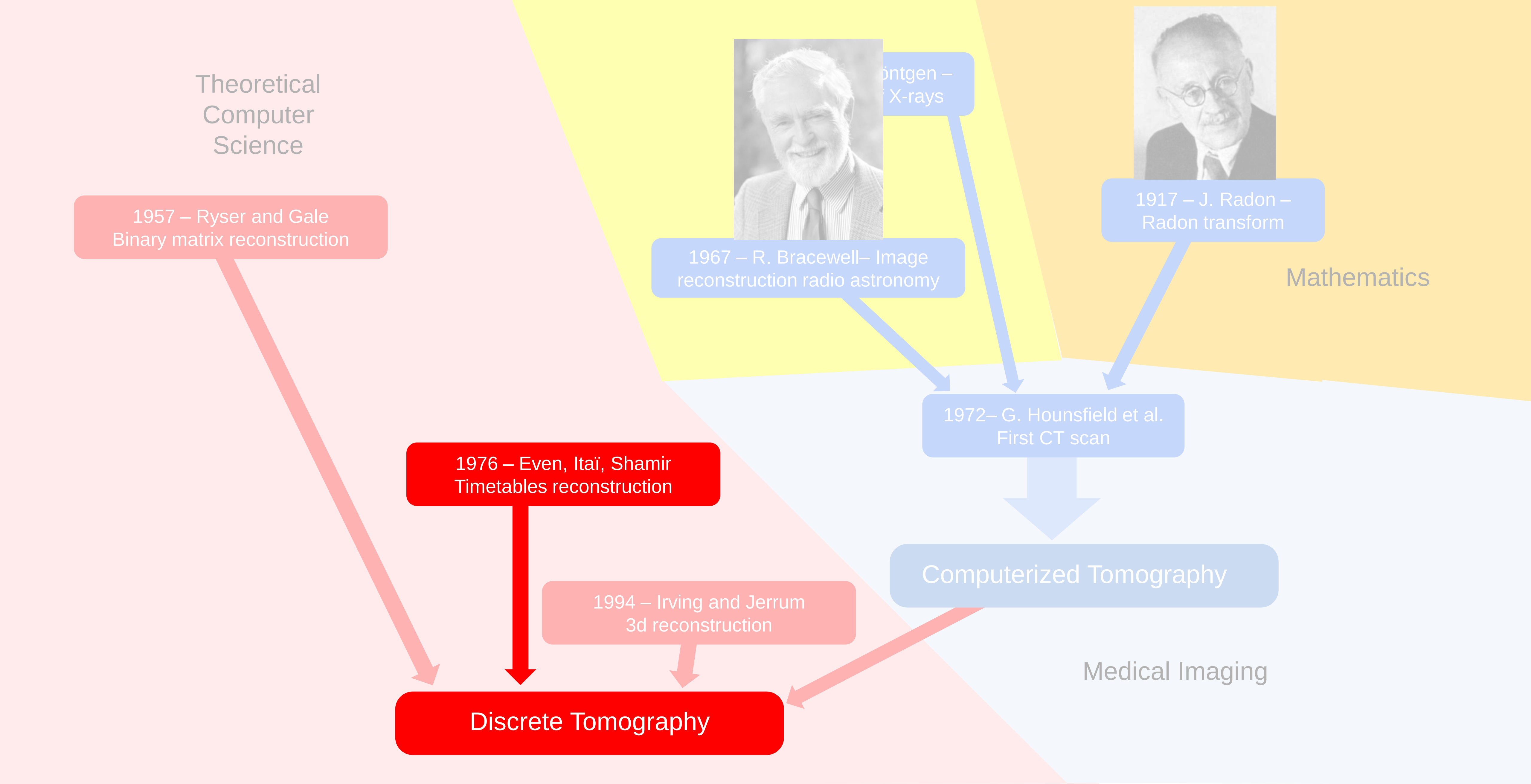
Theorem (Even, Itaï, Shamir– 1976):
Timetable problems are NP-complete.



A restricted Timetable instance.

Theorem (Even, Itaï, Shamir– 1976):
Timetable problems are NP-complete.

The Origins of Discrete Tomography



Theoretical Computer Science

1957 – Ryser and Gale
Binary matrix reconstruction

öntgen –
X-rays

1917 – J. Radon –
Radon transform

1967 – R. Bracewell– Image reconstruction radio astronomy

Mathematics

1972– G. Hounsfield et al.
First CT scan

1976 – Even, Itaï, Shamir
Timetables reconstruction

Computerized Tomography

1994 – Irving and Jerrum
3d reconstruction

Medical Imaging

Discrete Tomography

The Origins of Discrete Tomography

Theoretical
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Science

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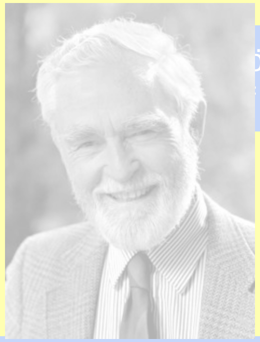

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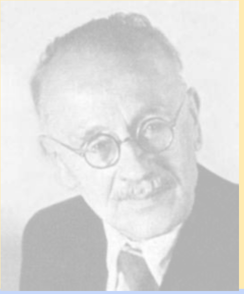

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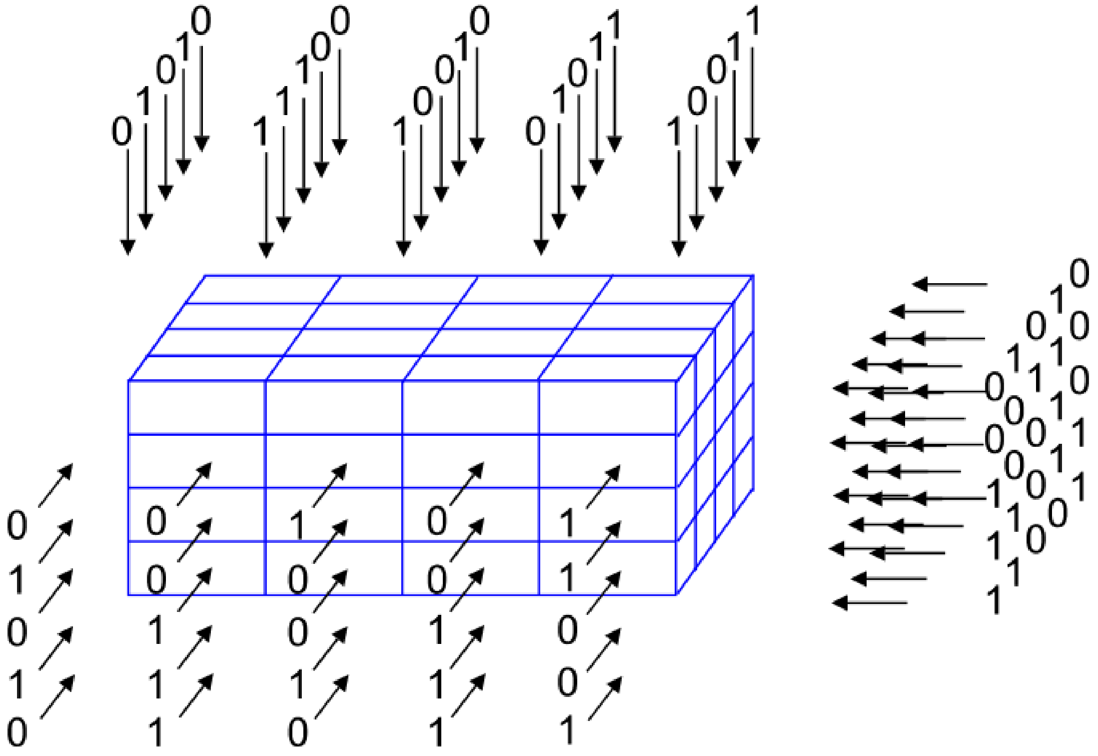
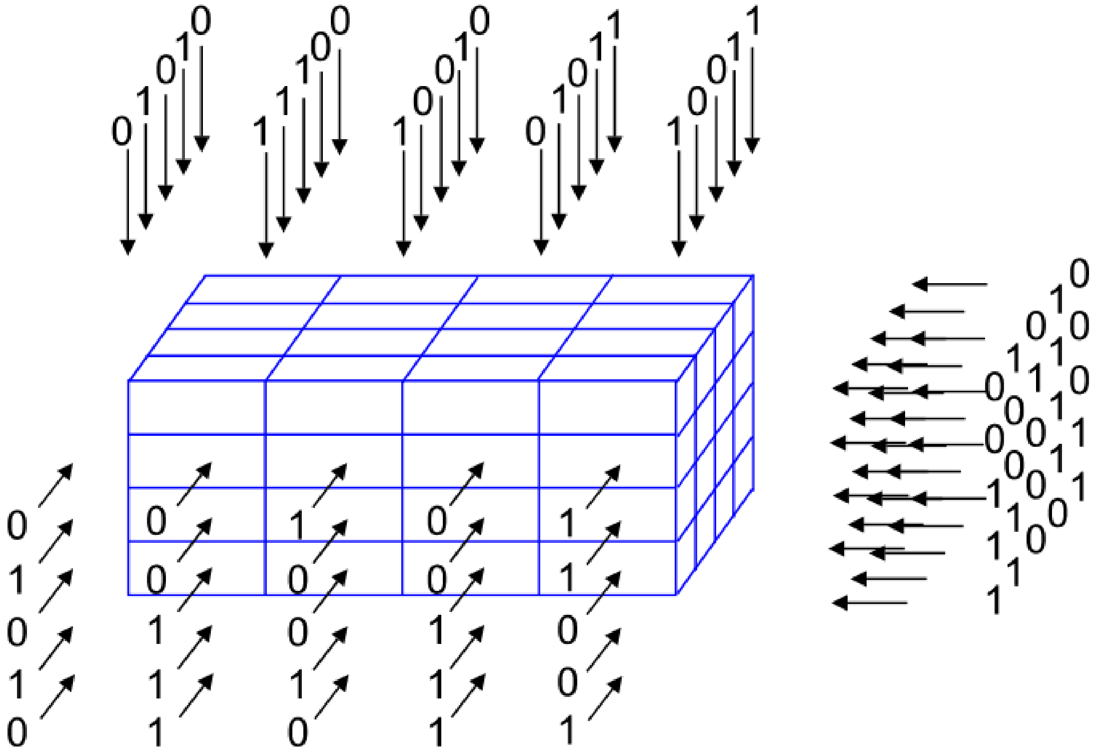
Medical Imaging

Cryptography

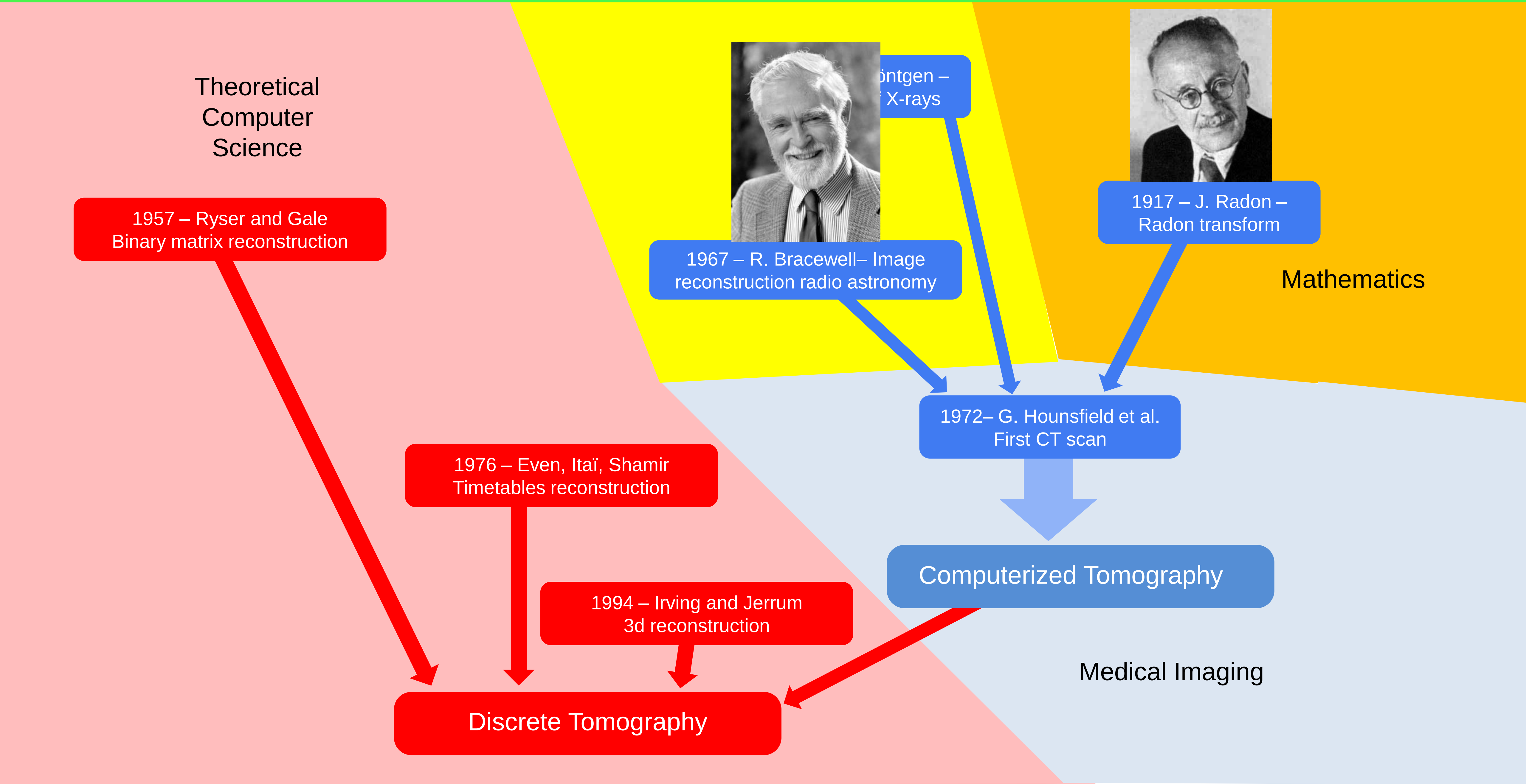




Theorem (Irving–Jerrum 1994):

Discrete Tomography with 3 directions in 3D is NP-complete.

The Origins of Discrete Tomography



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Computerized Tomography

Medical Imaging

A 1h talk in 30 minutes!

What are we talking about ?

0 Discrete Tomography in brief

Overview of the story…

1 The Origins of Discrete Tomography

One result

2 Uniqueness

One open problem

3 Alon’s open problem

A 1h talk in 30 minutes!

What are we talking about ?

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Overview of the story…

1 The Origins of Discrete Tomography

One result

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One open problem

3 Alon’s open problem

②

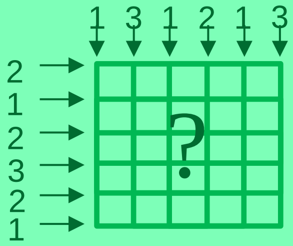
Uniqueness

No uniqueness in general

2 Uniqueness

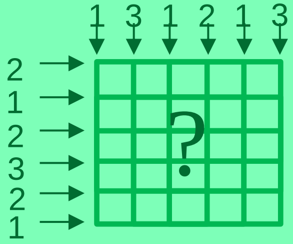
State of the art

1957 – Ryser and Gale
Binary matrix reconstruction

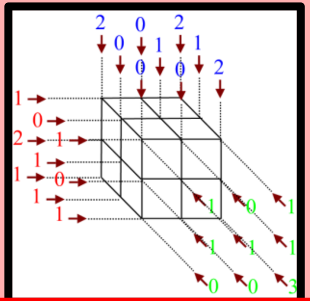


Polynomial time

1957 – Ryser and Gale
Binary matrix reconstruction



Polynomial time



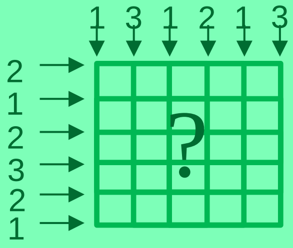
NP-hard

1994 – Irving and Jerrum
3d reconstruction

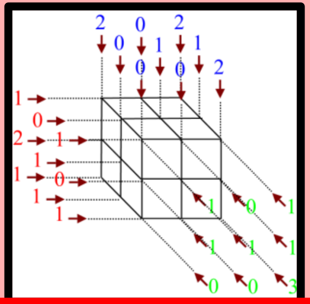
② A Uniqueness Result

State of the art

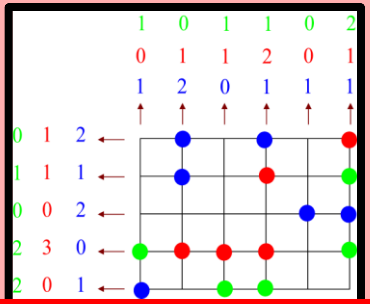
1957 – Ryser and Gale
Binary matrix reconstruction



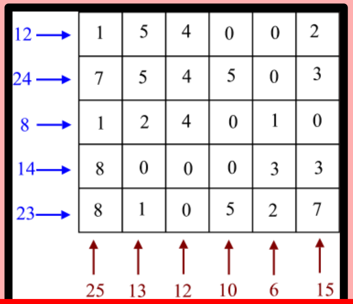
Polynomial time



1994 – Irving and Jerrum
3d reconstruction



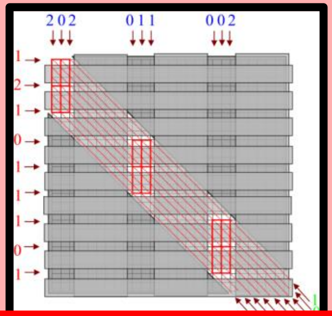
2009 – Dürr, Guinez Matamala
3 colors



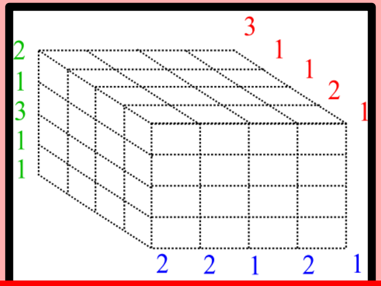
12 →	1	5	4	0	0	2
24 →	7	5	4	5	0	3
8 →	1	2	4	0	1	0
14 →	8	0	0	0	3	3
23 →	8	1	0	5	2	7
	25	13	12	10	6	15

2008 – G.
Prescribed coefficients

NP-hard



1999 – Gardner Gritzmann Prangenberg
2d with 3 directions



2001 – Brunetti, Del Lungo, G.
3d with planar X-rays

2001 – Picouleau
Domino tilings

State of the art

1957 – Ryser and Gale
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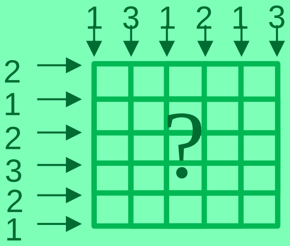
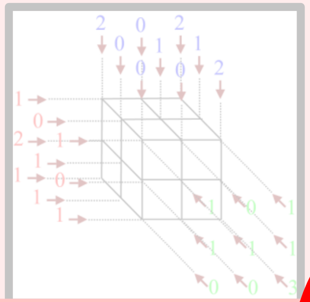


Polynomial time

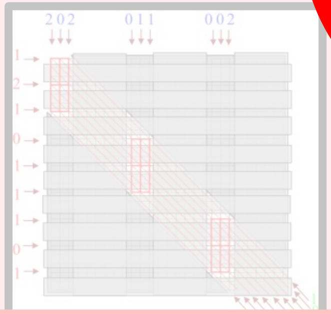


1994 – Irving and Jerr
3d reconstruction

2008 – G.
bed coefficients

NP-hard

So many hardness results!!!!!



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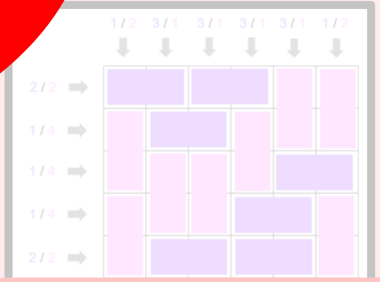

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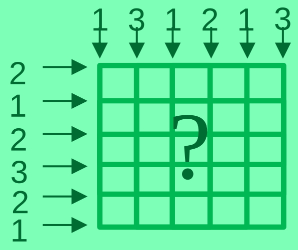
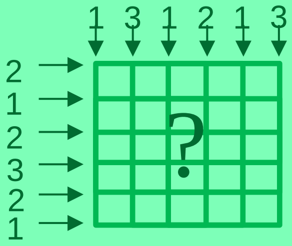


Polynomial time

What happens if we search for
a solution
with some wanted geometric properties ?

1957 – Ryser and Gale
Binary matrix reconstruction



Polynomial time

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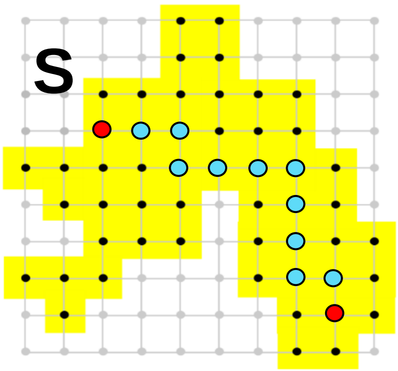
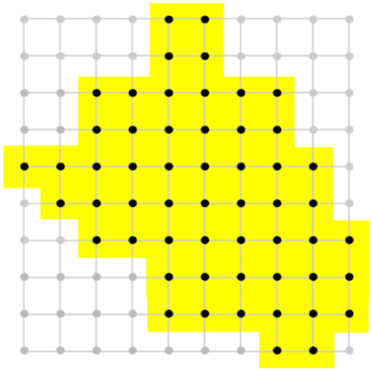
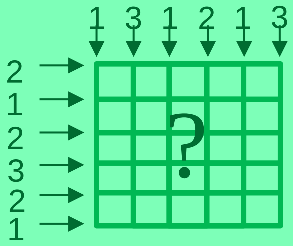


Polyomino



Convex lattice set

1957 – Ryser and Gale
Binary matrix reconstruction



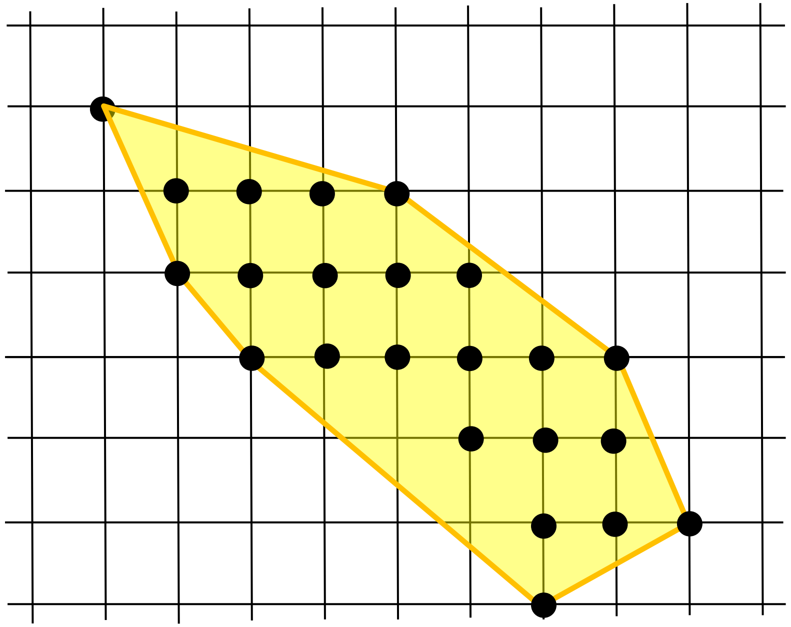
Polynomial time

What happens if we search for
a solution
with some wanted geometric properties ?

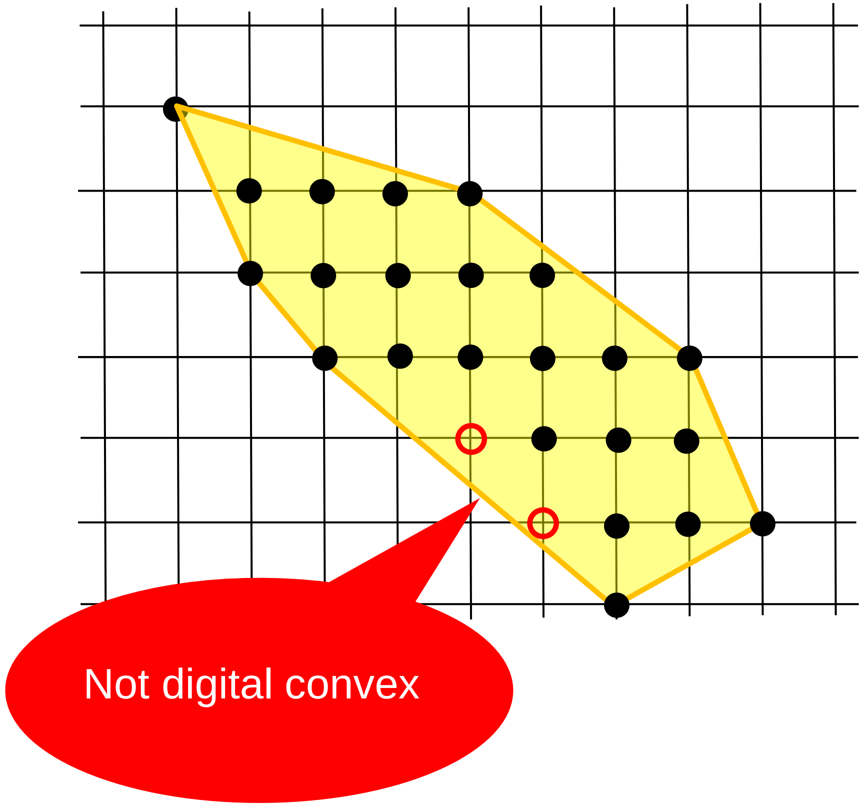
Connectivity

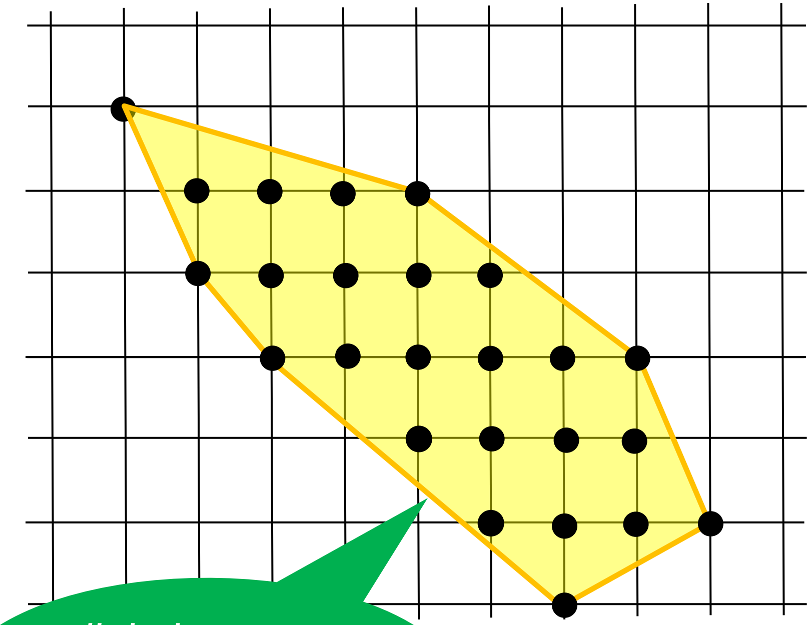
Convexity

A Uniqueness Result



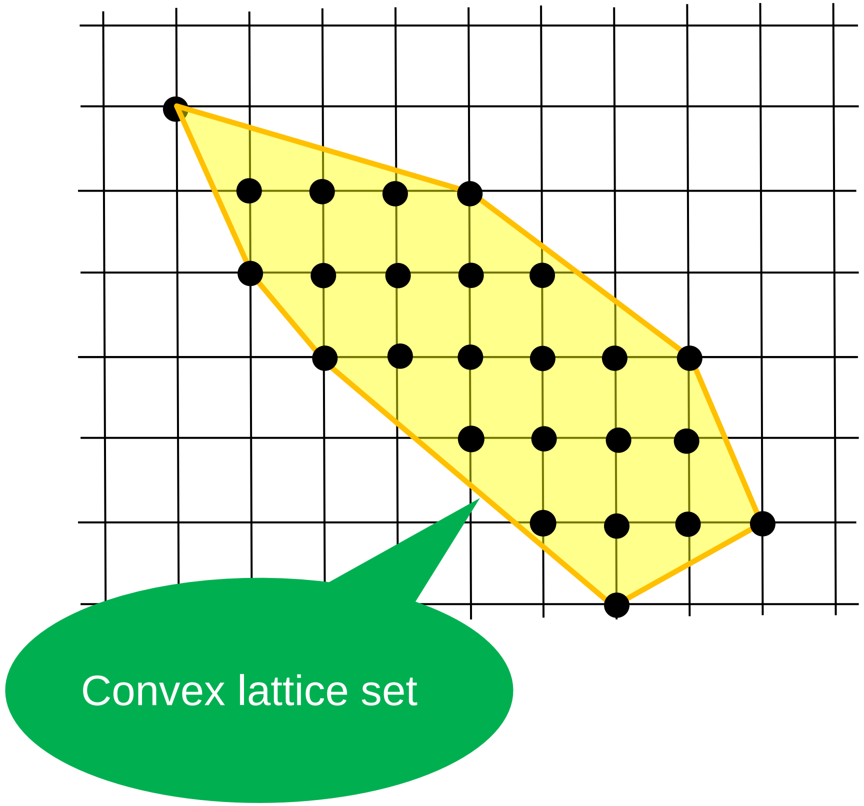
A Uniqueness Result





digital convex
or just
convex

A Uniqueness Result



GOOD NEWS

Theorem (Gardner Gritzmann1997 + Brunetti Daurat 2003):
For some directions, convex lattice sets
are uniquely determined by their X-rays
and can be reconstructed in polynomial time.

② A Uniqueness Result

For some directions ?

GOOD NEWS

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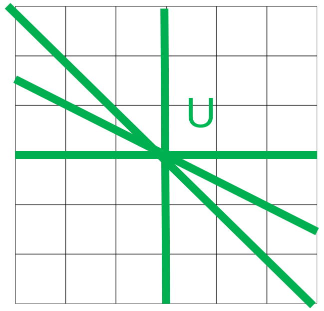

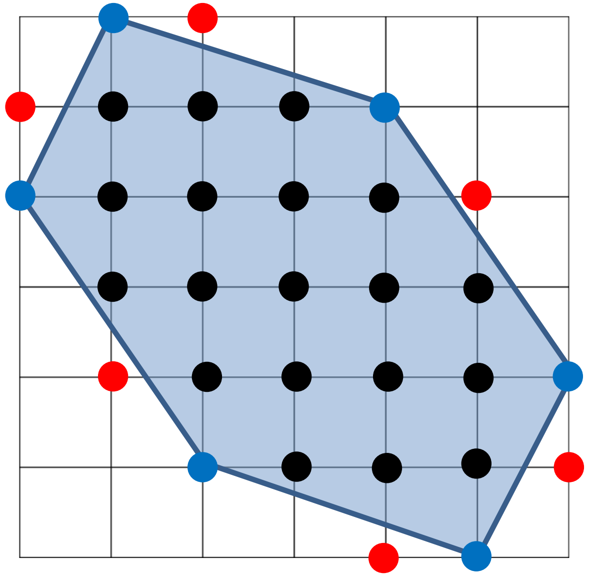

For some directions ?

Theorem (Gardner Gritzmann1997 + Brunetti Daurat 2003):
For some directions, convex lattice sets
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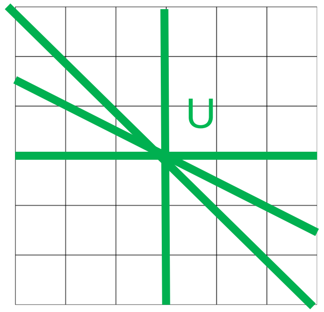
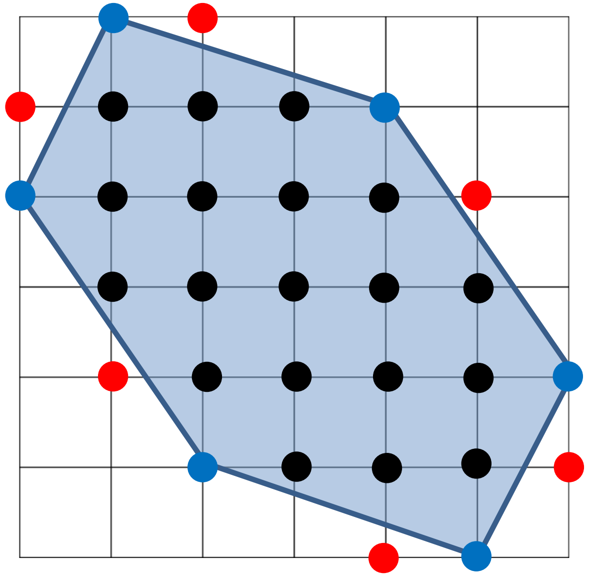
② A Uniqueness Result

General Uniqueness result is false





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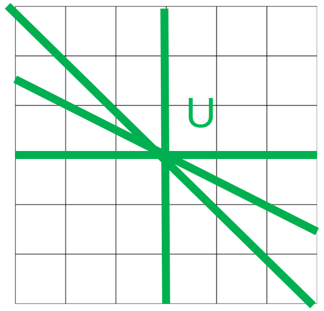
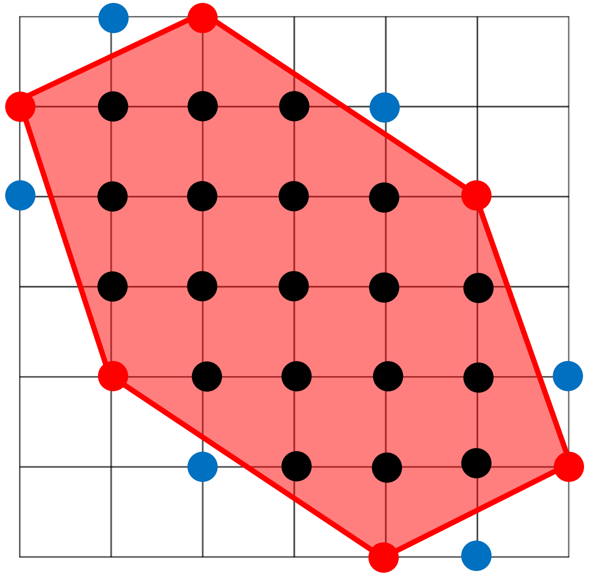
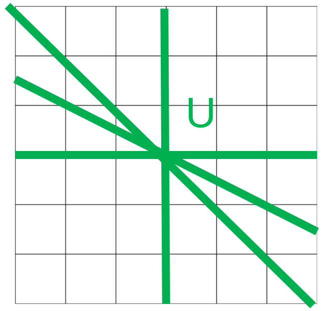
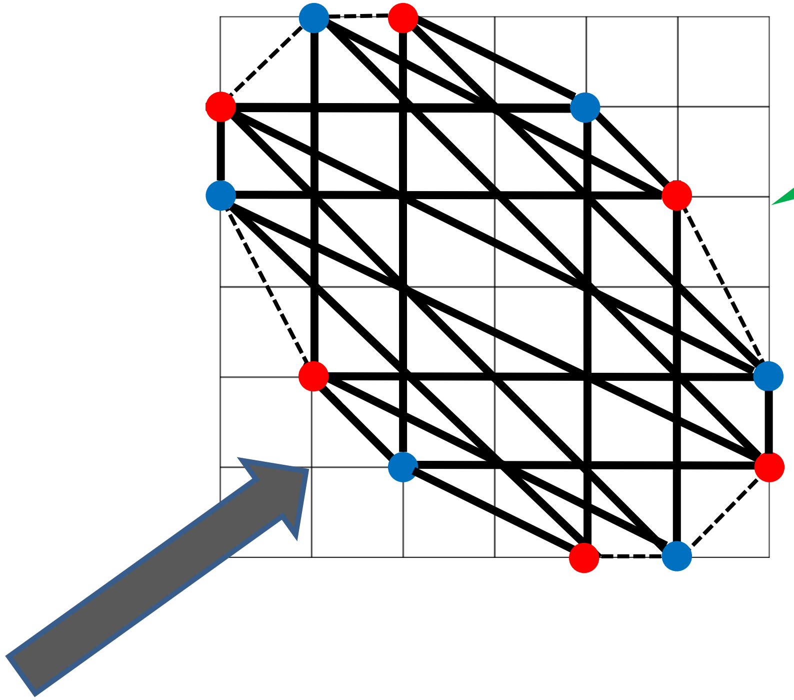




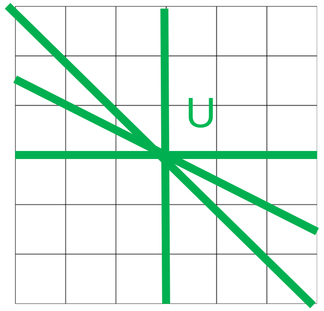


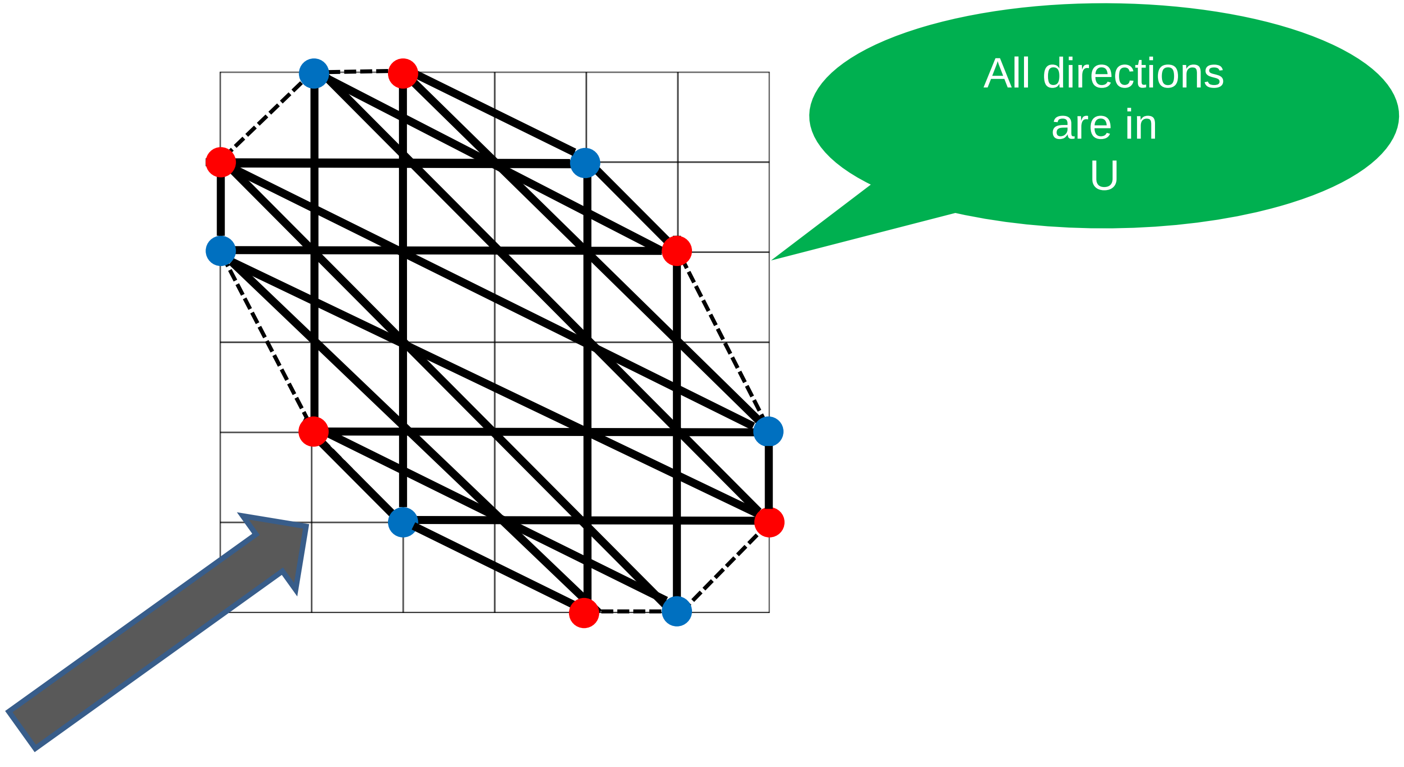


All directions
are in
U

U-polygon

Uniqueness result?



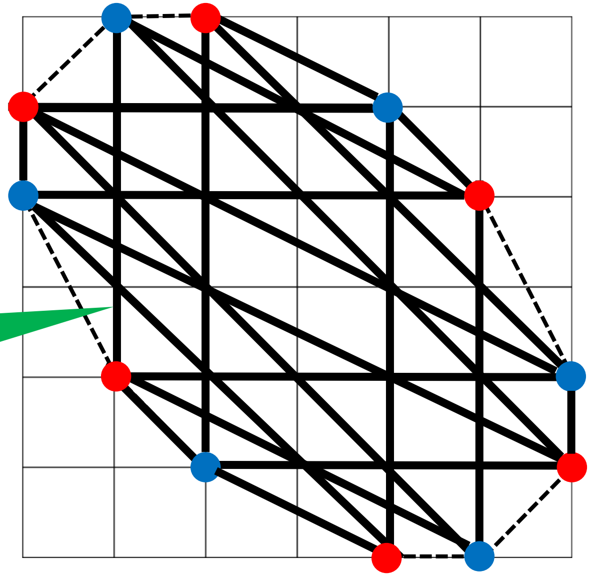


Uniqueness result for the set of directions U

if and only if

There exists NO U-polygon

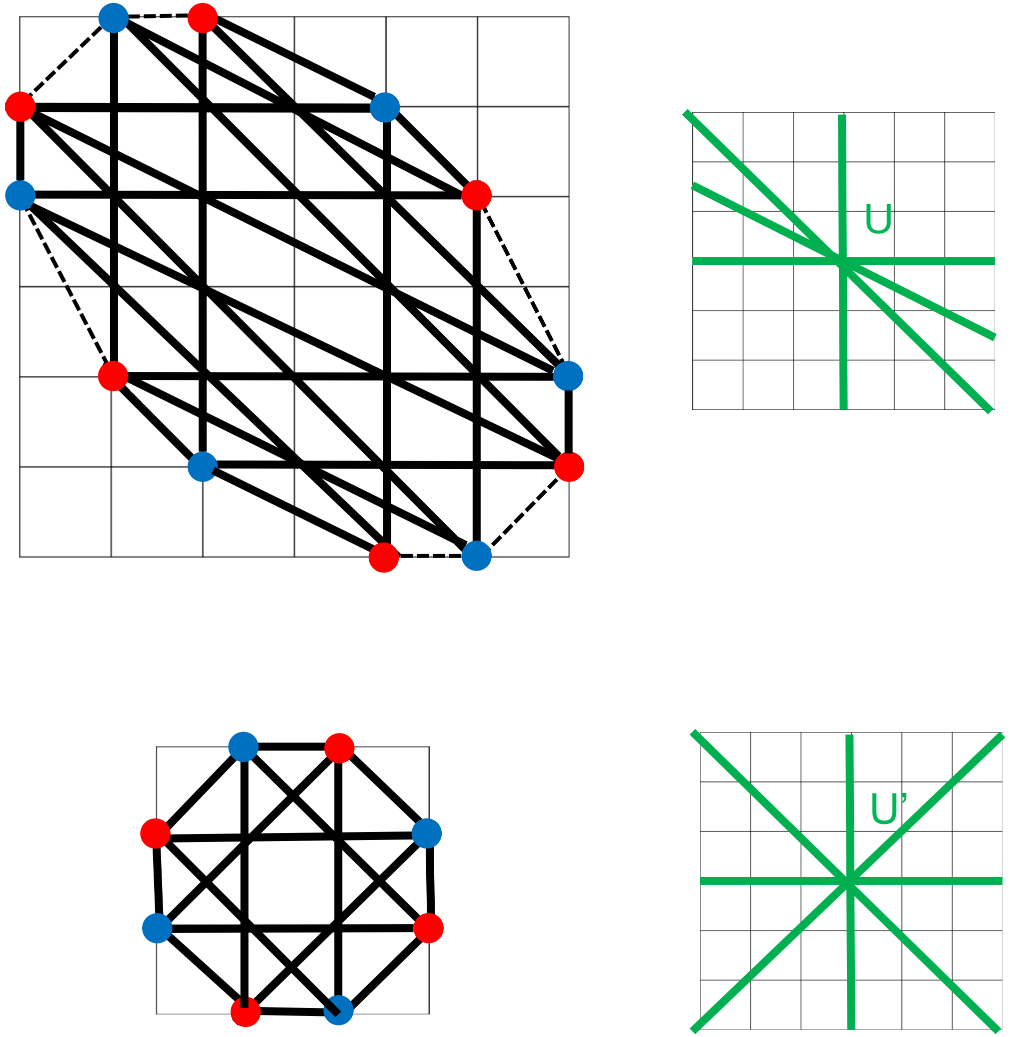
Uniqueness result?



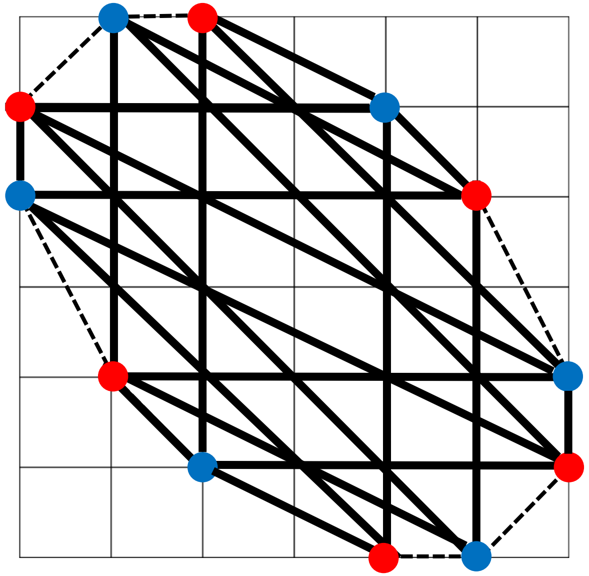
When does it exist ?

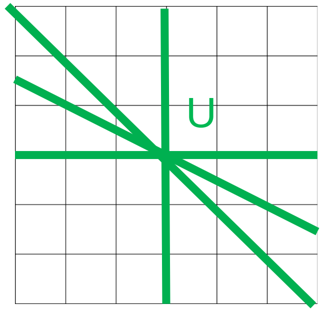
There exists NO U-polygon

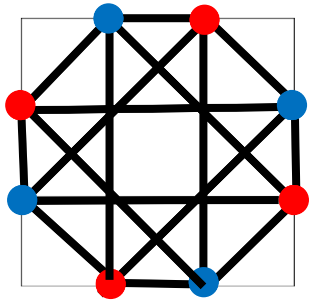
Uniqueness result?

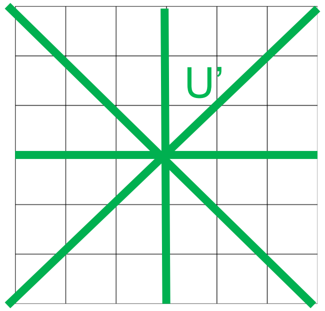


For which sets of directions U does there exist *U-polygons* ?



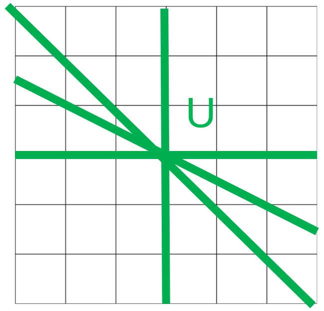


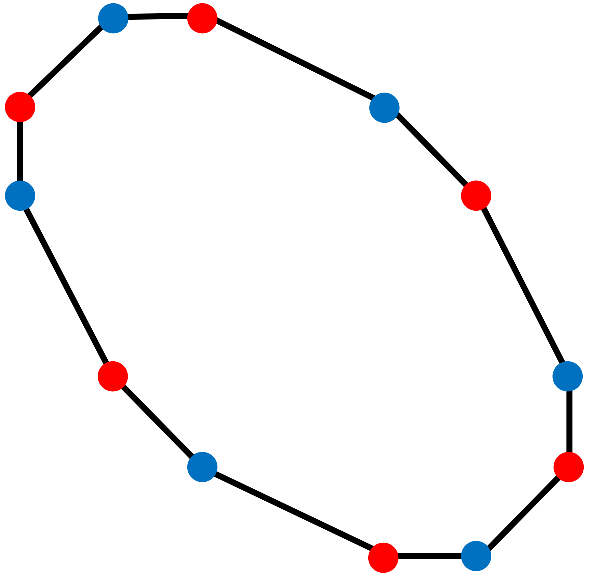




A Uniqueness Result

U-polygon





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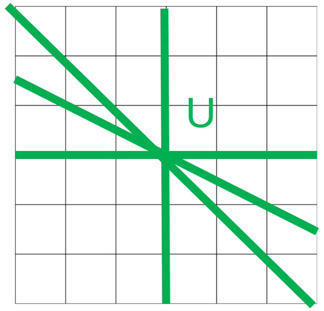
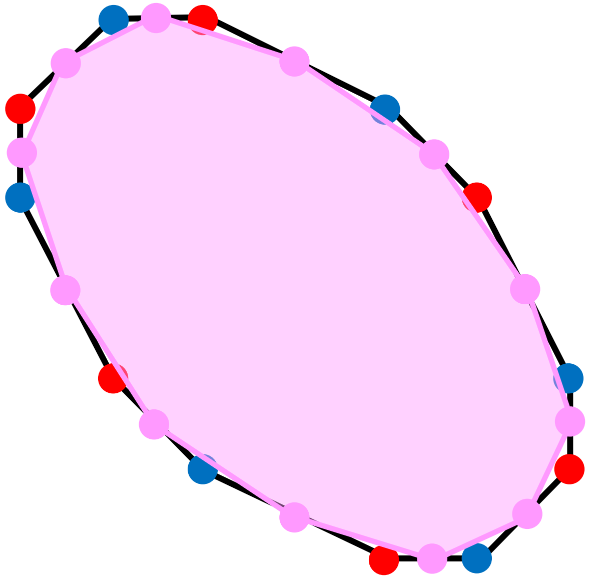
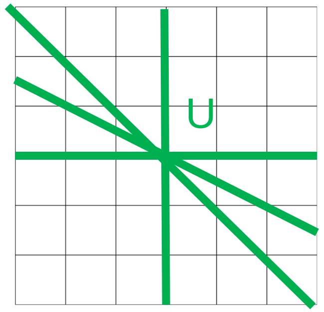
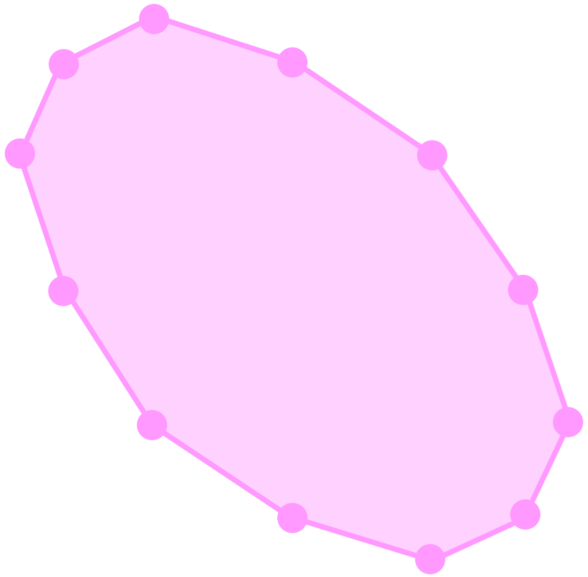




② A Uniqueness Result

Midpoint polygon

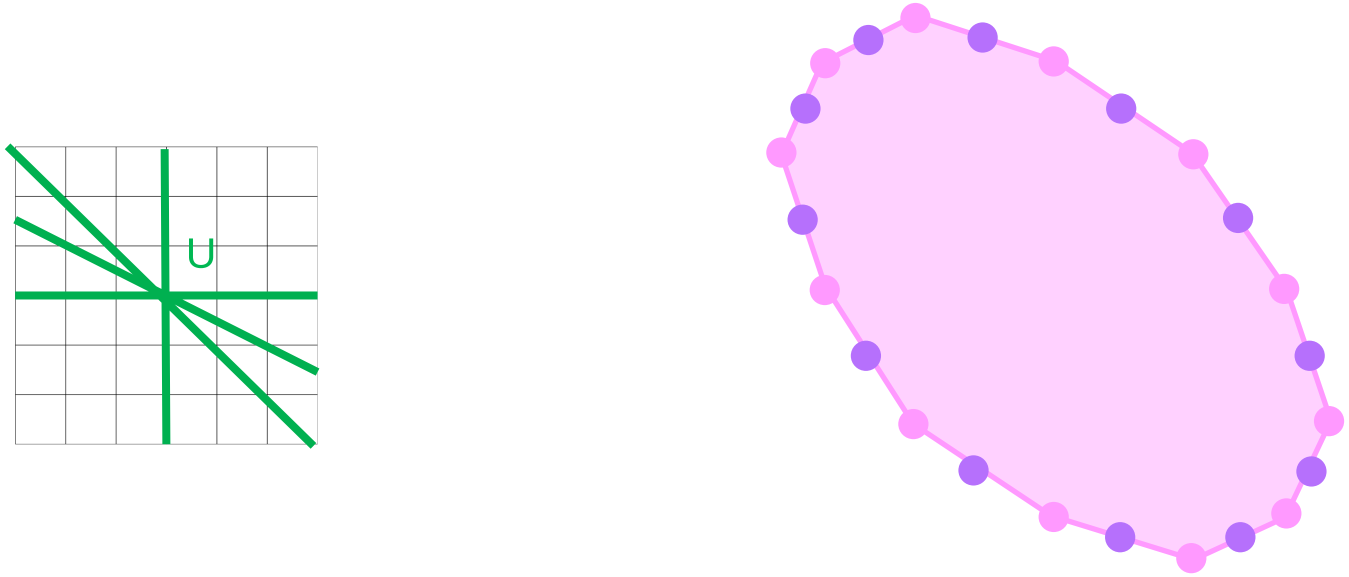




②

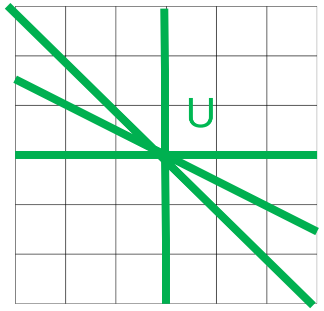
A Uniqueness Result

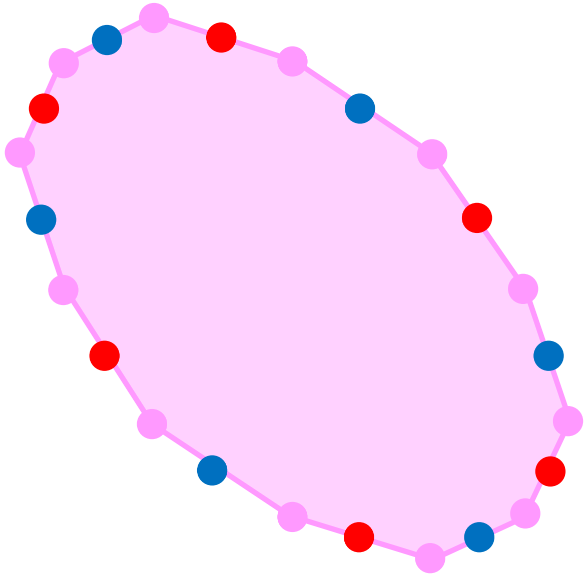
Second iteration



A Uniqueness Result

Second iteration





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A Uniqueness Result

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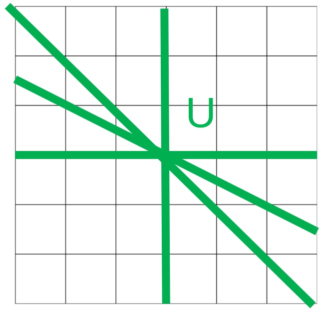
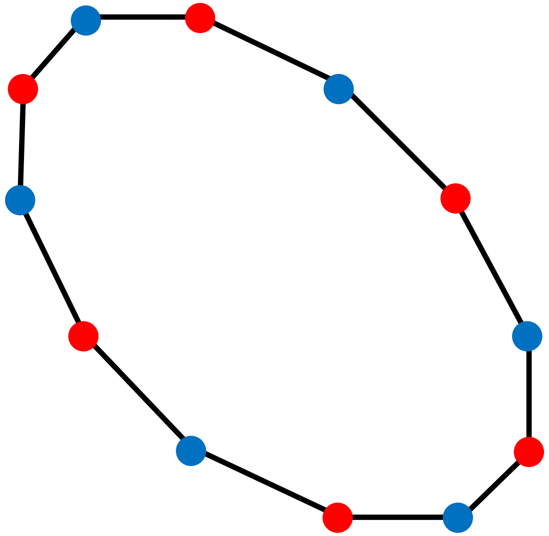
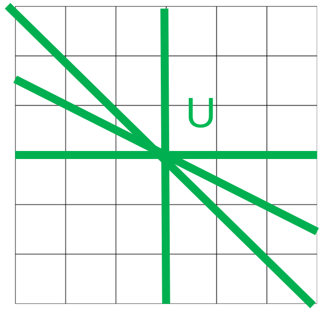


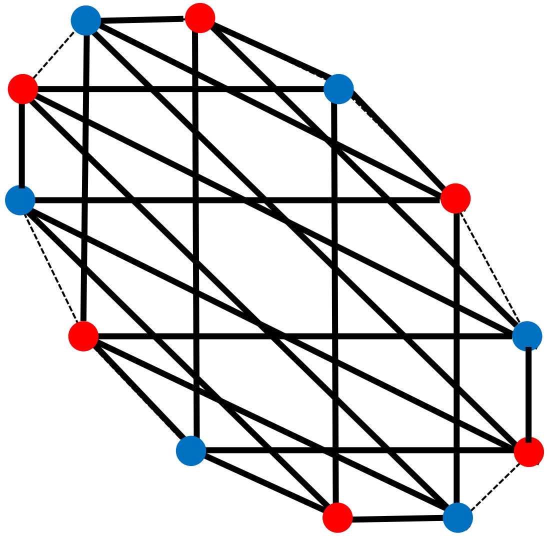


②

A Uniqueness Result

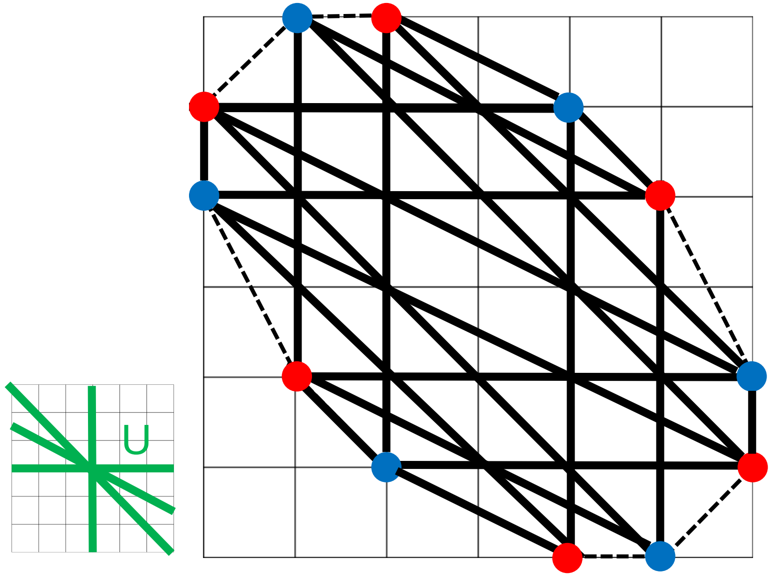
New U-polygon!





② A Uniqueness Result

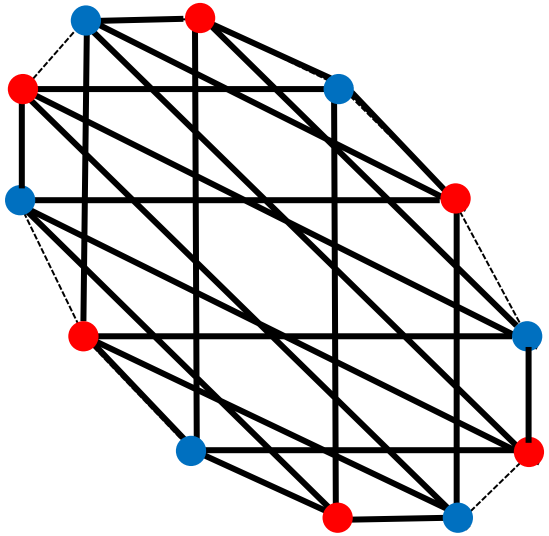
Repeat the process



U-polygon

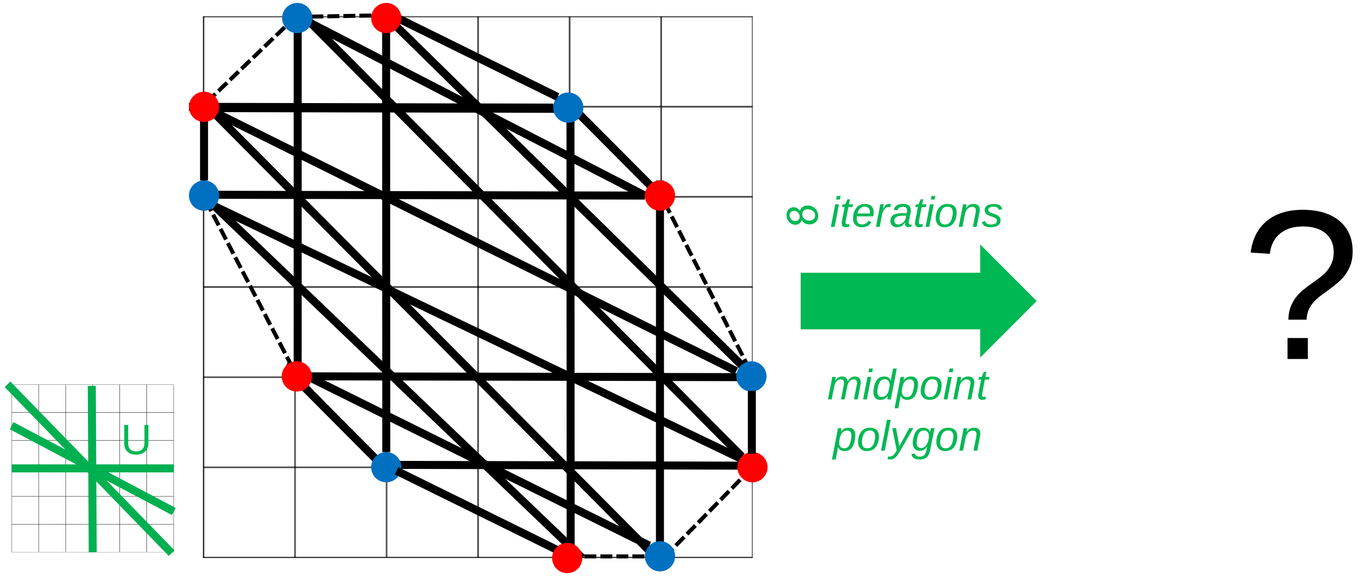
2 iterations

midpoint polygon



U-polygon

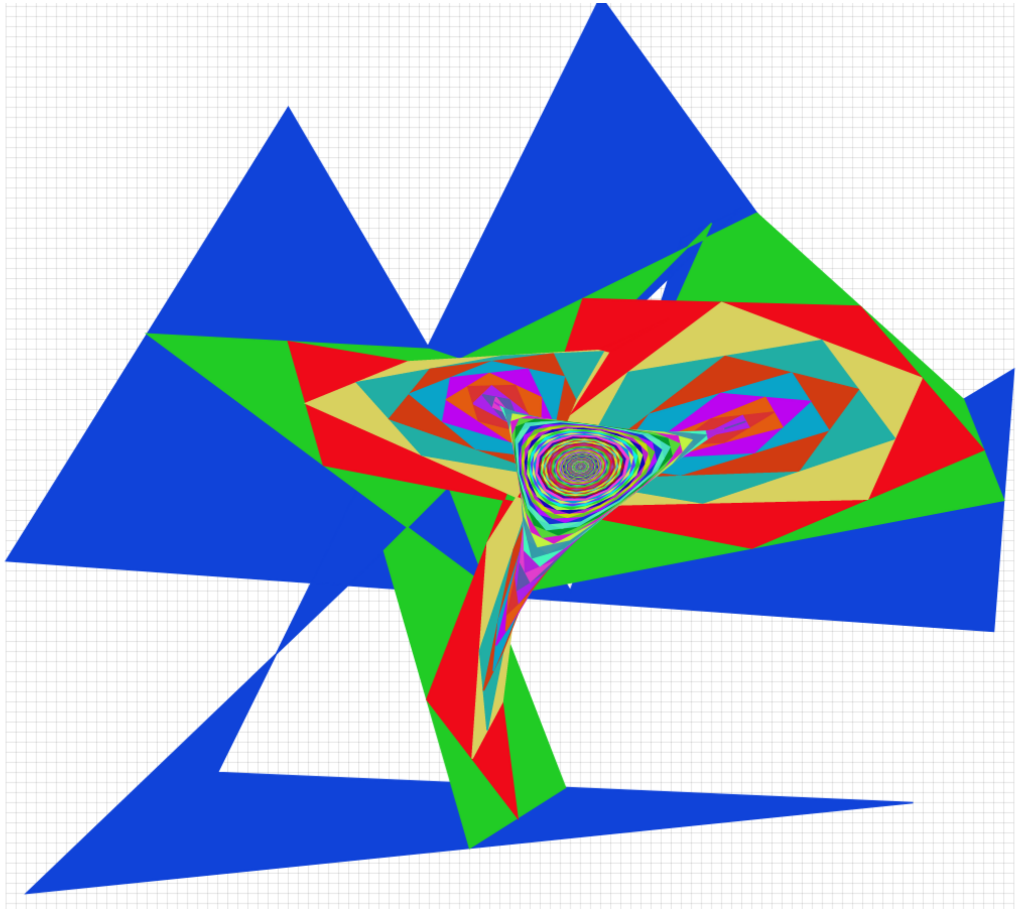
Repeat the process



∞ iterations

midpoint polygon

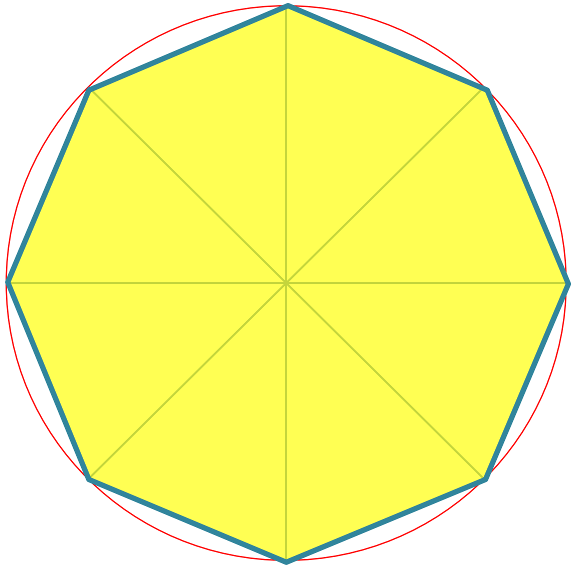
U-polygon



By iterating the midpoint transformation, the limit is an affinely regular polygon…

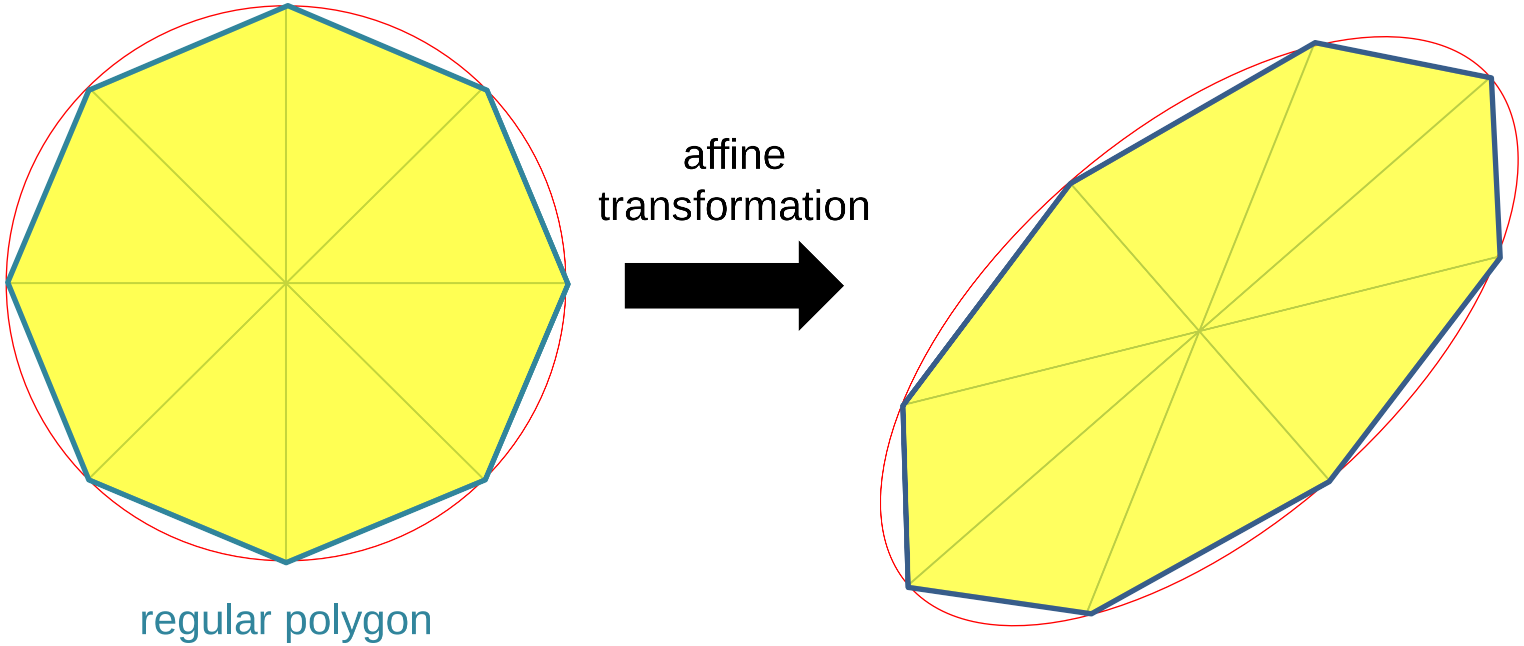
A Uniqueness Result

Repeat the process



regular polygon

By iterating the midpoint transformation, the limit is an affinely regular polygon…

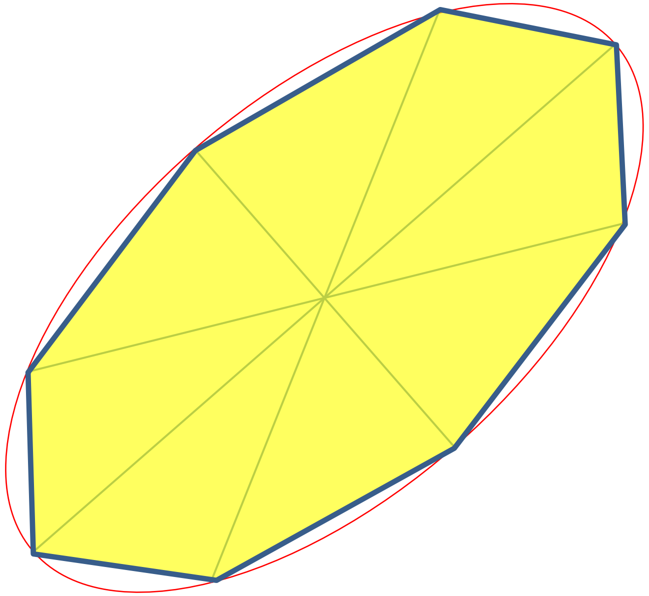


affine transformation

regular polygon

affinely regular polygon

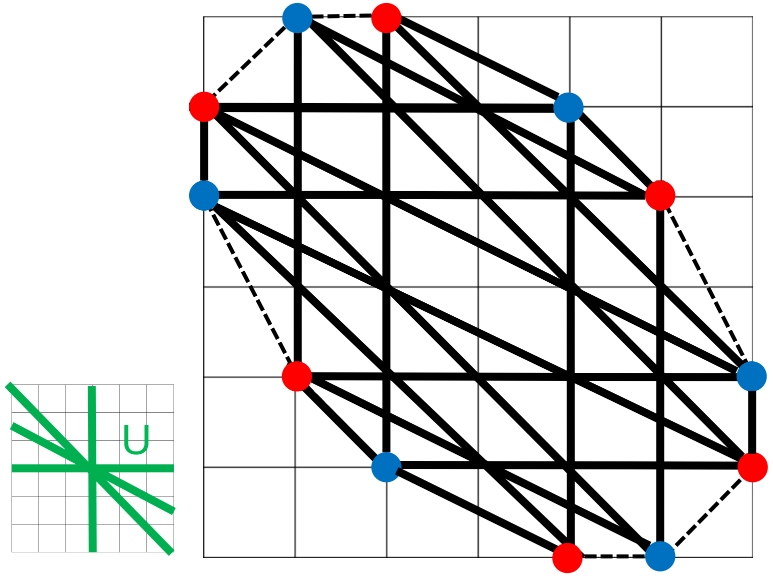
By iterating the midpoint transformation, the limit is an affinely regular polygon…



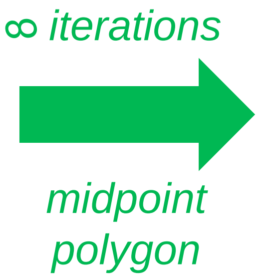
affinely regular polygon

By iterating the midpoint transformation, the limit is an affinely regular polygon…

A Uniqueness Result

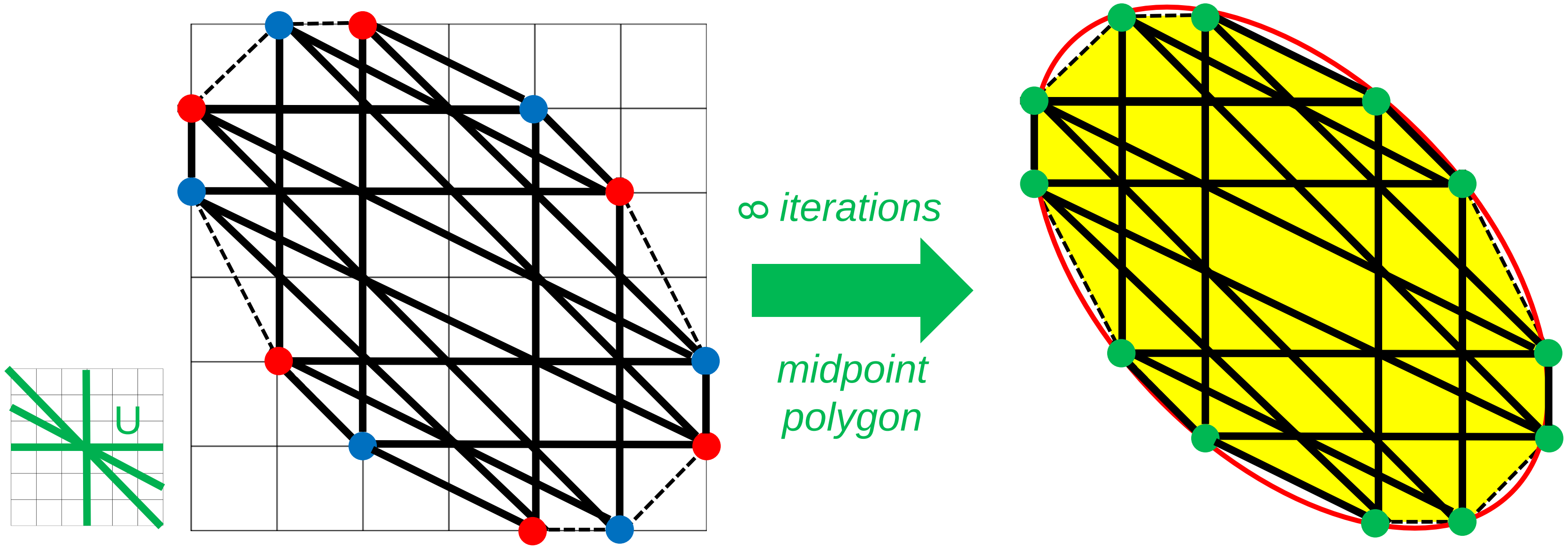


U-polygon



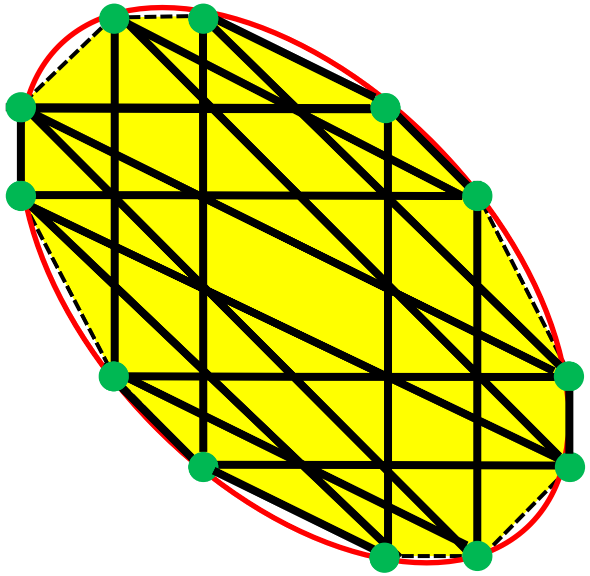
?

Convergence



U-polygon

affinely regular U-polygon



affinely regular U-polygon

Theorem (Gardner Gritzmann1997):

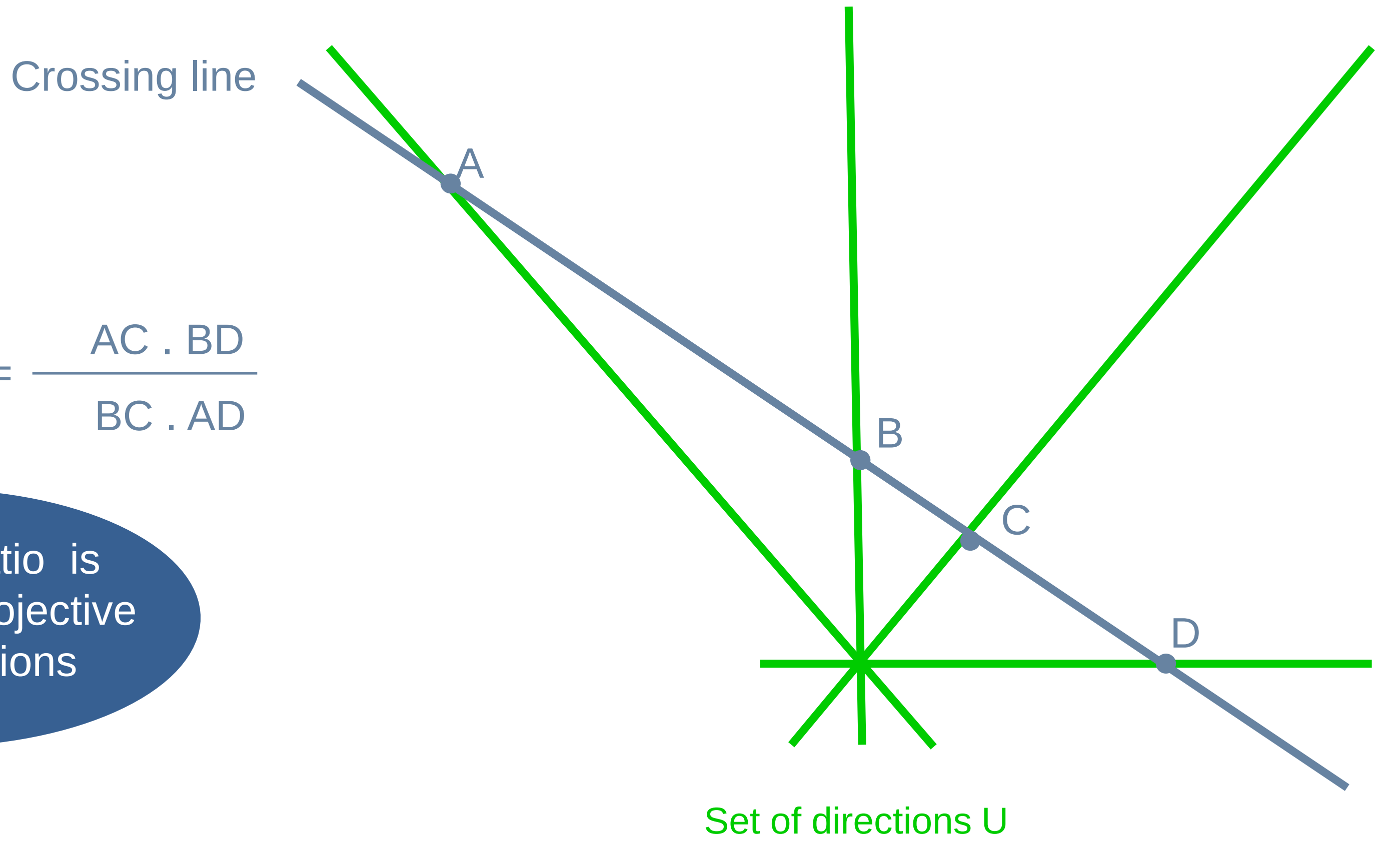
If *U* has 2 or 3 directions : affinely regular *U*-polygons always exist

If *U* has 4 directions : affinely regular *U*-polygons exist iff their cross-ratio is in { 4/3 ,3/2 ,2 ,3 ,4 }

If *U* has 7 directions : affinely regular *U*-polygons never exist

② A Uniqueness Result

Cross-ratio



$$\text{cross-ratio} = \frac{AC \cdot BD}{BC \cdot AD}$$

The cross-ratio is invariant by projective transformations

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② A Uniqueness Result

Non trivial proof with:

Trignometry

Cyclotomic polynomials

P-adic numbers

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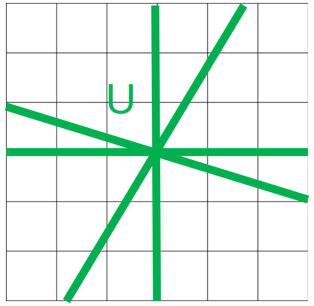

For cross-ratio not in { 4/3 ,3/2 ,2 ,3 ,4 }

No affinely regular U-polygon

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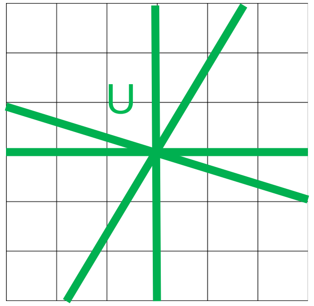
Consequences



For cross-ratio not in { 4/3 ,3/2 ,2 ,3 ,4 }

No affinely regular U-polygon

⬇

No U-polygon

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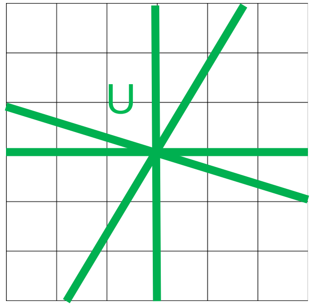

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No affinely regular U-polygon

⇓

No U-polygon

⇓

Convex lattice sets are uniquely determined by their X-rays in direction U

Theorem (Gardner Gritzmann1997):

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Theorem (Gardner Gritzmann1997 + Brunetti Daurat 2003):
For some directions, convex lattice sets
are uniquely determined by their X-rays
and can be reconstructed in polynomial time.

GOOD NEWS

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For some directions, convex lattice sets
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and can be reconstructed in polynomial time.

GOOD NEWS

Theorem (Barcucci-Del Lungo-Nivat-Pinzani 1996):
HV convex polyominoes can be reconstructed
from their horizontal and vertical X-rays
in polynomial time.

A 1h talk in 30 minutes!

What are we talking about ?

0 Discrete Tomography in brief

Overview of the story…

1 The Origins of Discrete Tomography

One result

2 Uniqueness

One open problem

3 Alon’s open problem

A 1h talk in 30 minutes!

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③ Alon’s Combinatorial Open Problem

Puzzle

n Integers from 1 to n

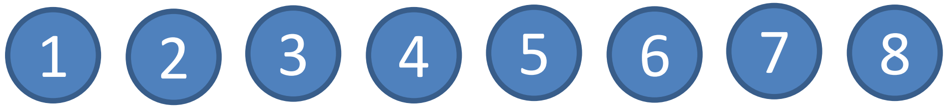
here $n=8$

1 2 3 4 5 6 7 8

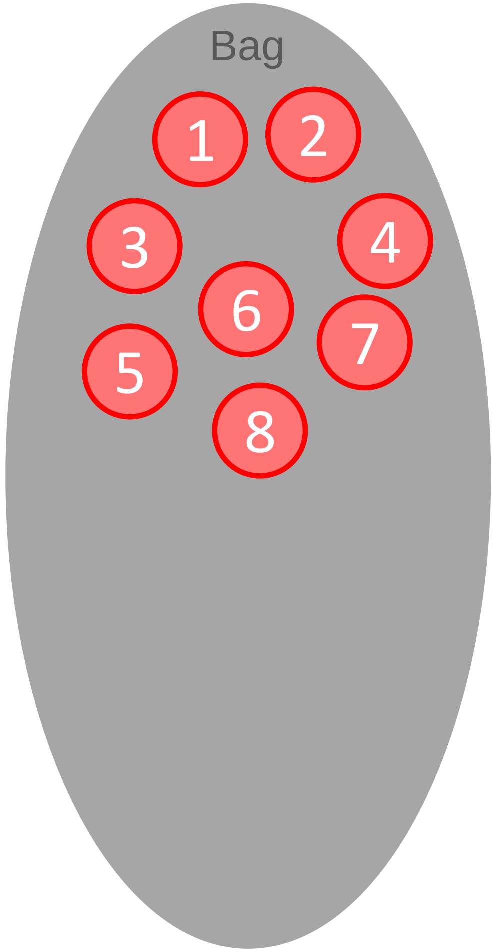
③ Alon’s Combinatorial Open Problem

Puzzle

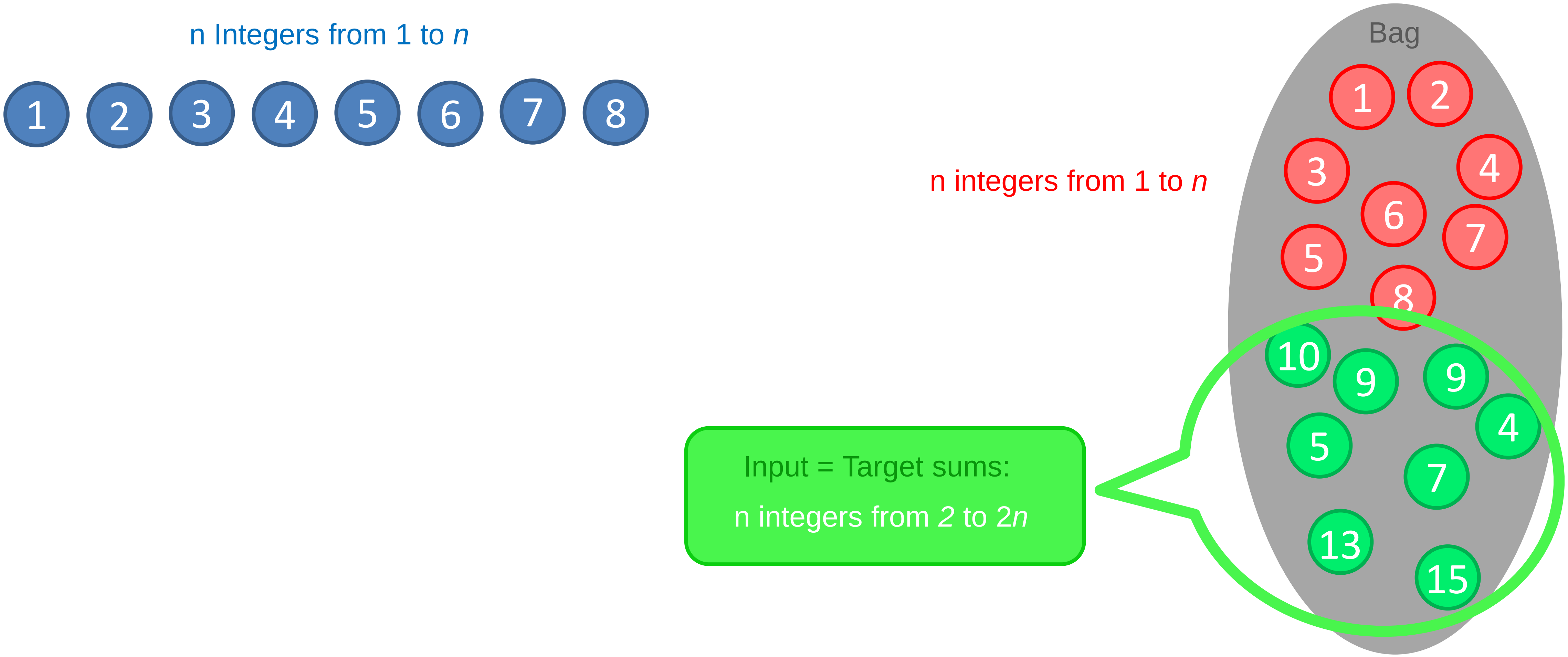
n Integers from 1 to *n*



n integers from 1 to *n*



Alon’s Combinatorial Open Problem



n Integers from 1 to *n*

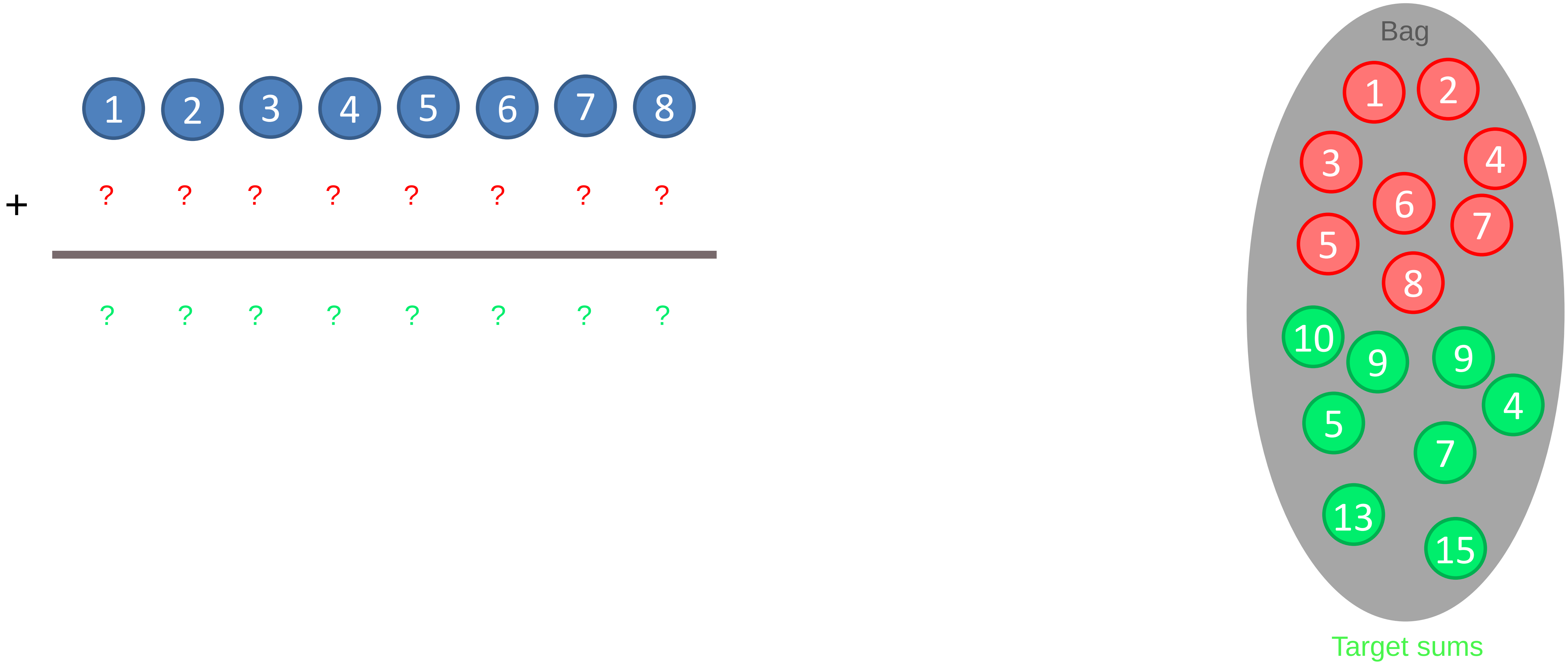
n integers from 1 to *n*

Input = Target sums:

n integers from *2* to 2*n*

③ Alon’s Combinatorial Open Problem

Puzzle



1 2 3 4 5 6 7 8

\+ ? ? ? ? ? ? ? ?

? ? ? ? ? ? ? ?

Bag

1 2 3 4 6 7 5 8

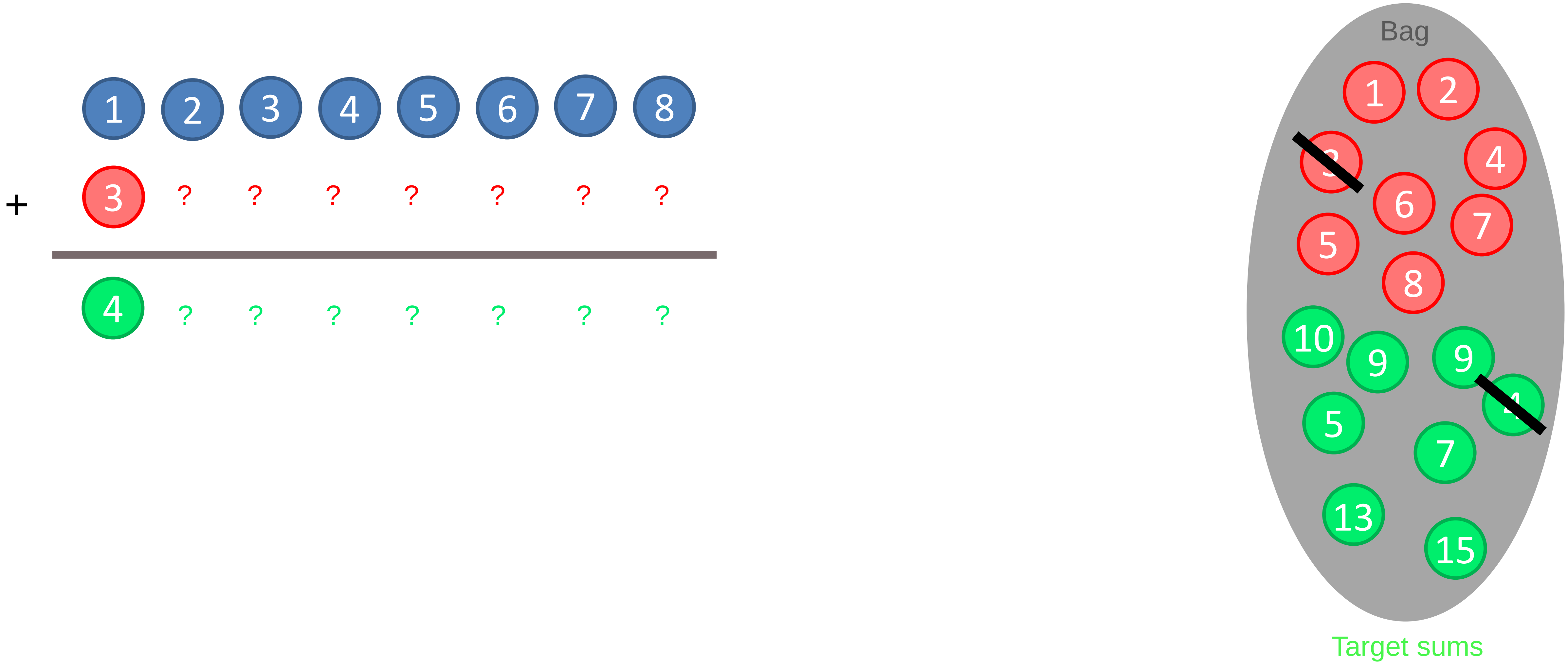
10 9 9 4 5 7 13 15

Target sums

③

Alon’s Combinatorial Open Problem

Puzzle



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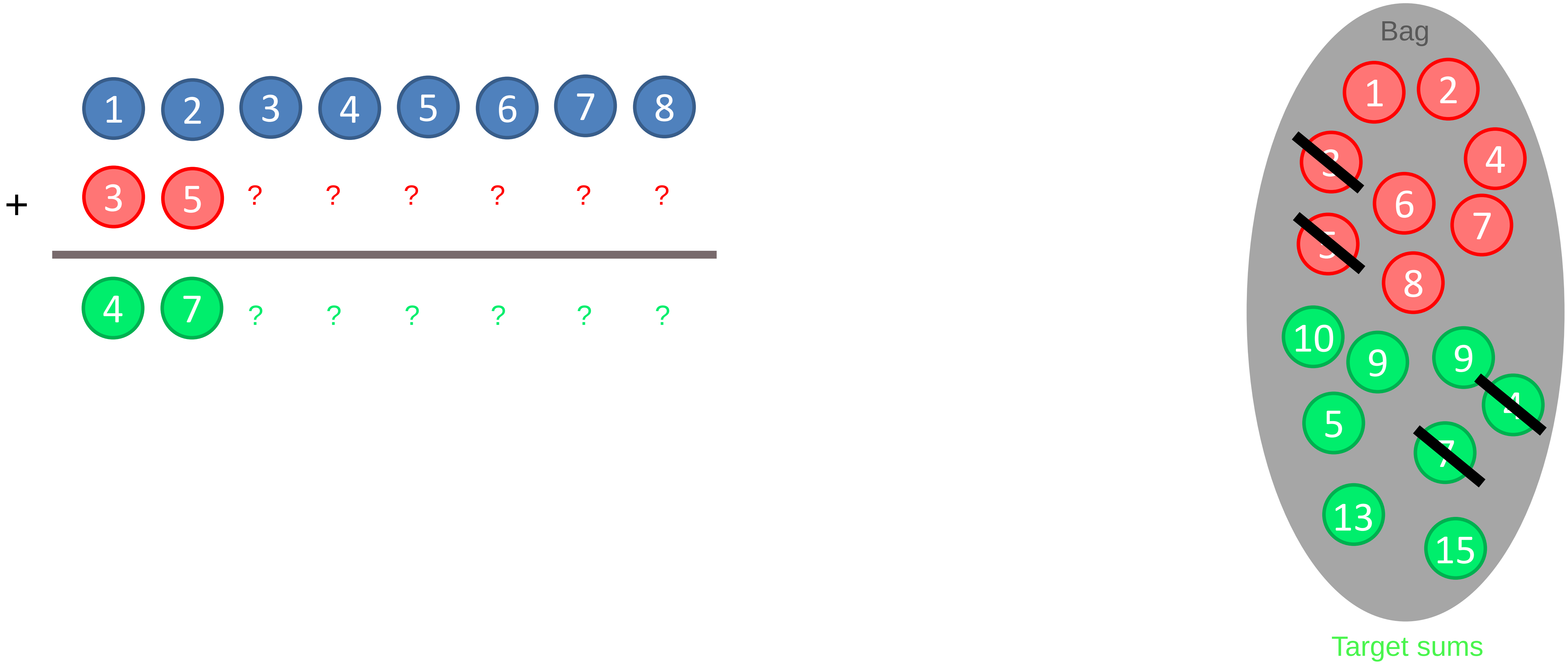
Alon’s Combinatorial Open Problem

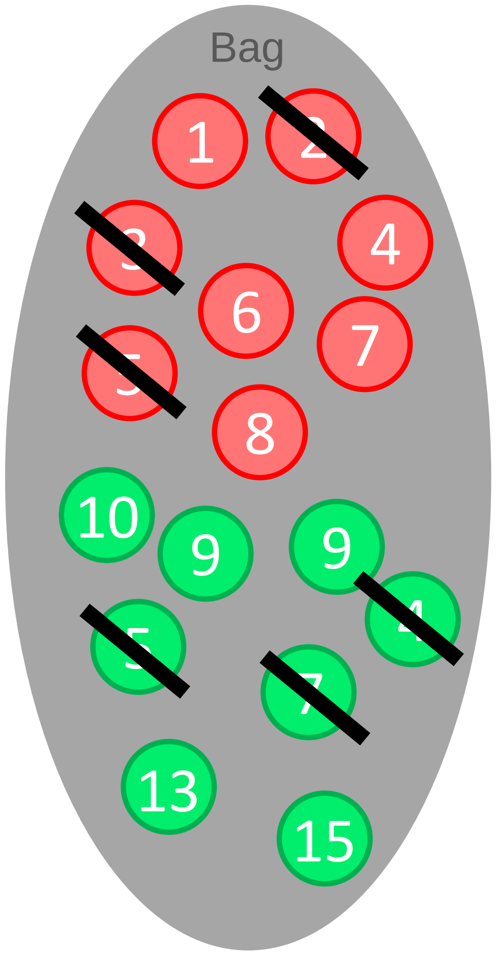
Puzzle



③ Alon's Combinatorial Open Problem

Puzzle

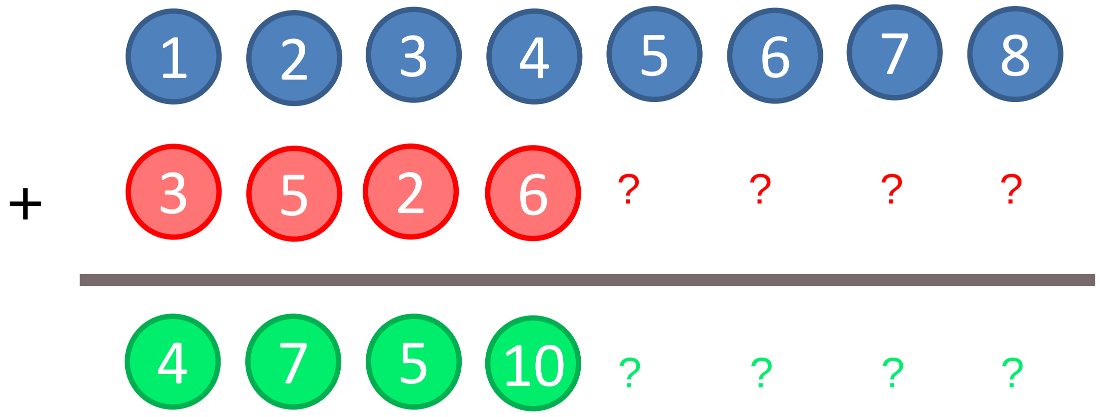
	1	2	3	4	5	6	7	8
+	3	5	2	?	?	?	?	?
	4	7	5	?	?	?	?	?

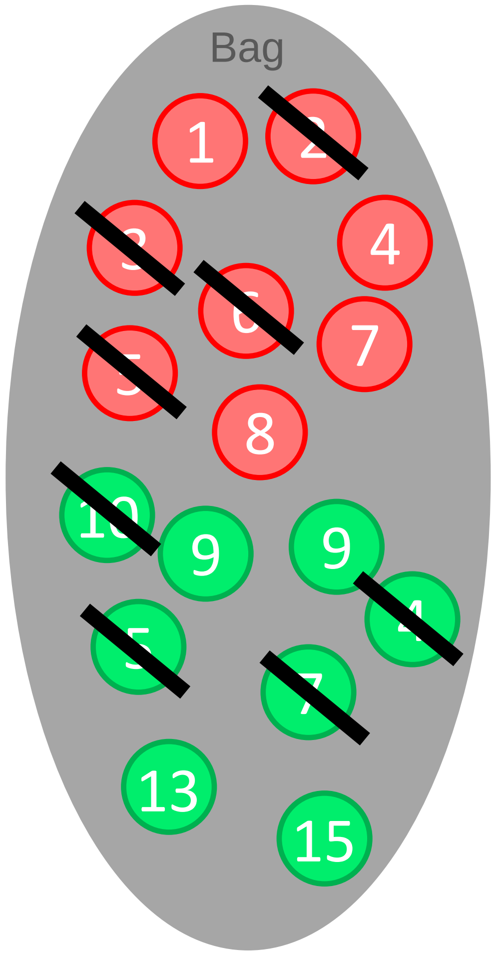


Target sums

③ Alon’s Combinatorial Open Problem

Puzzle

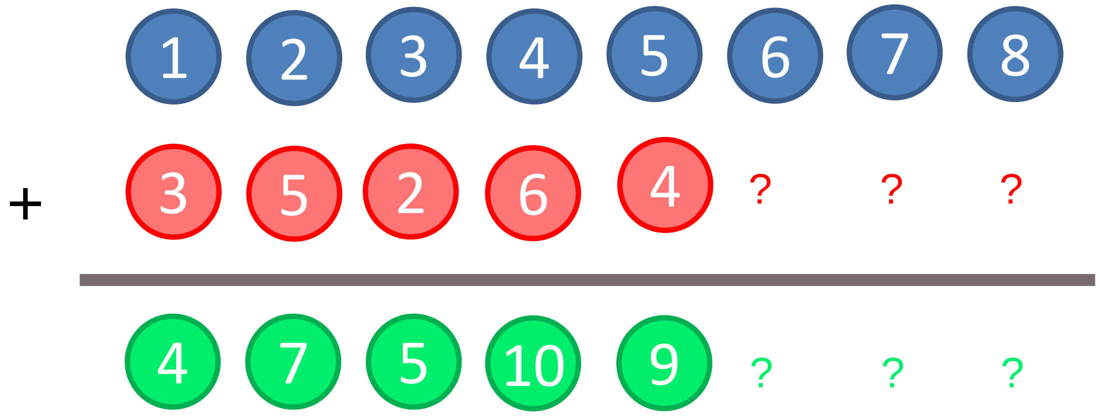


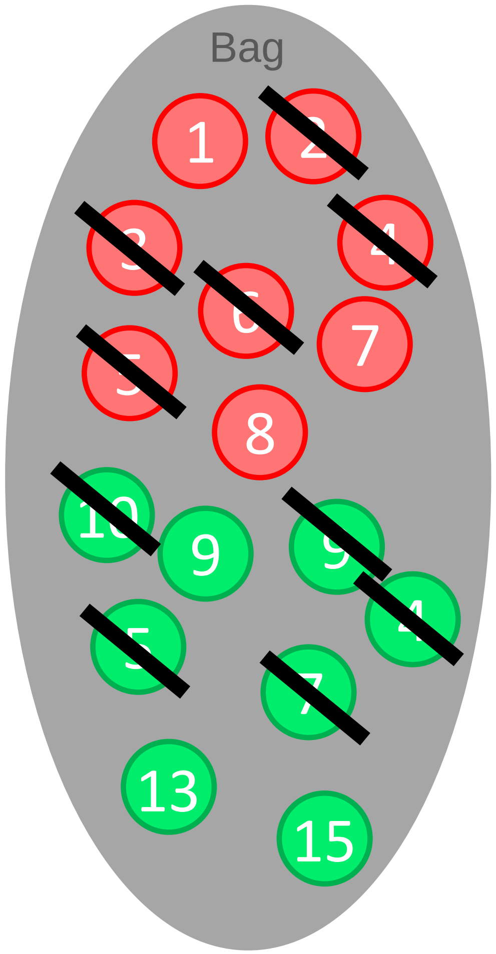


Target sums

③ Alon's Combinatorial Open Problem

Puzzle





Target sums

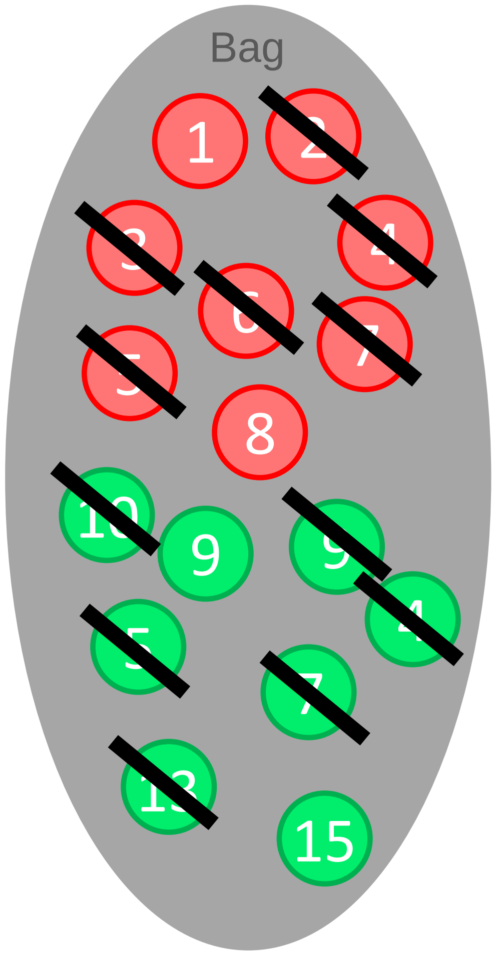
③ Alon’s Combinatorial Open Problem

Puzzle

1 2 3 4 5 6 7 8

\+ 3 5 2 6 4 7 ? ?

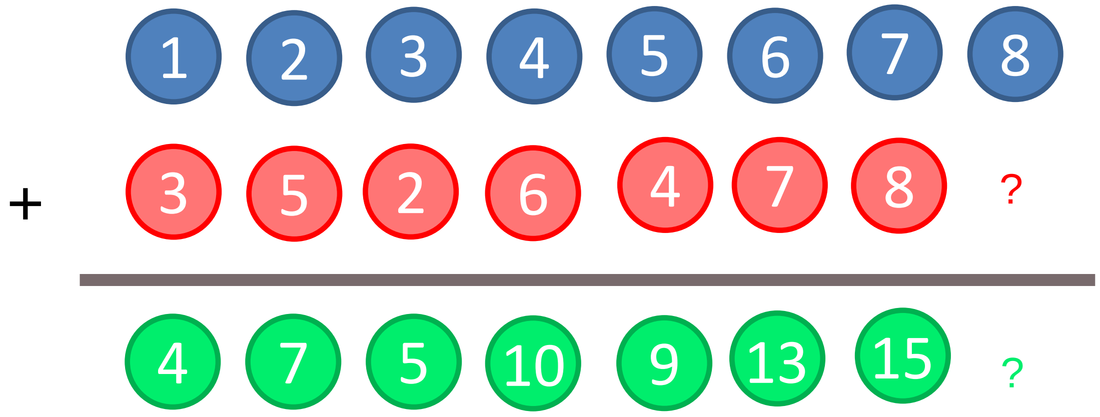
4 7 5 10 9 13 ? ?



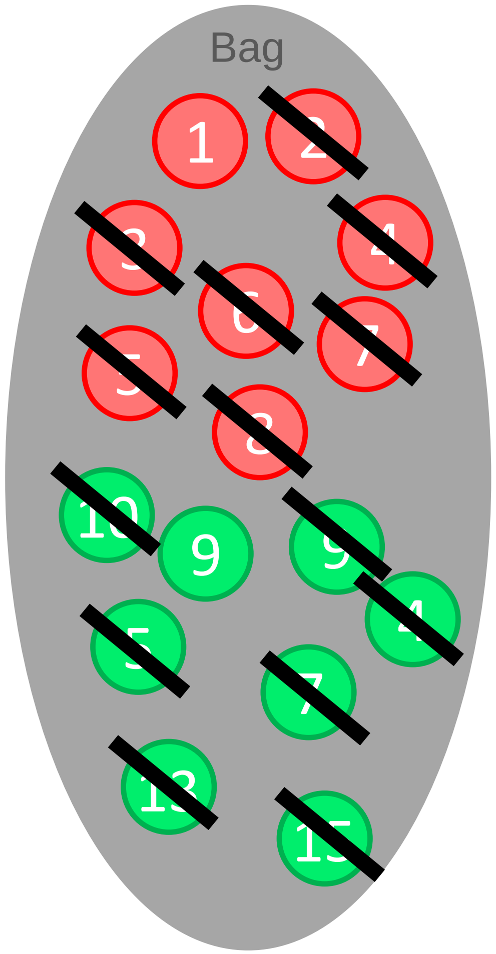
Target sums

③ Alon's Combinatorial Open Problem

Puzzle

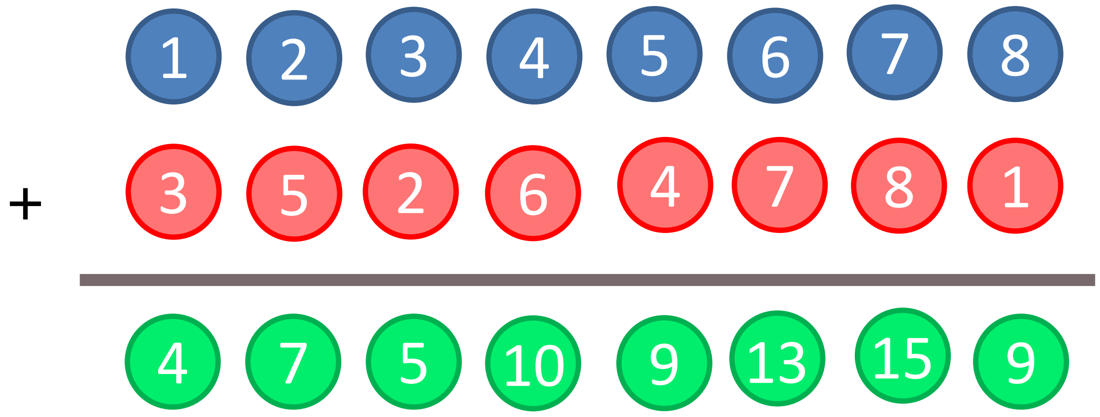


	1	2	3	4	5	6	7	8
+	3	5	2	6	4	7	8	?
	4	7	5	10	9	13	15	?

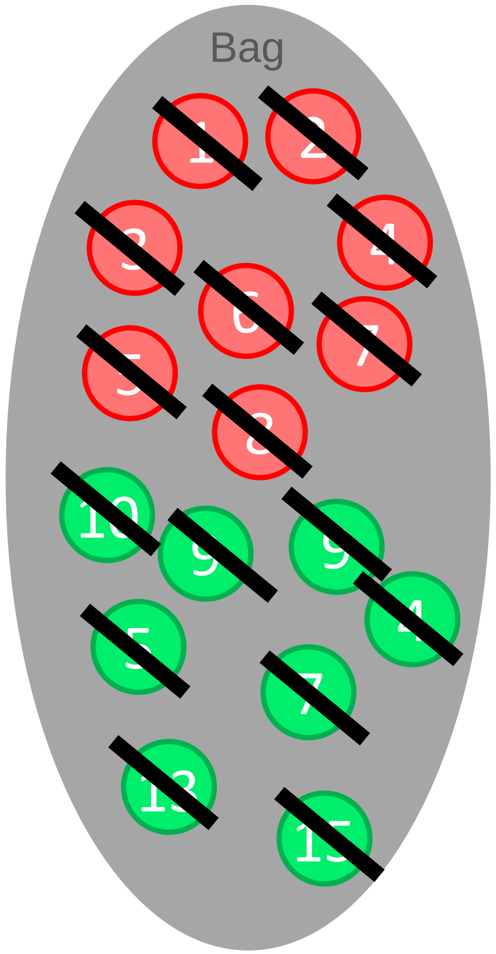


Target sums

③ Alon’s Combinatorial Open Problem



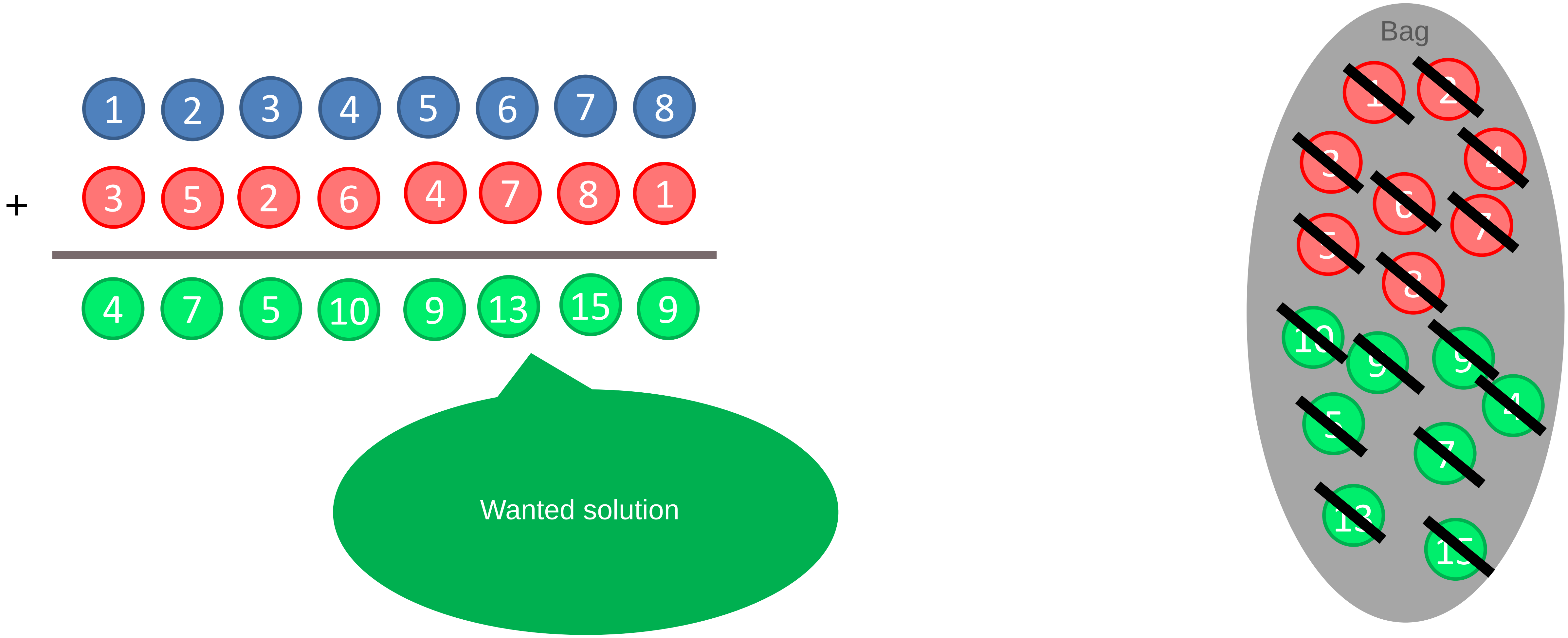
	1	2	3	4	5	6	7	8
+	3	5	2	6	4	7	8	1
	4	7	5	10	9	13	15	9



Target sums
(there sum is 2(n+1))

③ Alon's Combinatorial Open Problem

Puzzle



Wanted solution

Target sums
(there sum is 2(n+1))

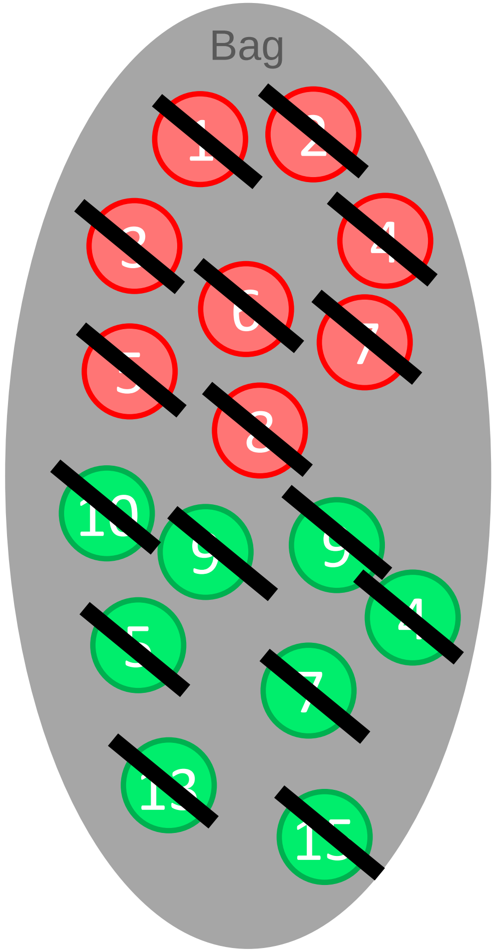
③ Alon's Combinatorial Open Problem

Puzzle

	1	2	3	4	5	6	7	8
+	3	5	2	6	4	7	8	1
	4	7	5	10	9	13	15	9

Wanted solution

Not necessarily easy to find!



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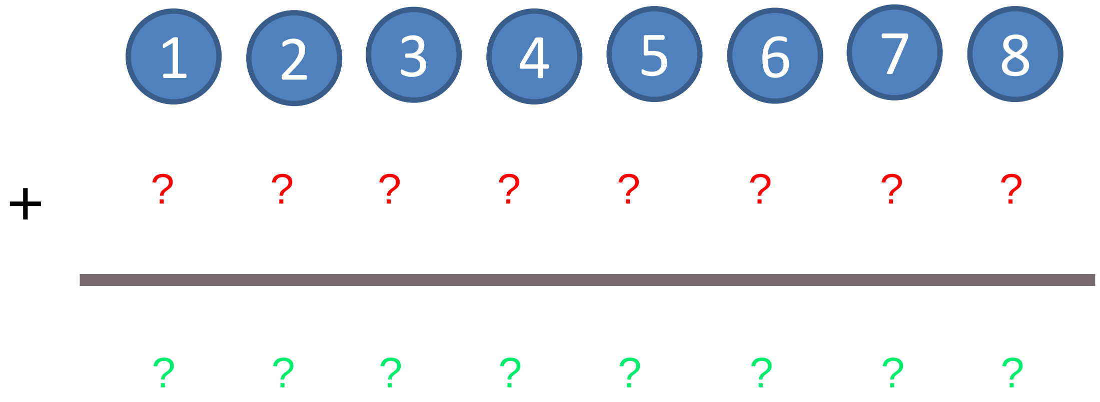
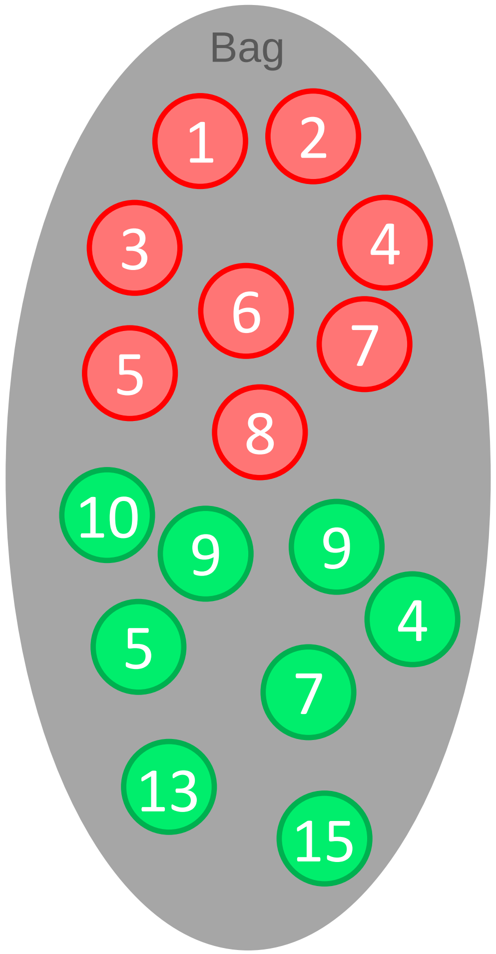
Alon’s Combinatorial Open Problem

Puzzle





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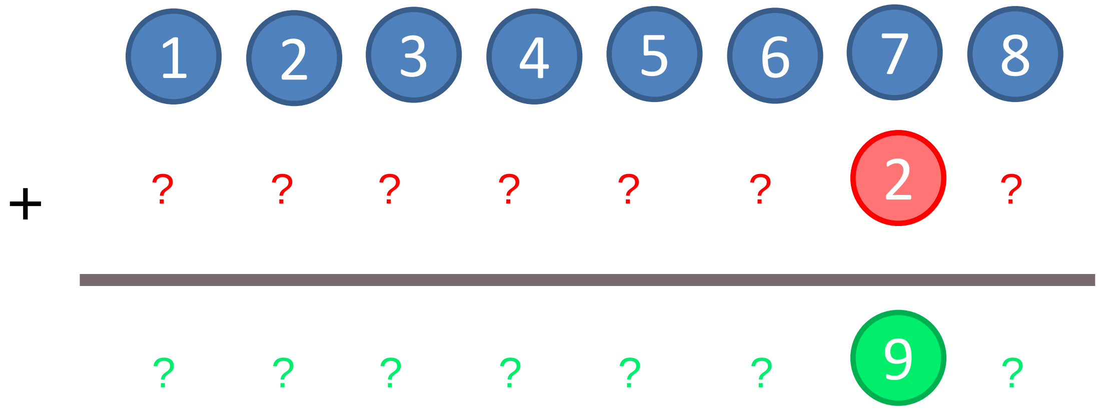
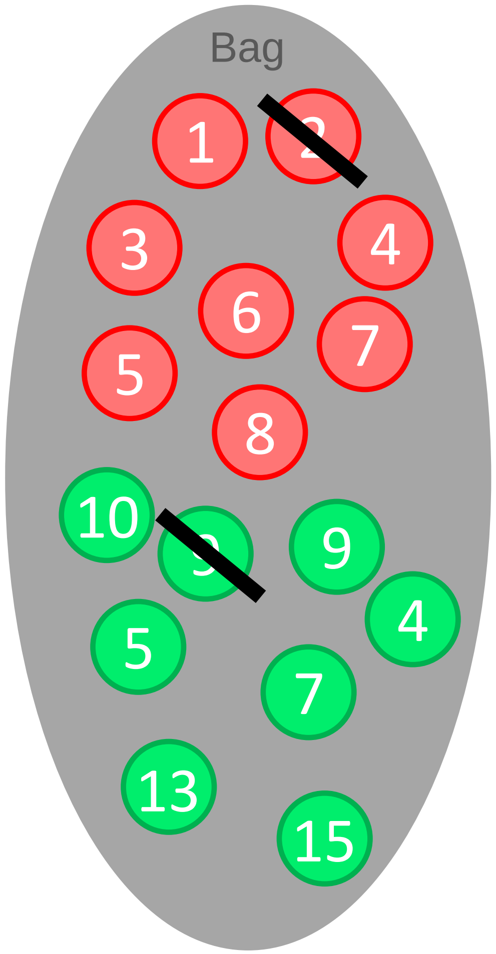
③ Alon's Combinatorial Open Problem

Puzzle





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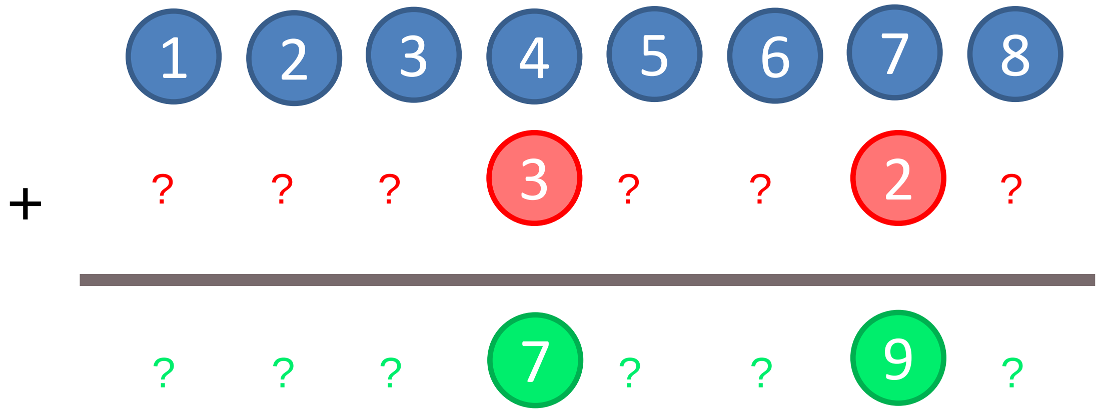
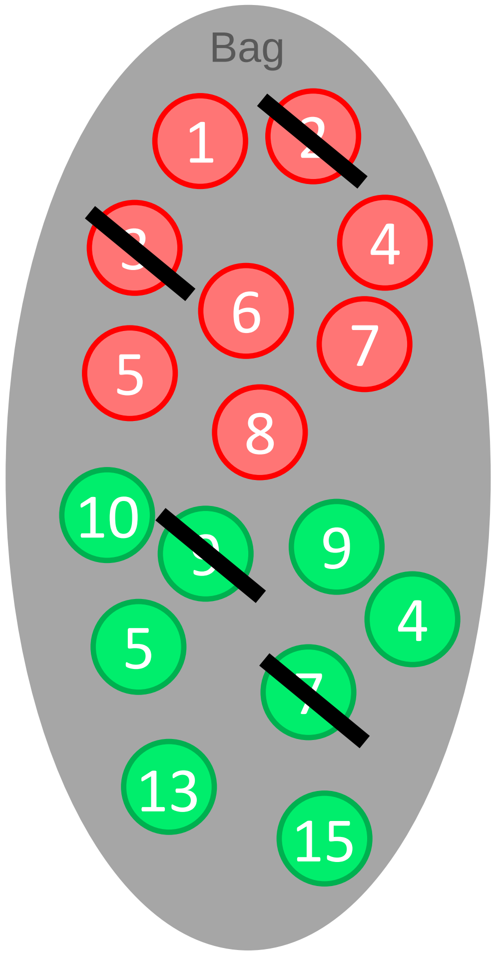
③ Alon's Combinatorial Open Problem

Puzzle





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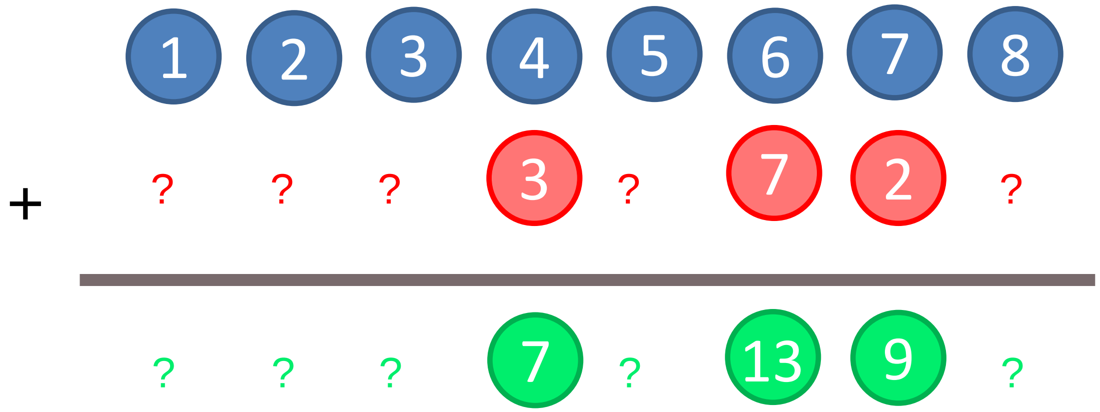
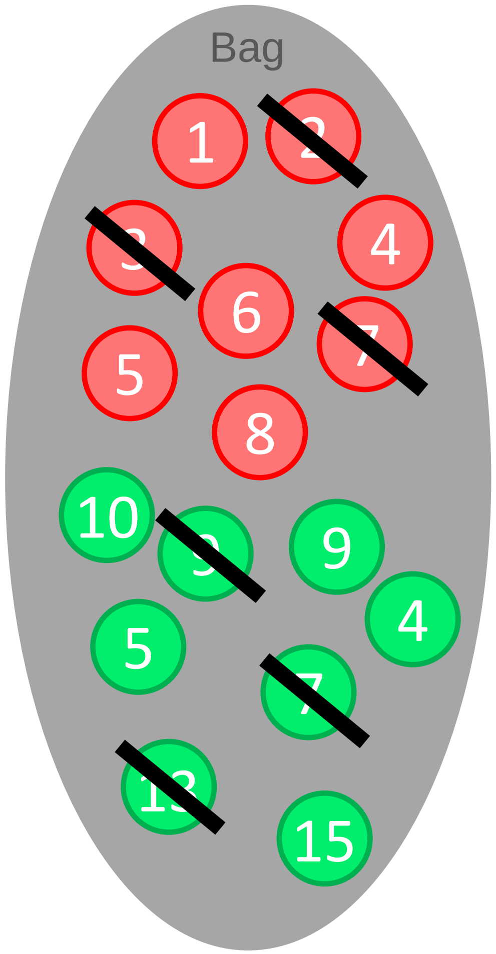
③ Alon’s Combinatorial Open Problem

Puzzle





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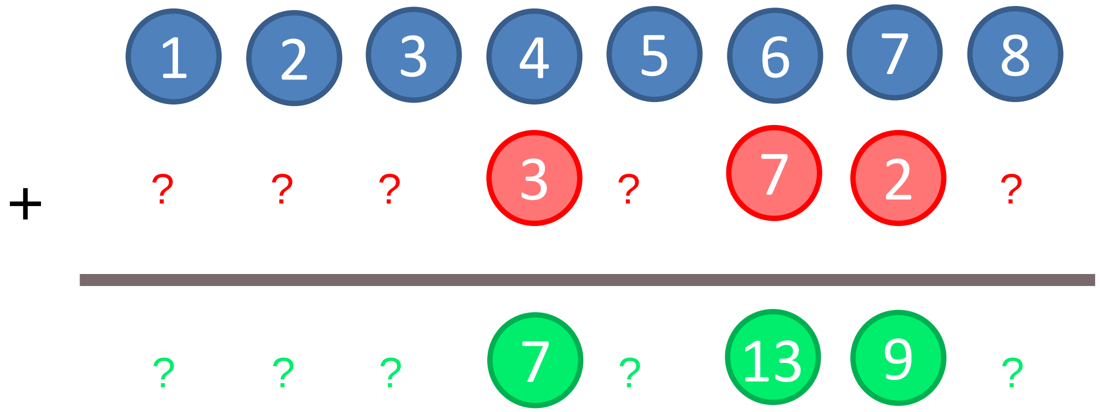
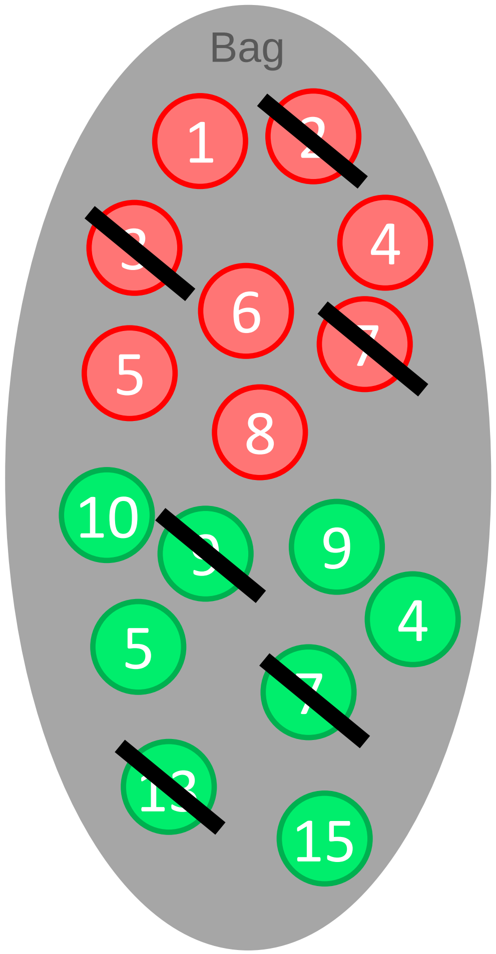
Alon’s Combinatorial Open Problem

Puzzle



No more able to do 15



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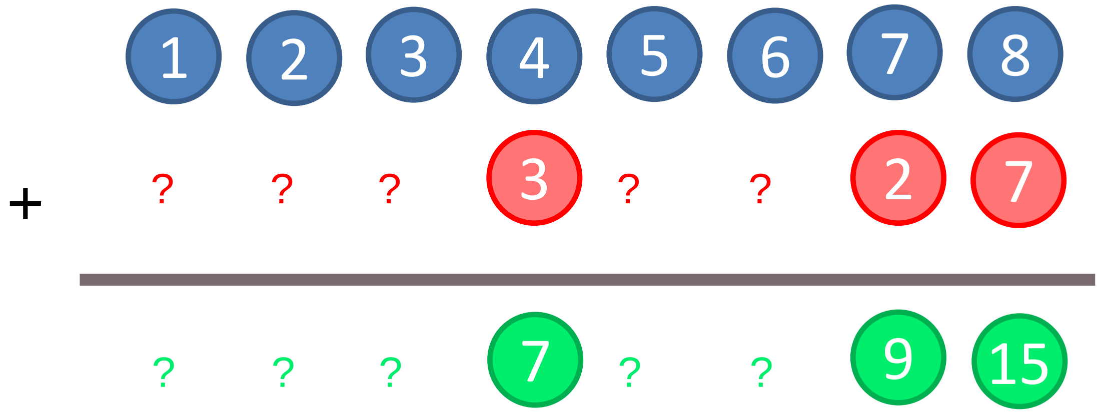
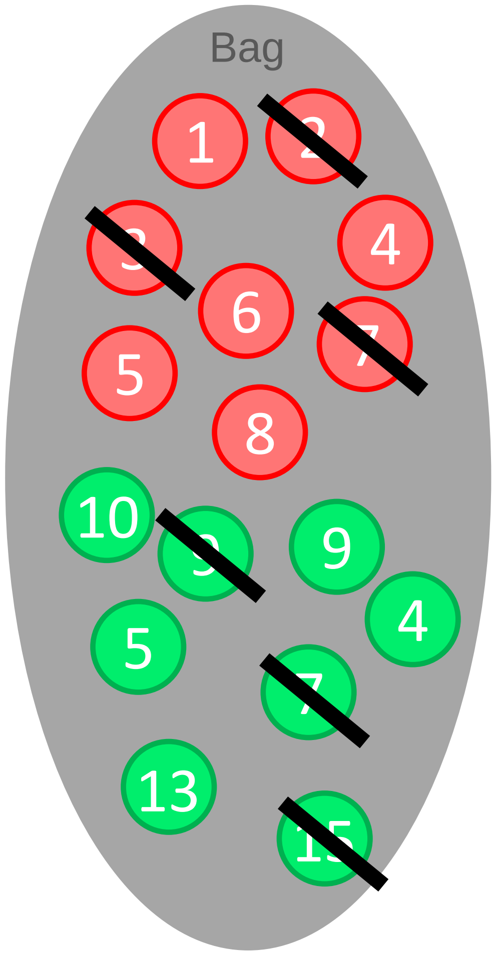
③ Alon’s Combinatorial Open Problem

Puzzle





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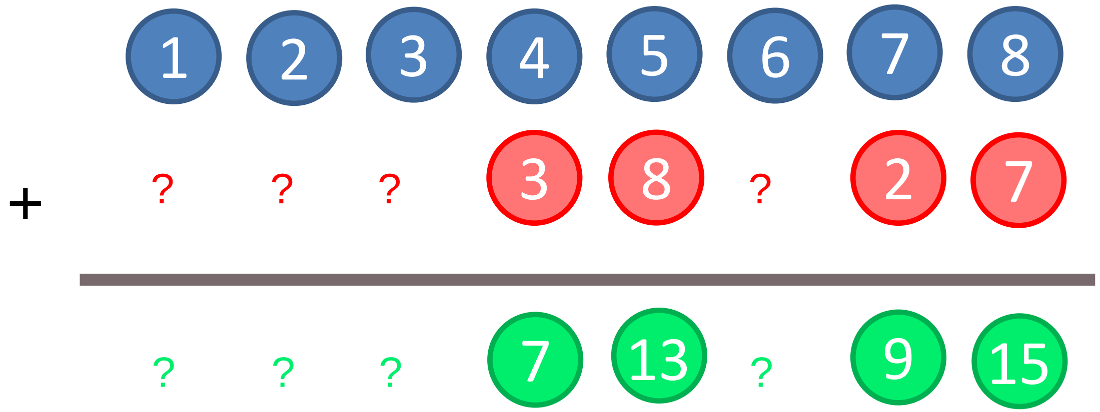
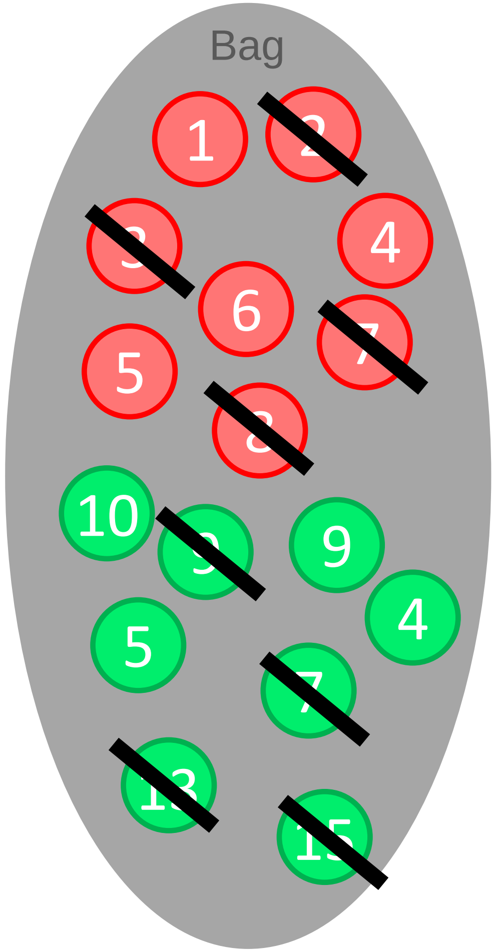
③ Alon's Combinatorial Open Problem





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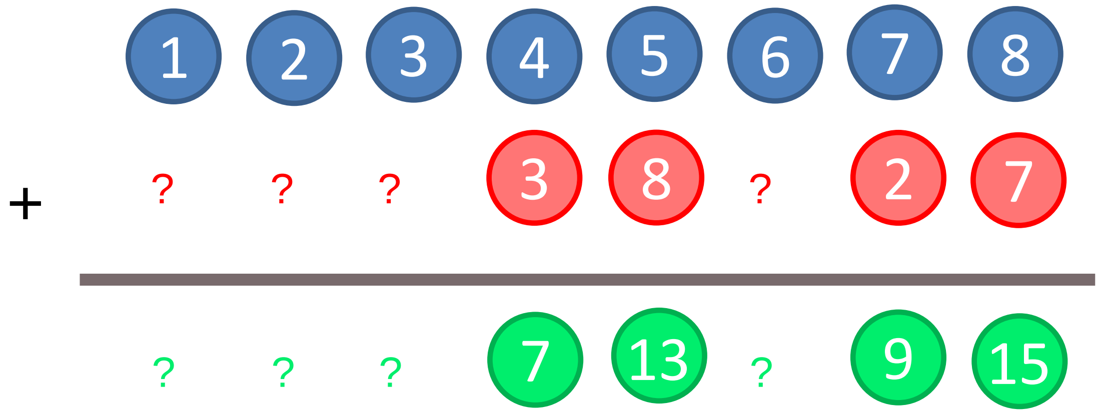
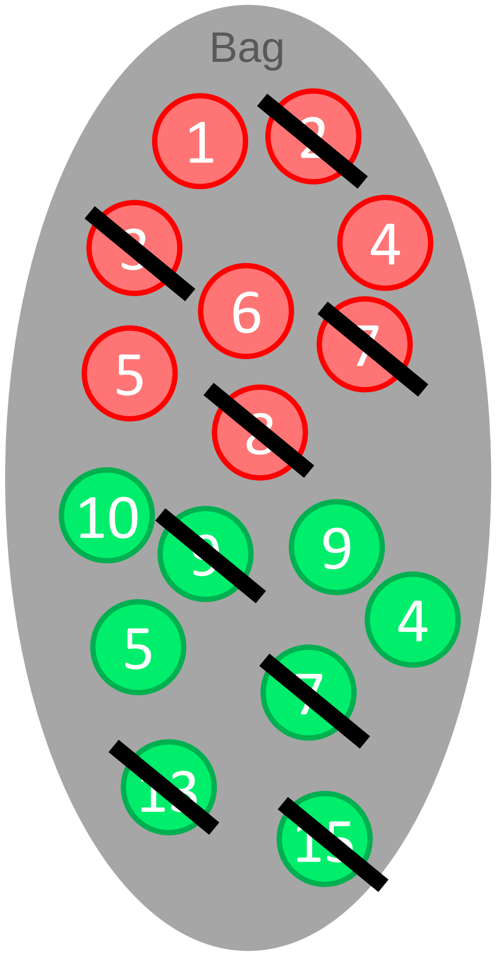
③ Alon’s Combinatorial Open Problem

Puzzle



No more able to use 5



Target sums
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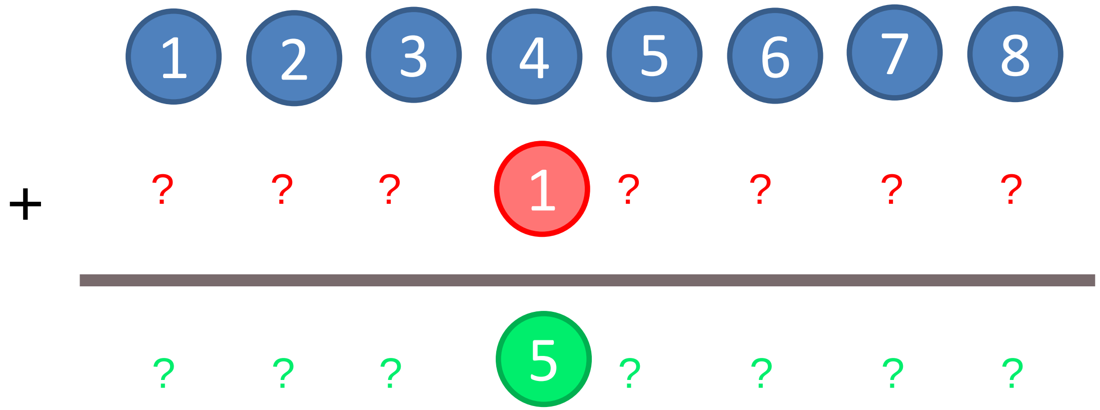
③

Alon's Combinatorial Open Problem

Puzzle



Target sums
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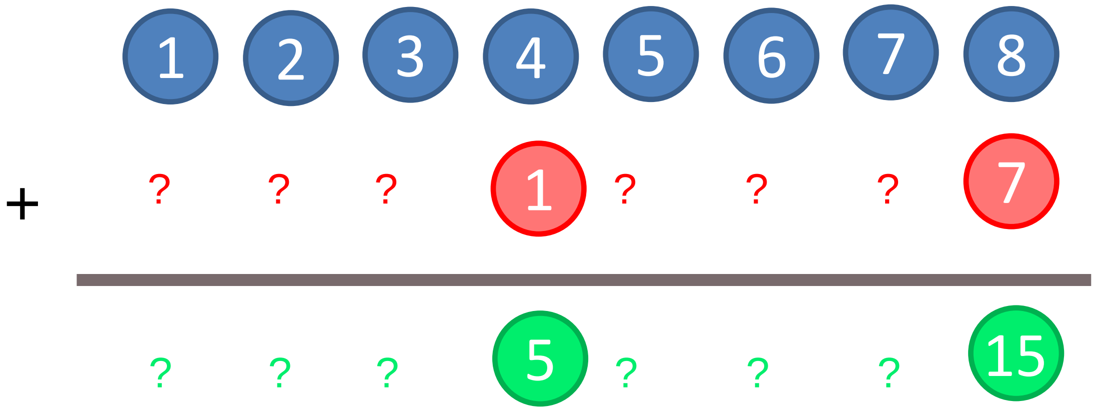
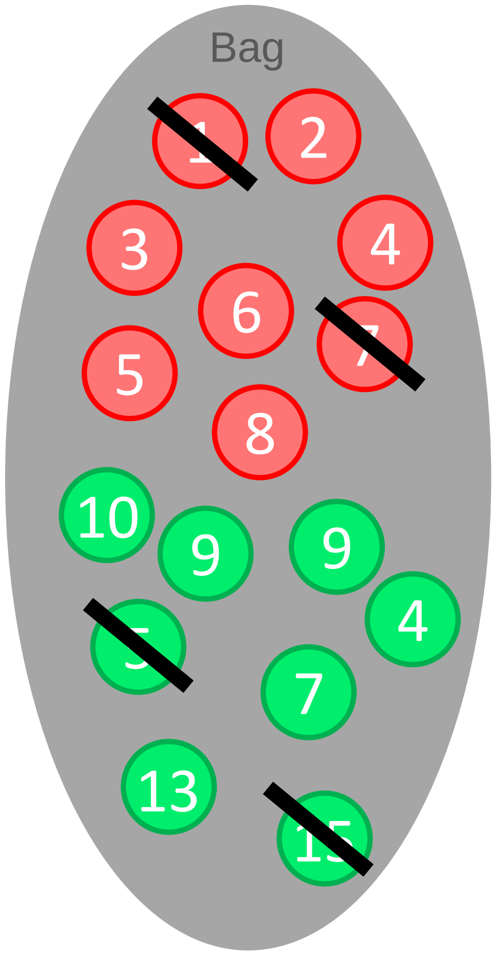
③

Alon’s Combinatorial Open Problem

Puzzle





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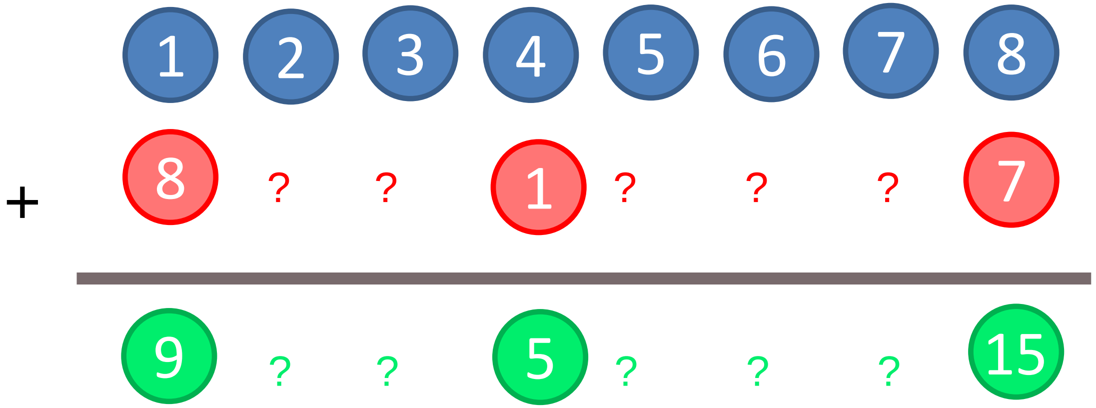
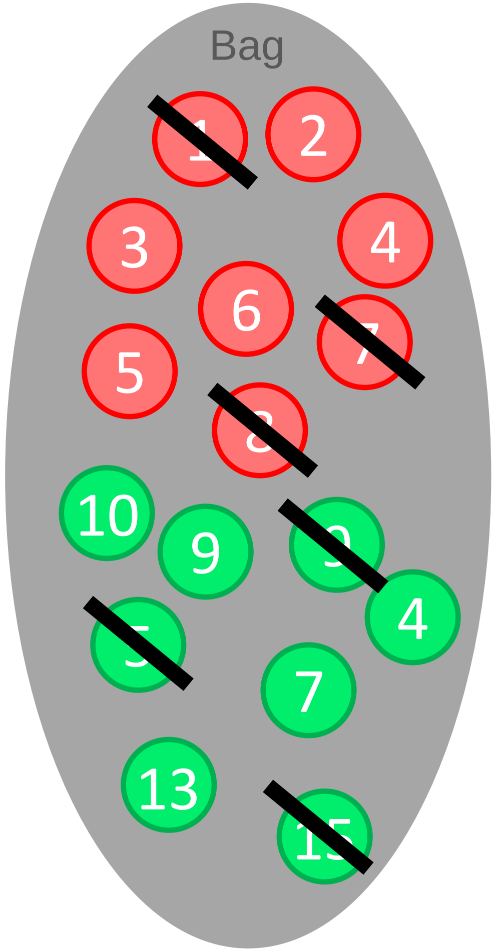
Alon’s Combinatorial Open Problem





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Alon’s Combinatorial Open Problem

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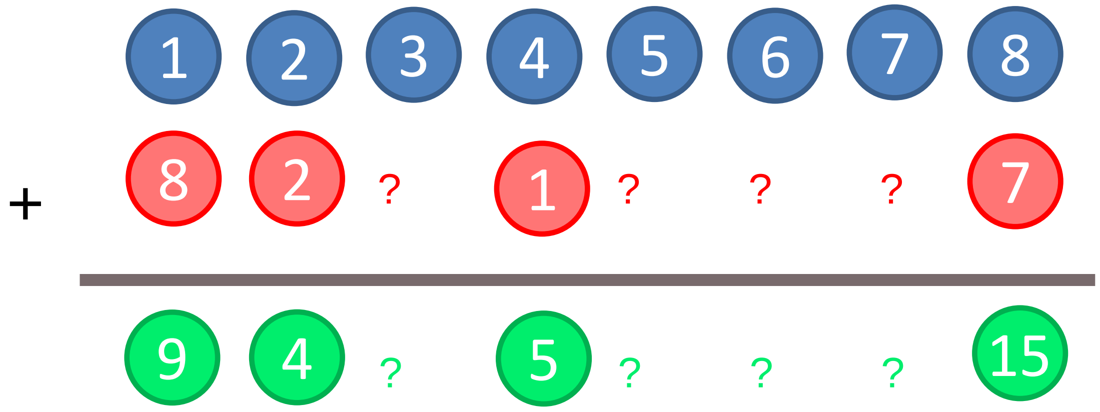

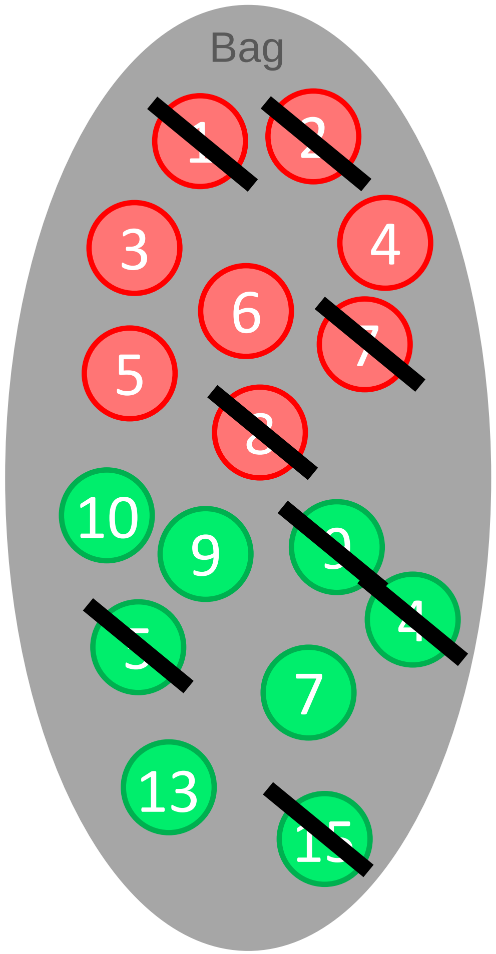

Target sums
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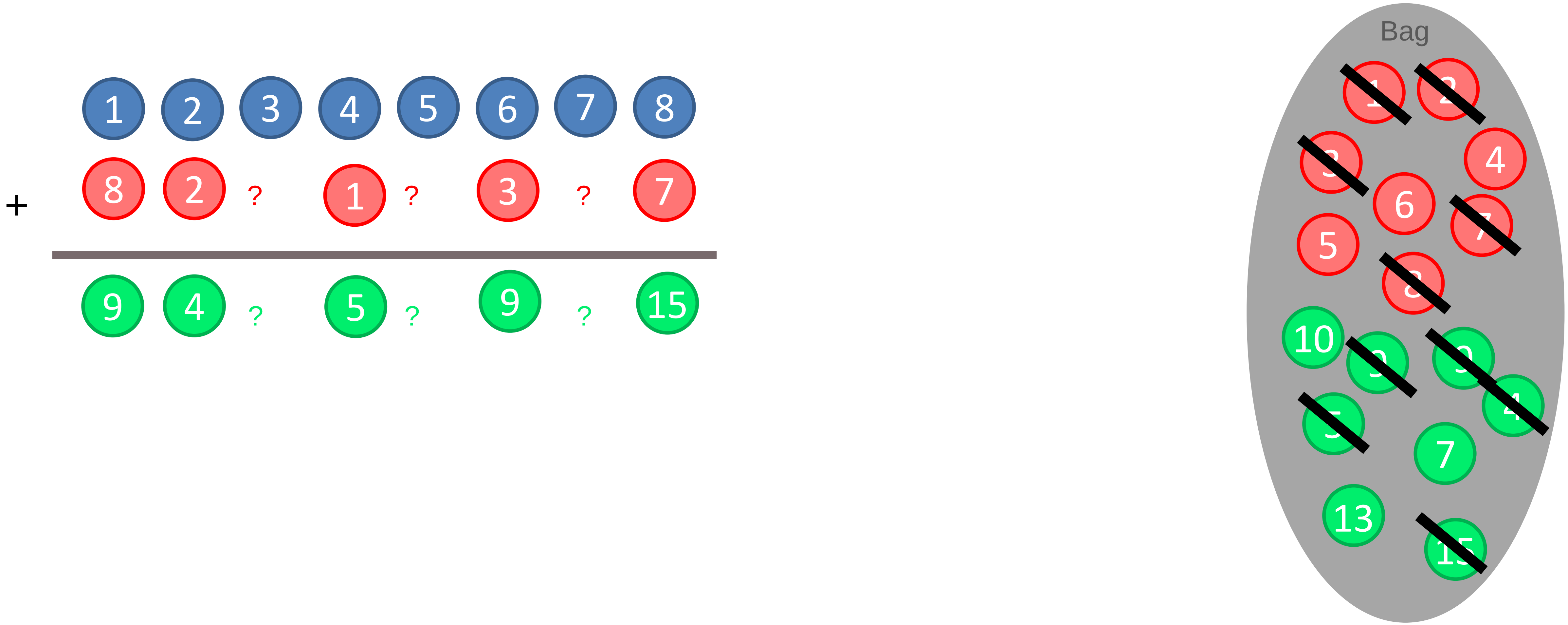
③ Alon's Combinatorial Open Problem

Puzzle



Target sums
(there sum is 2(n+1))

③ Alon's Combinatorial Open Problem

Puzzle

	1	2	3	4	5	6	7	8
+	8	2	?	1	5	3	?	7
	9	4	?	5	10	9	?	15

Bag

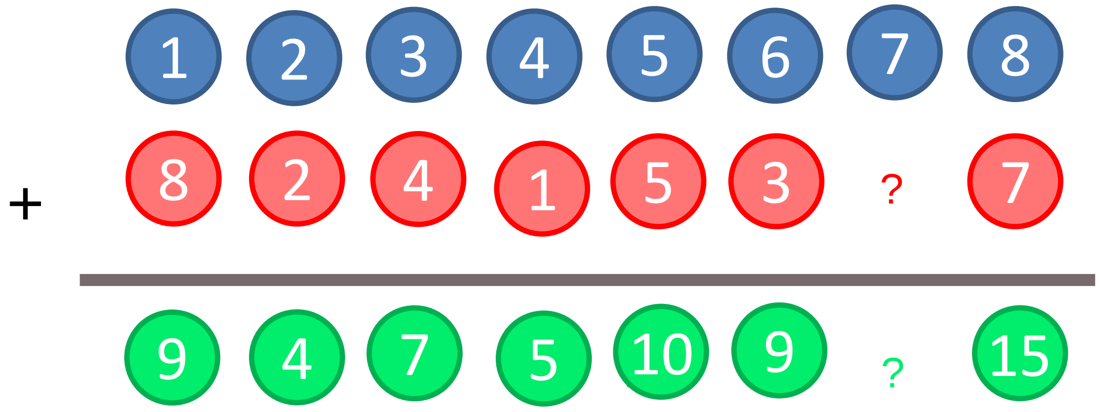
~~1~~ ~~2~~ ~~3~~ 4 6 ~~7~~ ~~5~~ ~~8~~

~~10~~ ~~9~~ ~~9~~ ~~4~~ ~~5~~ 7 13 ~~15~~

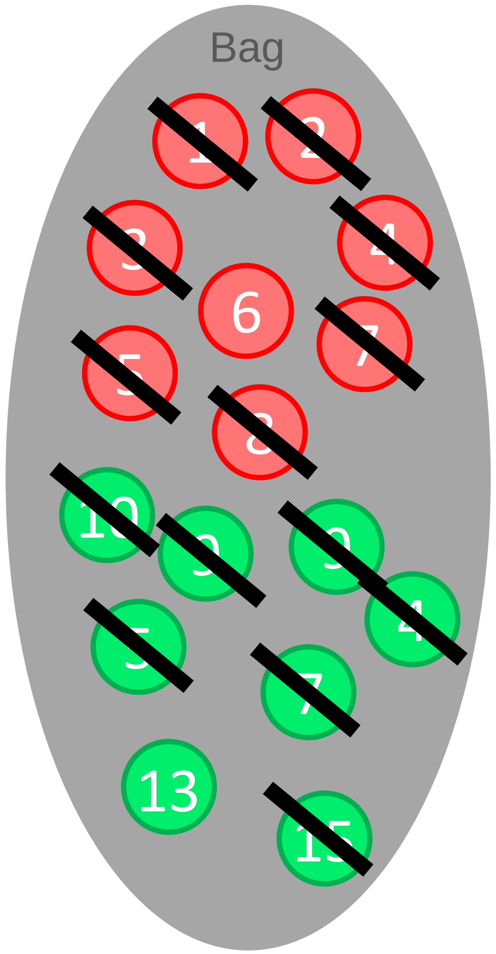
Target sums
(there sum is 2(n+1))

③ Alon's Combinatorial Open Problem

Puzzle



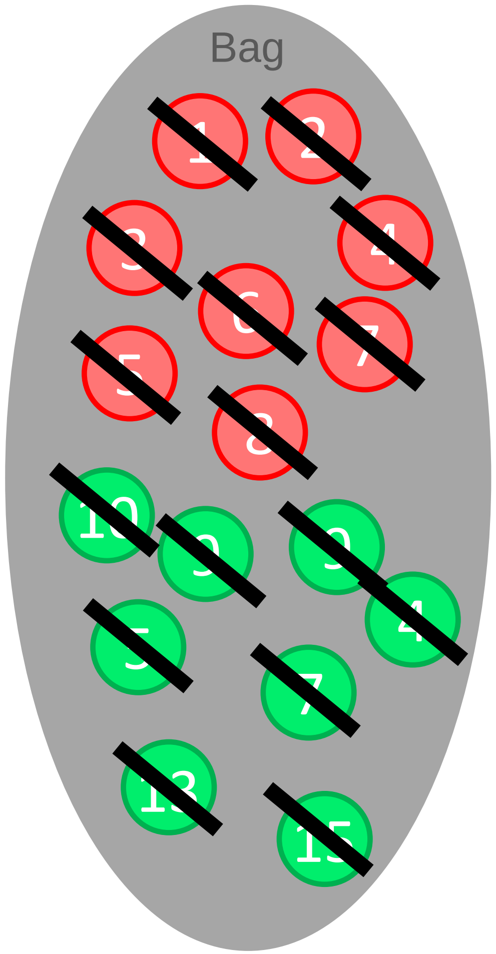
	1	2	3	4	5	6	7	8
+	8	2	4	1	5	3	?	7
	9	4	7	5	10	9	?	15



Target sums
(there sum is 2(n+1))

③ Alon’s Combinatorial Open Problem

	1	2	3	4	5	6	7	8
+	8	2	4	1	5	3	6	7
	9	4	7	5	10	9	13	15



Target sums
(there sum is 2(n+1))

③ Alon’s Combinatorial Open Problem

Why this ?

	1	2	3	4	5	6	7	8
+	8	2	4	1	5	3	6	7
	9	4	7	5	10	9	13	15

Relation
to
Discrete Tomography ?

③ Alon's Combinatorial Open Problem

1 2 3 4 5 6 7 8

\+ ? ? ? ? ? ? ? ?

? ? ? ? ? ? ? ?

Relation
to
Discrete Tomography ?

	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
3	4	5	6	7	8	9	10	11
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

③ Alon’s Combinatorial Open Problem

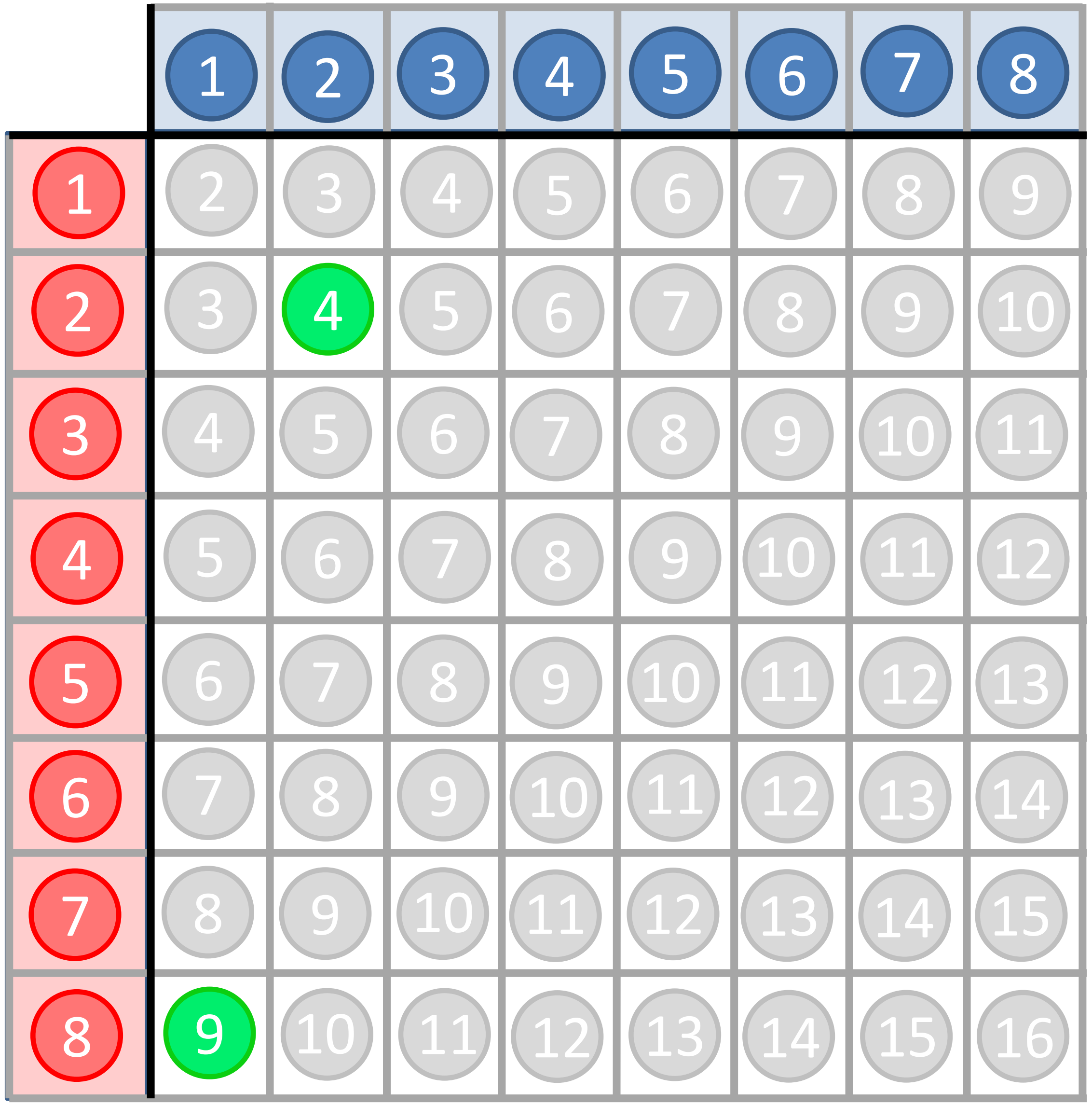
	1	2	3	4	5	6	7	8
+	8	?	?	?	?	?	?	?
	9	?	?	?	?	?	?	?

	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
3	4	5	6	7	8	9	10	11
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

③ Alon's Combinatorial Open Problem

Puzzle

	1	2	3	4	5	6	7	8
+	8	2	?	?	?	?	?	?
	9	4	?	?	?	?	?	?



	1	2	3	4	5	6	7	8
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2	3	4	5	6	7	8	9	10
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4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

③

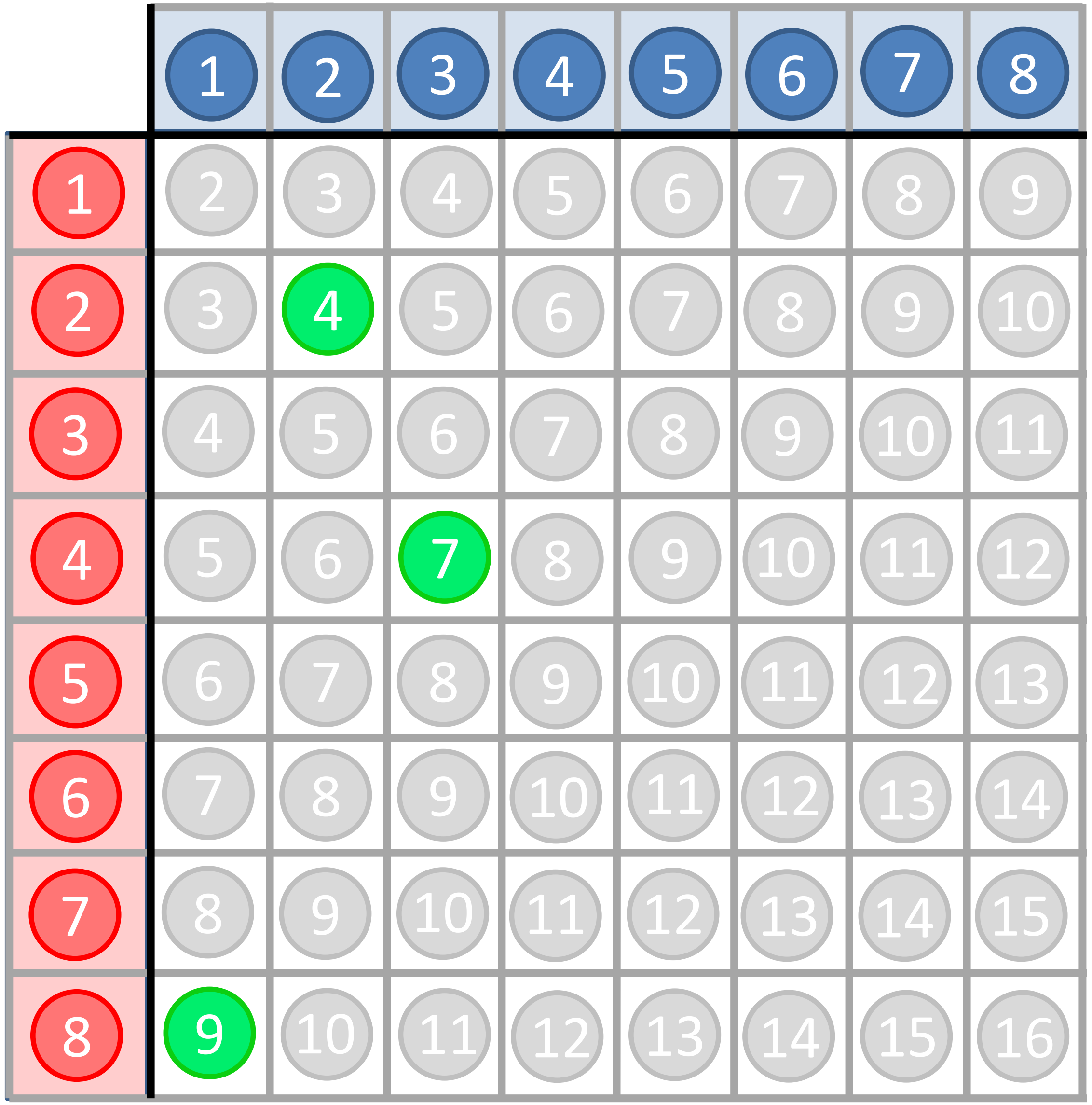
Alon’s Combinatorial Open Problem

Puzzle

1 2 3 4 5 6 7 8

\+ 8 2 4 ? ? ? ? ?

9 4 7 ? ? ? ? ?



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2	3	4	5	6	7	8	9	10
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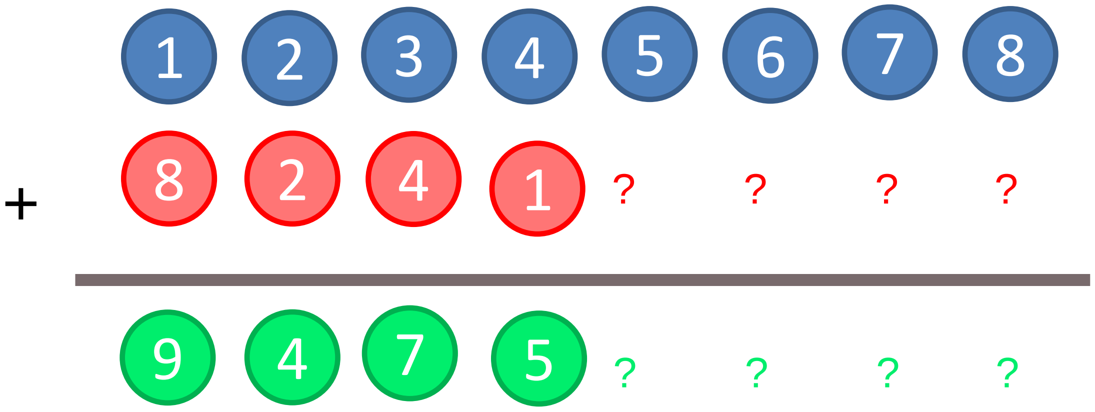
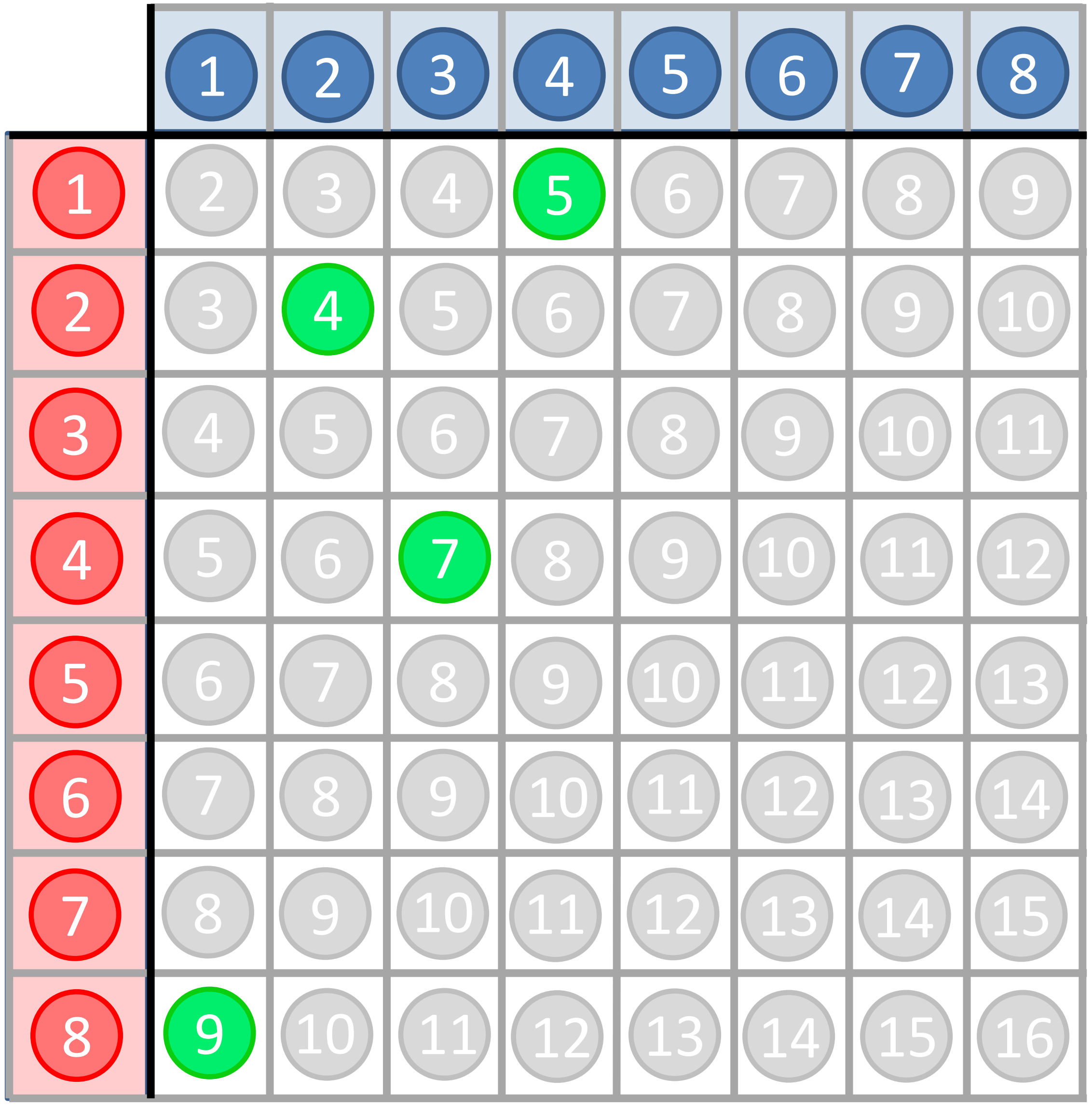
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

③ Alon's Combinatorial Open Problem

Puzzle



	1	2	3	4	5	6	7	8
+	8	2	4	1	?	?	?	?
	9	4	7	5	?	?	?	?



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2	3	4	5	6	7	8	9	10
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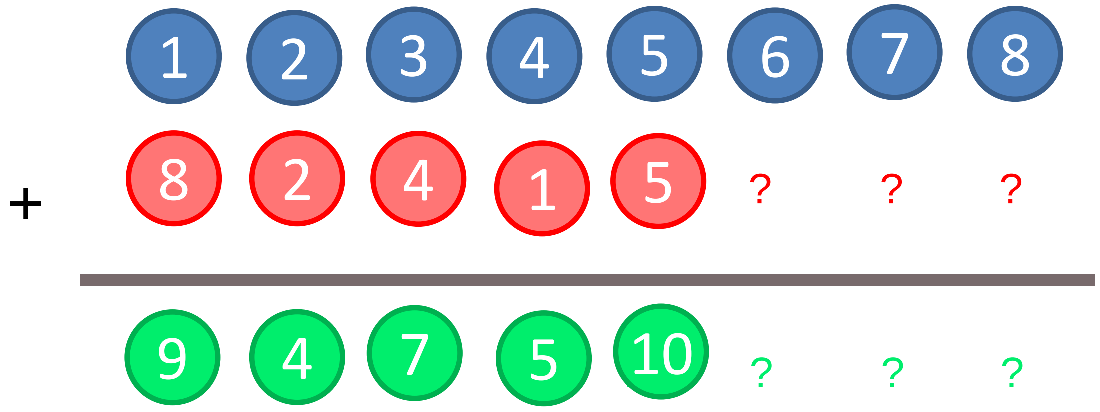
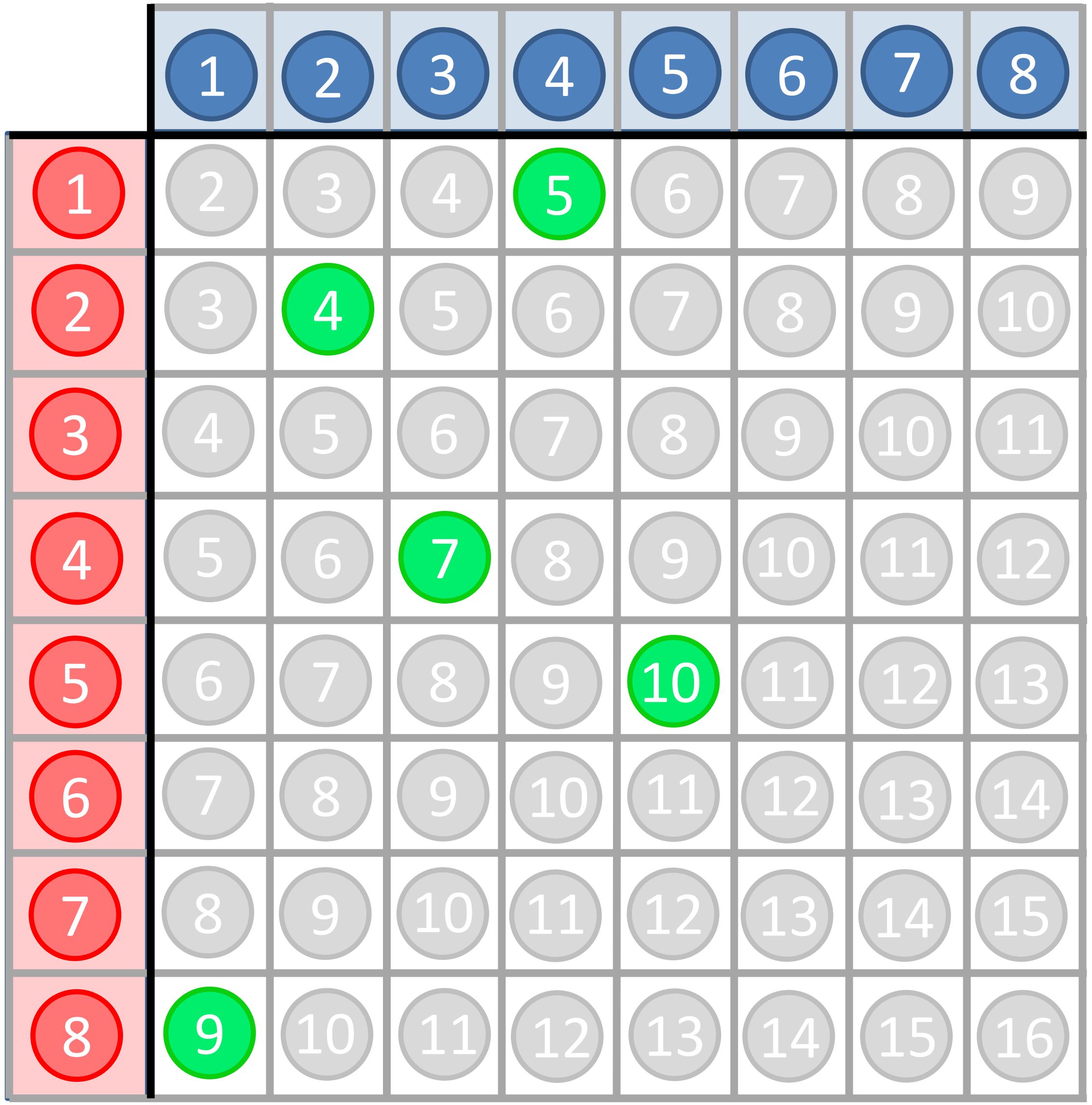
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

③ Alon's Combinatorial Open Problem

Puzzle



	1	2	3	4	5	6	7	8
+	8	2	4	1	5	?	?	?
	9	4	7	5	10	?	?	?



	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
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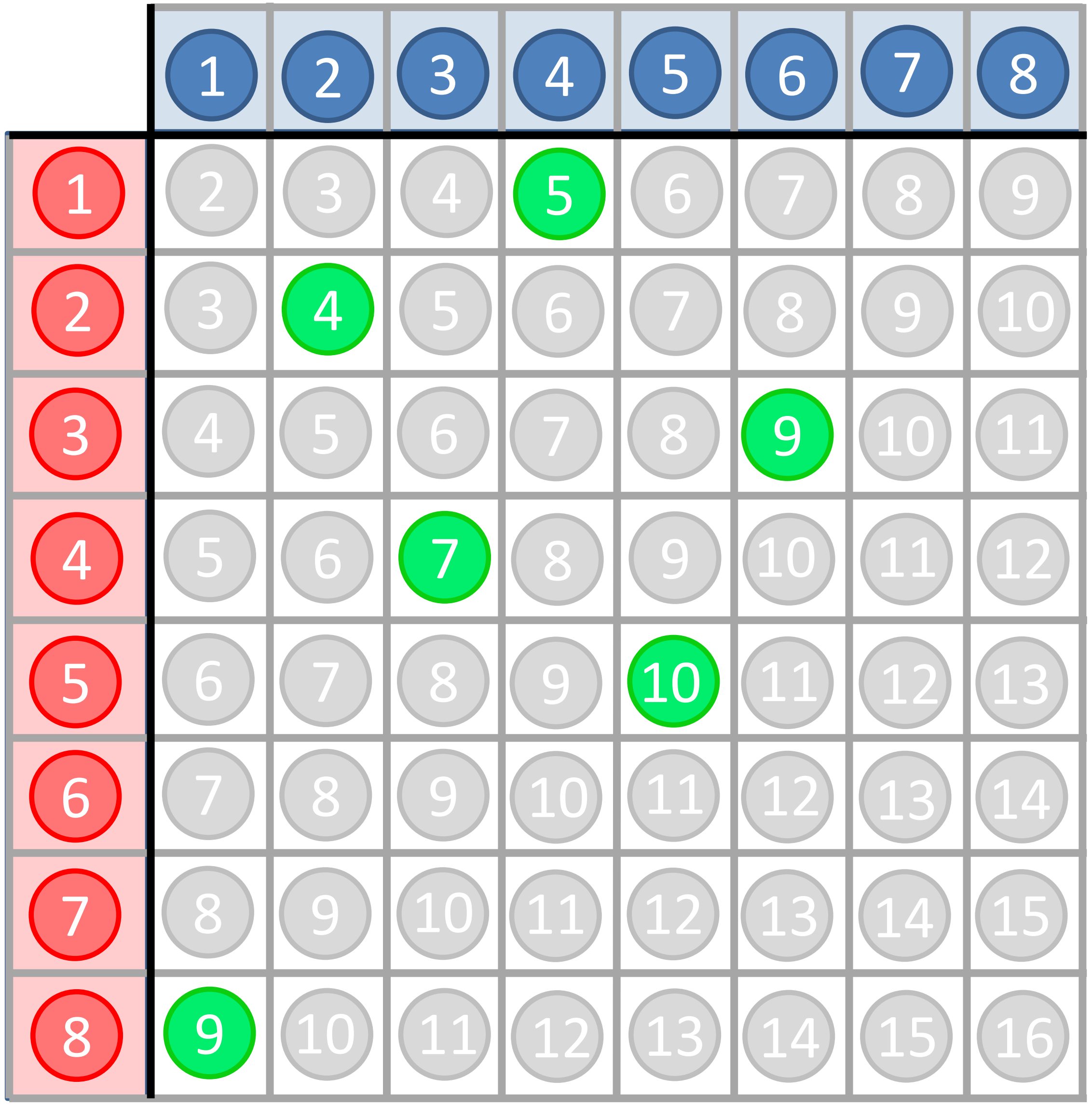
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

③

Alon’s Combinatorial Open Problem

Puzzle

	1	2	3	4	5	6	7	8
+	8	2	4	1	5	3	?	?
	9	4	7	5	10	9	?	?



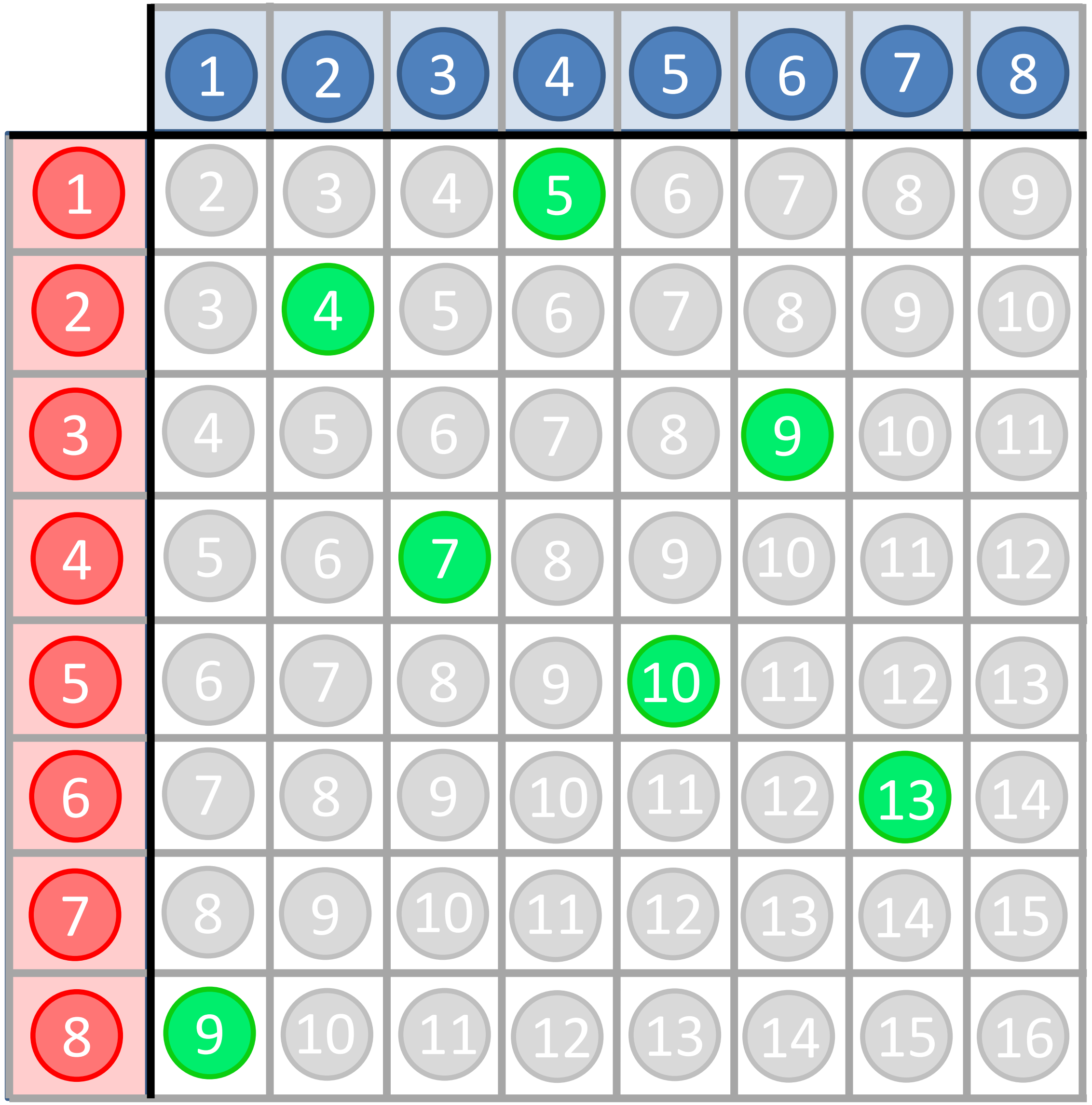
	1	2	3	4	5	6	7	8
1	2	3	4	**5**	6	7	8	9
2	3	**4**	5	6	7	8	9	10
3	4	5	6	7	8	**9**	10	11
4	5	6	**7**	8	9	10	11	12
5	6	7	8	9	**10**	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	**9**	10	11	12	13	14	15	16

③

Alon’s Combinatorial Open Problem

Puzzle

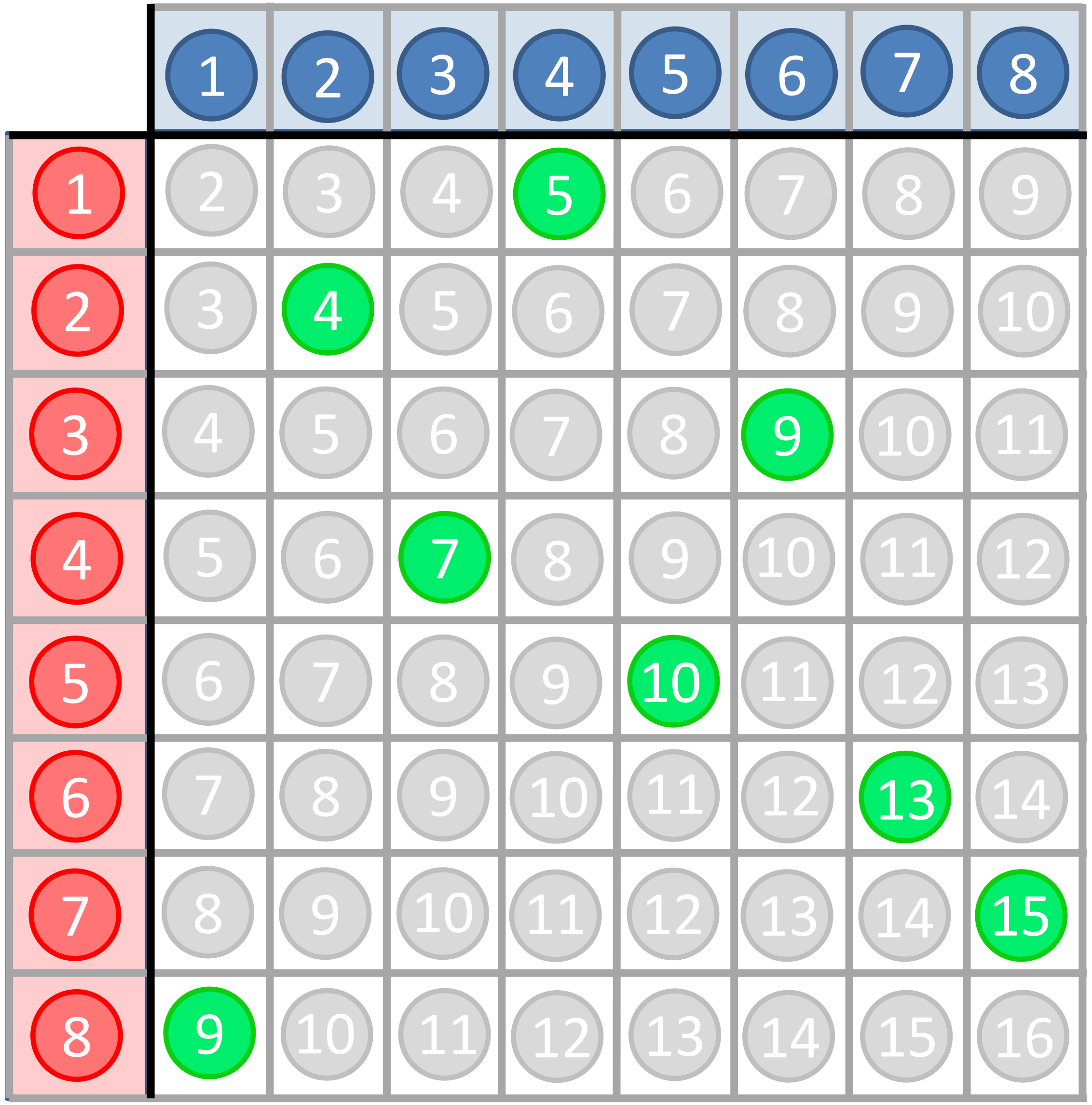
	1	2	3	4	5	6	7	8
+	8	2	4	1	5	3	6	?
	9	4	7	5	10	9	13	?



	1	2	3	4	5	6	7	8
1	2	3	4	**5**	6	7	8	9
2	3	**4**	5	6	7	8	9	10
3	4	5	6	7	8	**9**	10	11
4	5	6	**7**	8	9	10	11	12
5	6	7	8	9	**10**	11	12	13
6	7	8	9	10	11	12	**13**	14
7	8	9	10	11	12	13	14	15
8	**9**	10	11	12	13	14	15	16

③ Alon’s Combinatorial Open Problem

	1	2	3	4	5	6	7	8
+	8	2	4	1	5	3	6	7
	9	4	7	5	10	9	13	15



	1	2	3	4	5	6	7	8
1	2	3	4	**5**	6	7	8	9
2	3	**4**	5	6	7	8	9	10
3	4	5	6	7	8	**9**	10	11
4	5	6	**7**	8	9	10	11	12
5	6	7	8	9	**10**	11	12	13
6	7	8	9	10	11	12	**13**	14
7	8	9	10	11	12	13	14	**15**
8	**9**	10	11	12	13	14	15	16

③ Alon’s Combinatorial Open Problem

Puzzle

	1	2	3	4	5	6	7	8
+	8	2	4	1	5	3	6	7
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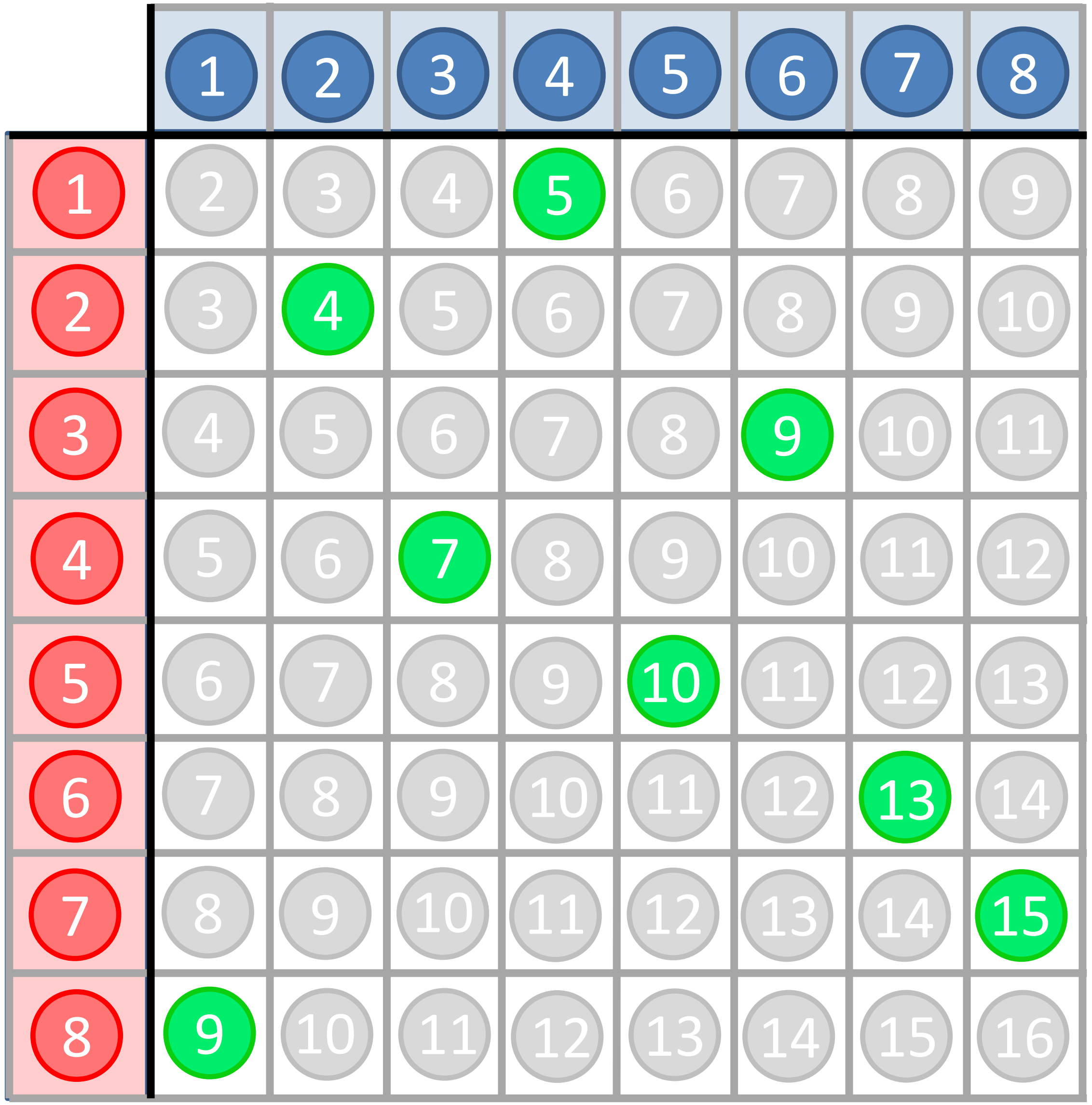


	1	2	3	4	5	6	7	8
1	2	3	4	**5**	6	7	8	9
2	3	**4**	5	6	7	8	9	10
3	4	5	6	7	8	**9**	10	11
4	5	6	**7**	8	9	10	11	12
5	6	7	8	9	**10**	11	12	13
6	7	8	9	10	11	12	**13**	14
7	8	9	10	11	12	13	14	**15**
8	**9**	10	11	12	13	14	15	16

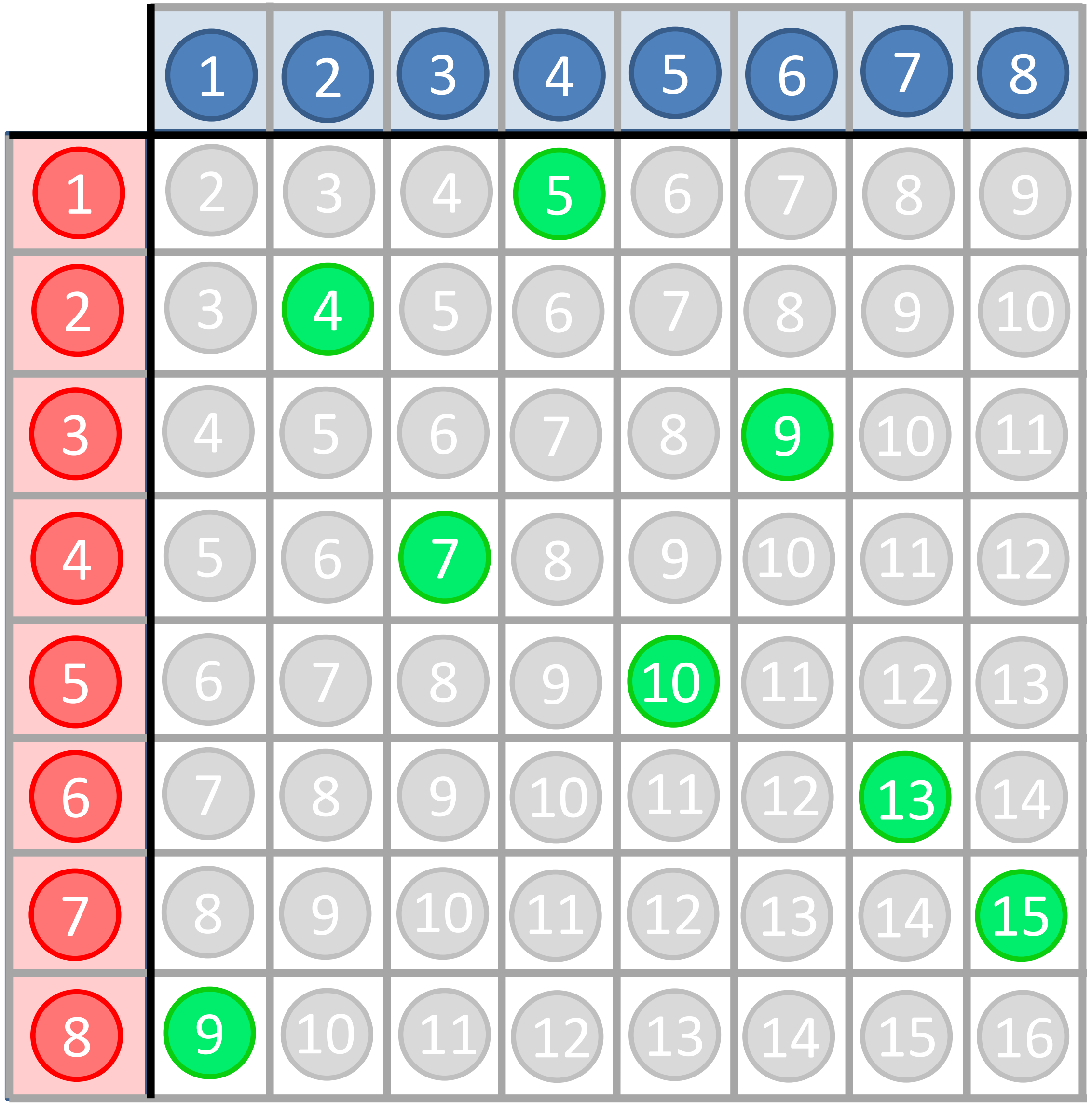
Solution
=
permutation matrix
with prescribed integers

③

Alon’s Combinatorial Open Problem

Puzzle

9 4 7 5 10 9 13 15



	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
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4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

Solution
=
permutation matrix
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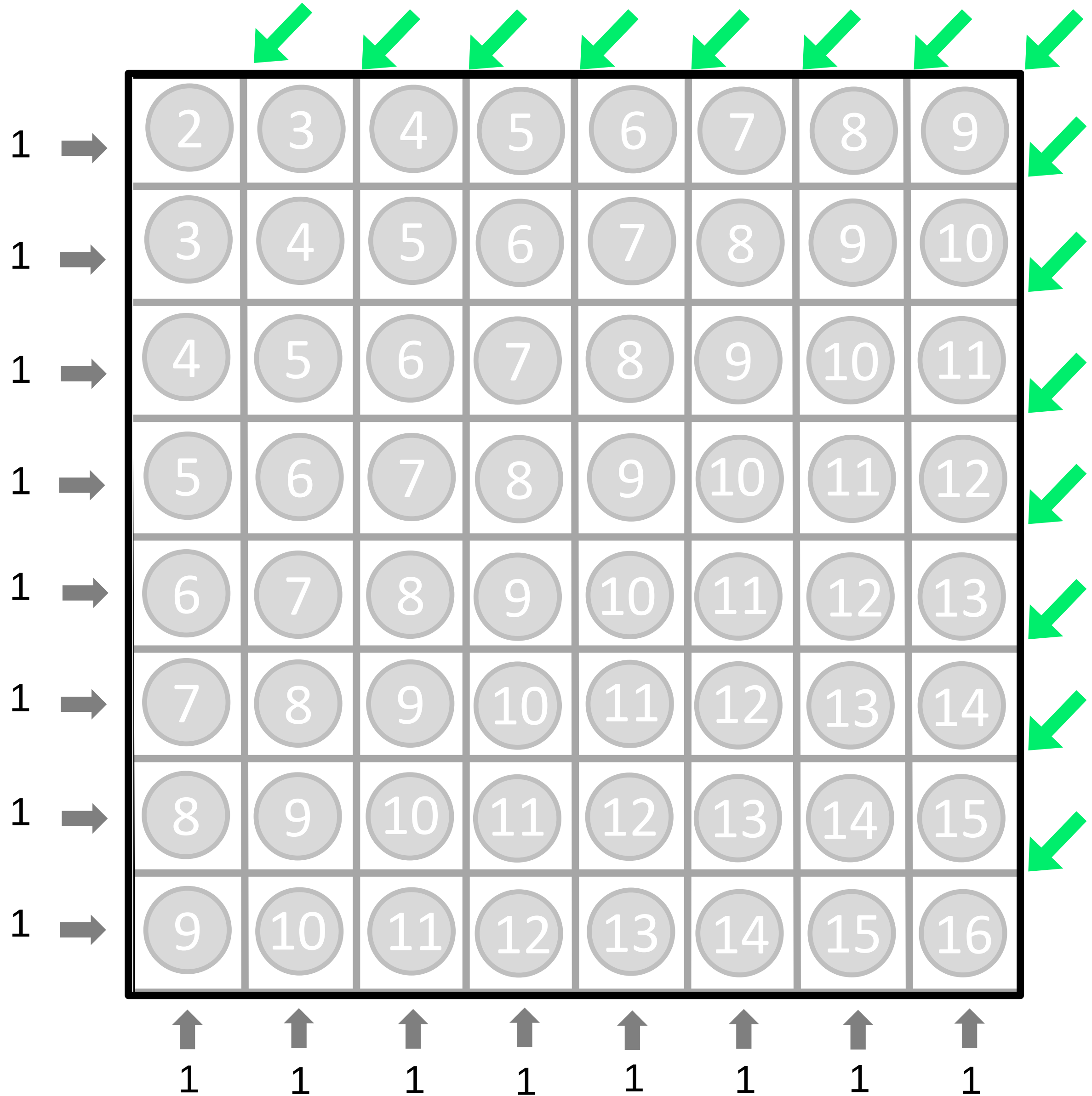
③ Alon’s Combinatorial Open Problem

9 4 7 5 10 9 13 15

Solution
=
permutation matrix
with prescribed integers



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1	3	4	5	6	7	8	9	10
1	4	5	6	7	8	9	10	11
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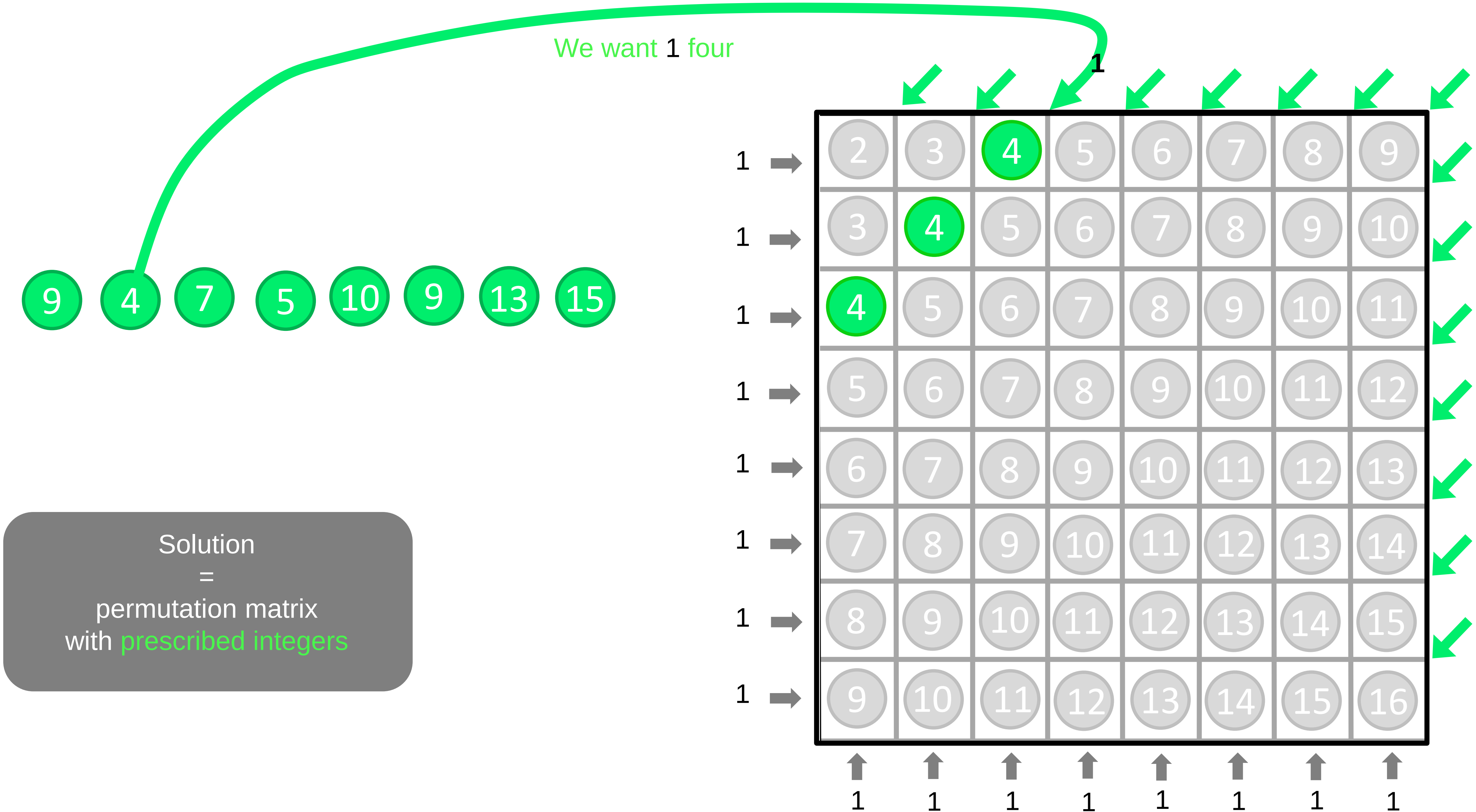
1	6	7	8	9	10	11	12	13
1	7	8	9	10	11	12	13	14
1	8	9	10	11	12	13	14	15
1	9	10	11	12	13	14	15	16
	1	1	1	1	1	1	1	1

③ Alon's Combinatorial Open Problem



1	2	3	4	5	6	7	8	9
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1	4	5	6	7	8	9	10	11
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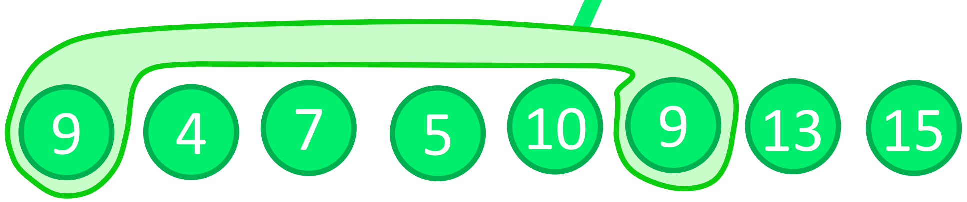
1	6	7	8	9	10	11	12	13
1	7	8	9	10	11	12	13	14
1	8	9	10	11	12	13	14	15
1	9	10	11	12	13	14	15	16
	1	1	1	1	1	1	1	1

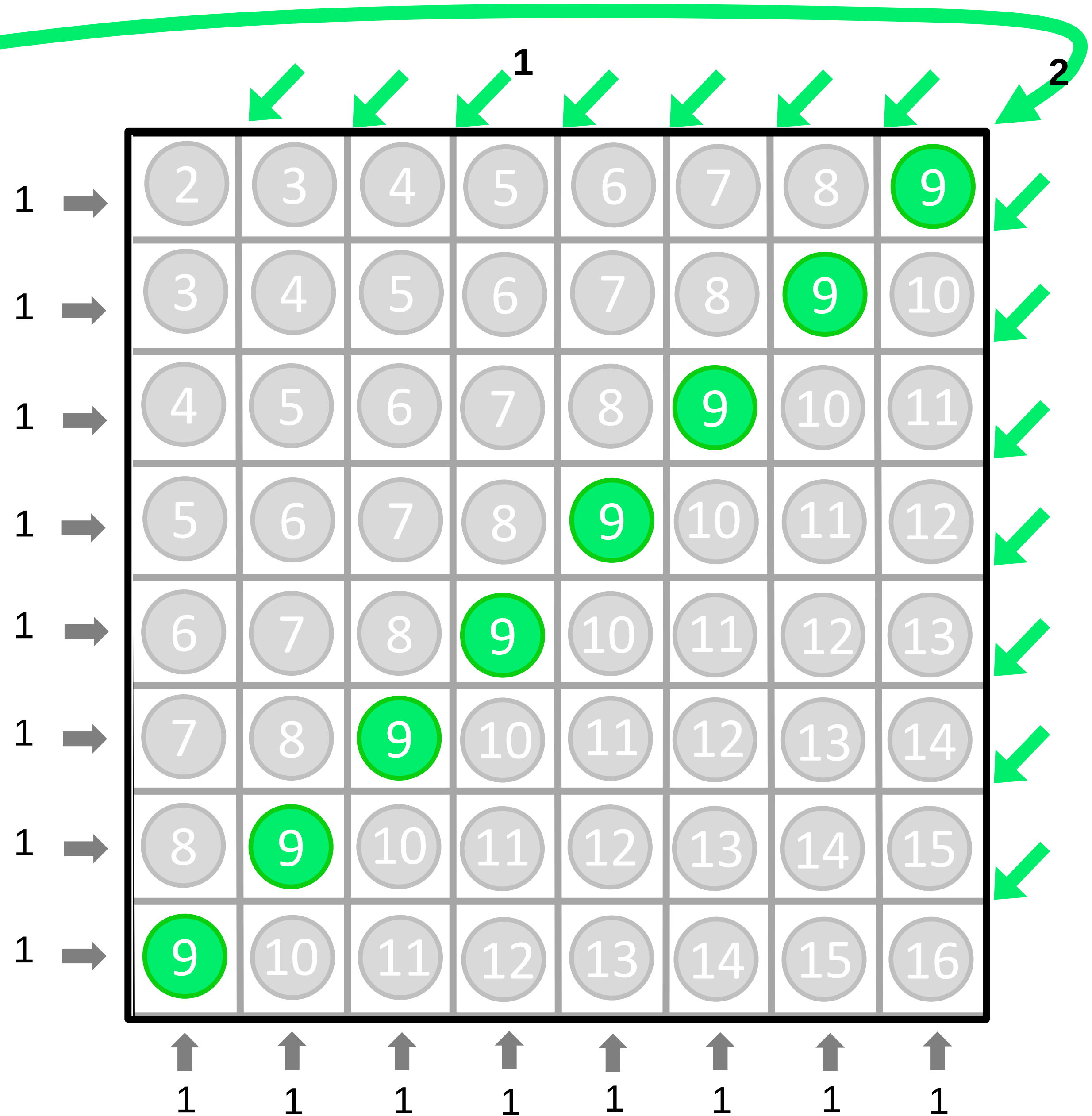
Solution
=
permutation matrix
with prescribed integers

③ Alon's Combinatorial Open Problem

Permutation Matrix

We want 2 nine



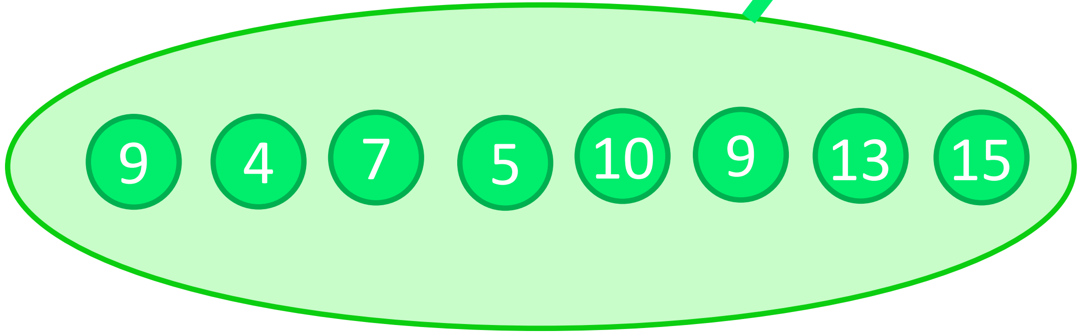


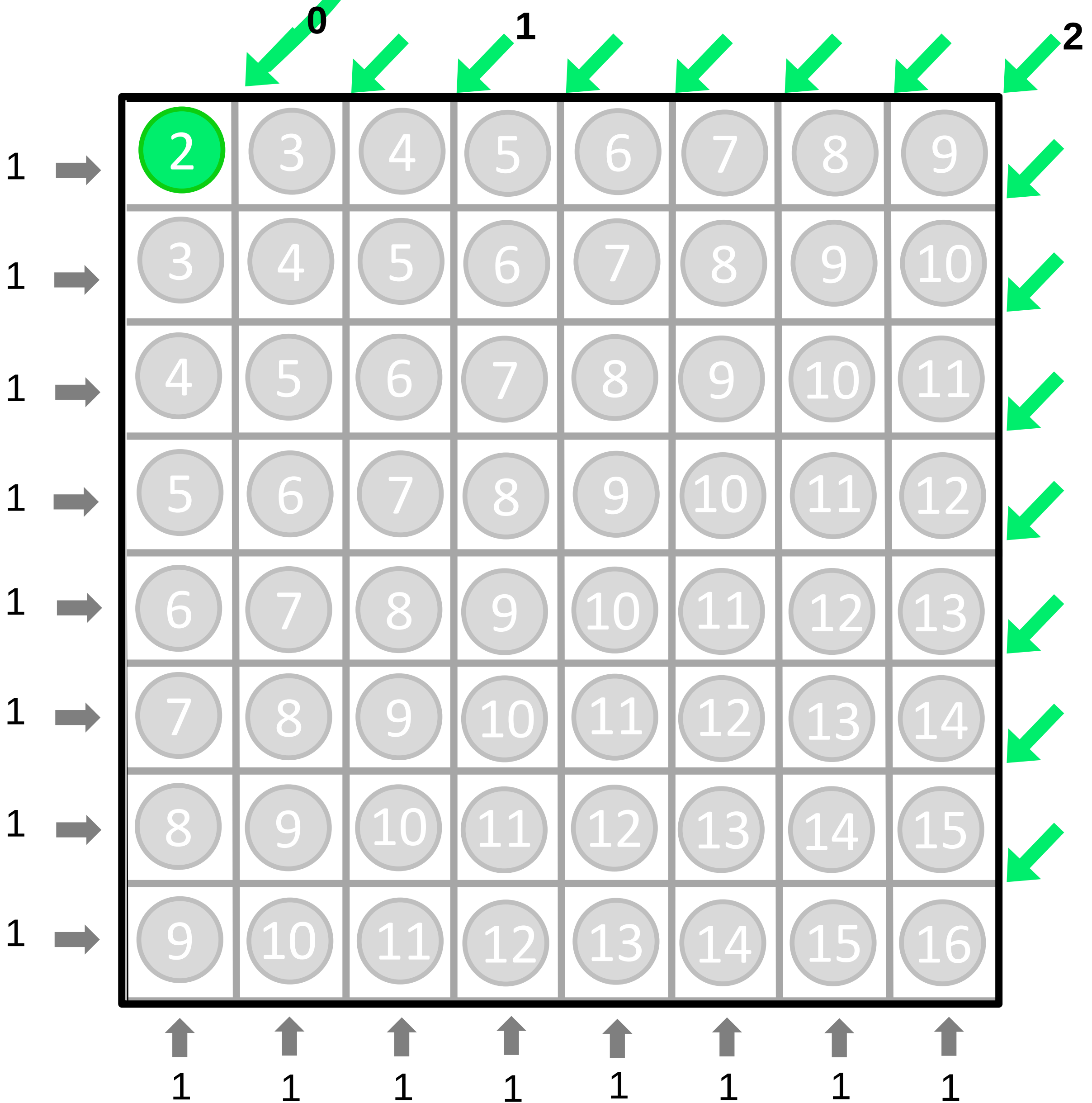
1	2	3	4	5	6	7	8	9
1	3	4	5	6	7	8	9	10
1	4	5	6	7	8	9	10	11
1	5	6	7	8	9	10	11	12
1	6	7	8	9	10	11	12	13
1	7	8	9	10	11	12	13	14
1	8	9	10	11	12	13	14	15
1	9	10	11	12	13	14	15	16
	1	1	1	1	1	1	1	1

Solution
=
permutation matrix
with prescribed integers

③ Alon’s Combinatorial Open Problem

We want 0 two





1	2	3	4	5	6	7	8	9
1	3	4	5	6	7	8	9	10
1	4	5	6	7	8	9	10	11
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1	6	7	8	9	10	11	12	13
1	7	8	9	10	11	12	13	14
1	8	9	10	11	12	13	14	15
1	9	10	11	12	13	14	15	16
	1	1	1	1	1	1	1	1

Solution
=
permutation matrix
with prescribed integers

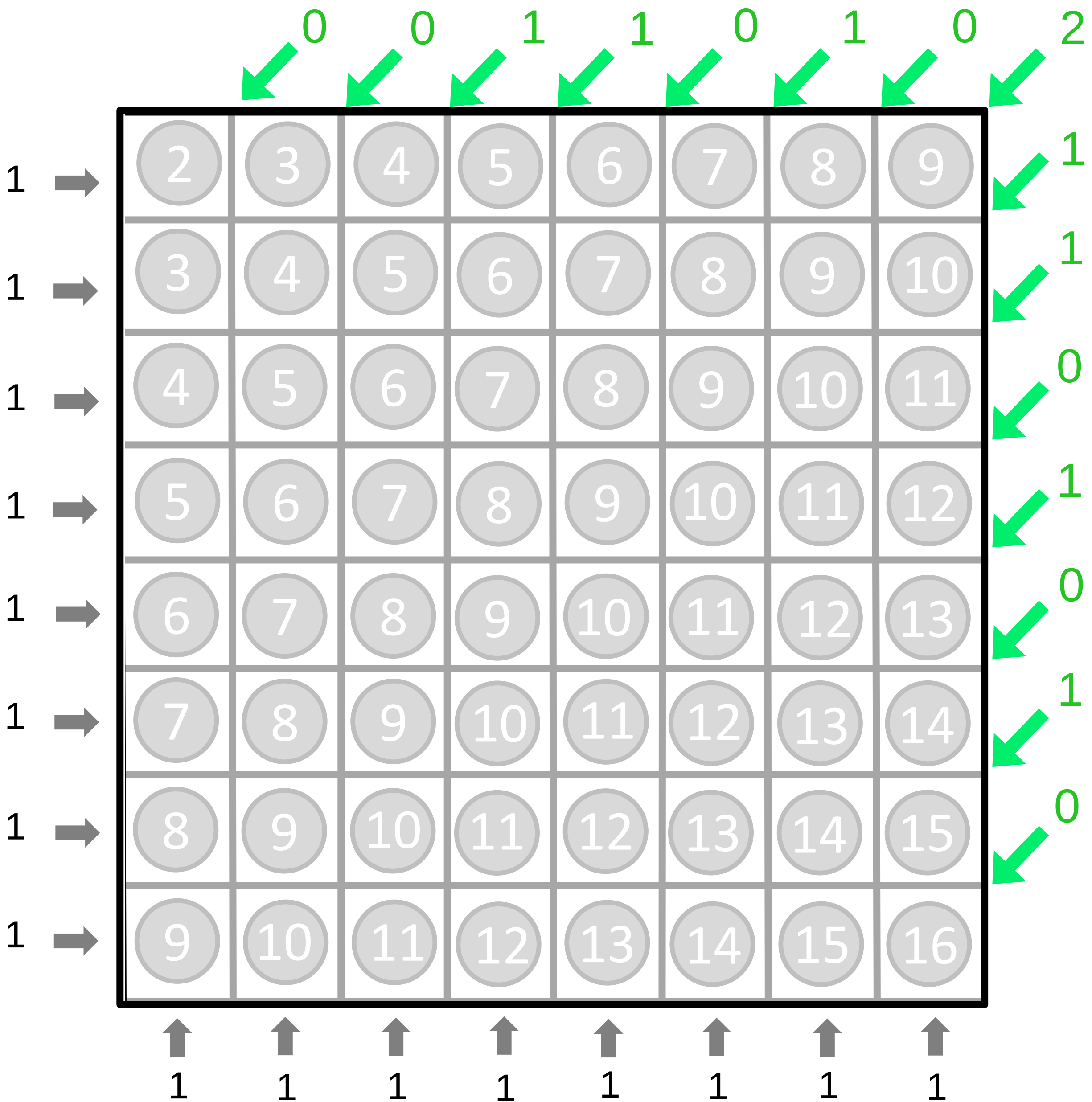
③ Alon’s Combinatorial Open Problem

Puzzle

Obtaining the target sums

9 4 7 5 10 9 13 15

is a Discrete Tomography puzzle



2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
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5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14
8	9	10	11	12	13	14	15
9	10	11	12	13	14	15	16

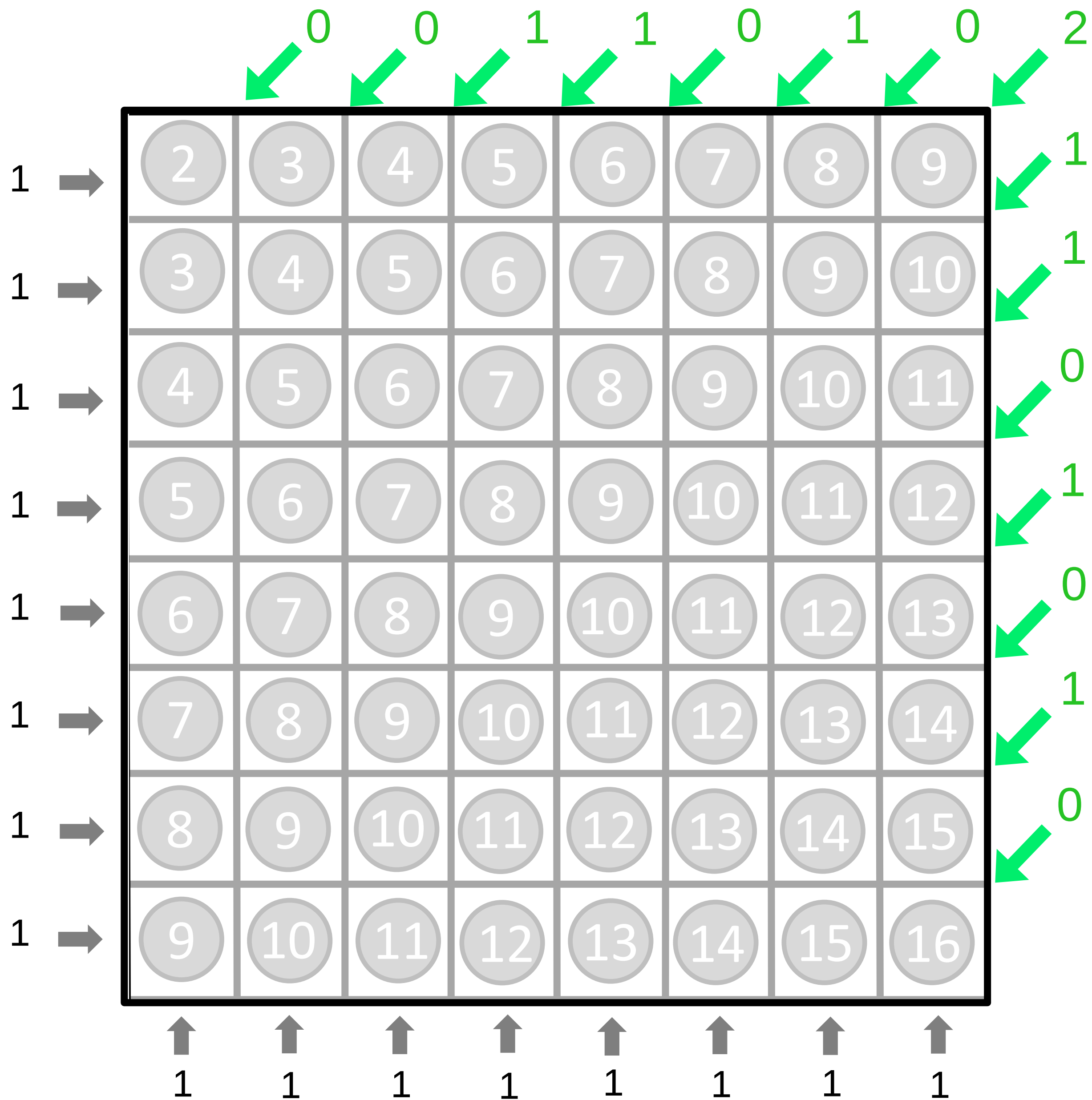
③ Alon’s Combinatorial Open Problem

Obtaining the target sums

9 4 7 5 10 9 13 15

is a Discrete Tomography puzzle

Still *NP-hard*….



2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14
8	9	10	11	12	13	14	15
9	10	11	12	13	14	15	16

③ Alon's Combinatorial Open Problem

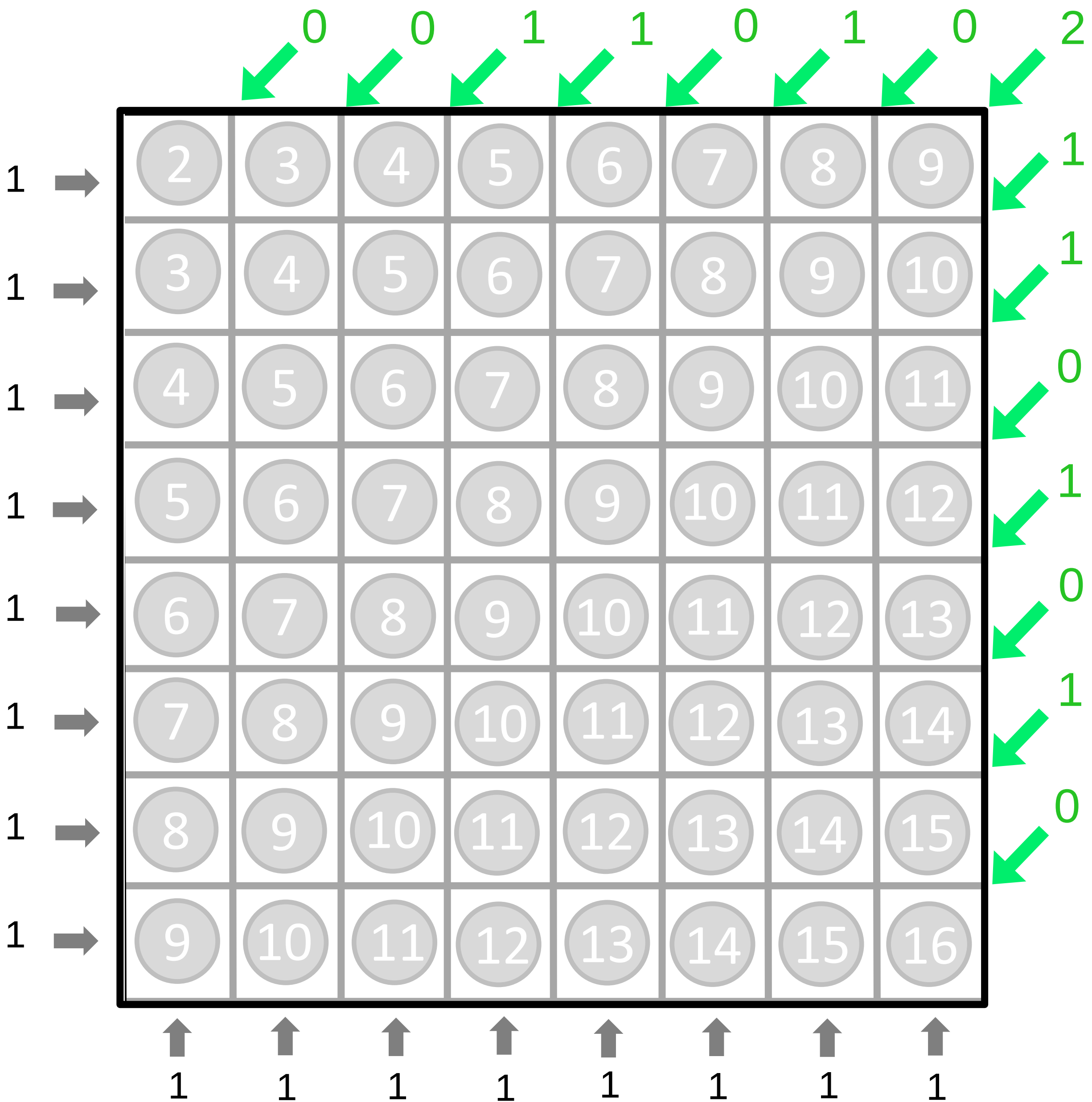
Obtaining the target sums

9 4 7 5 10 9 13 15

is a Discrete Tomography puzzle

Still *NP-hard*….

Change the rule of +



1	2	3	4	5	6	7	8	9
1	3	4	5	6	7	8	9	10
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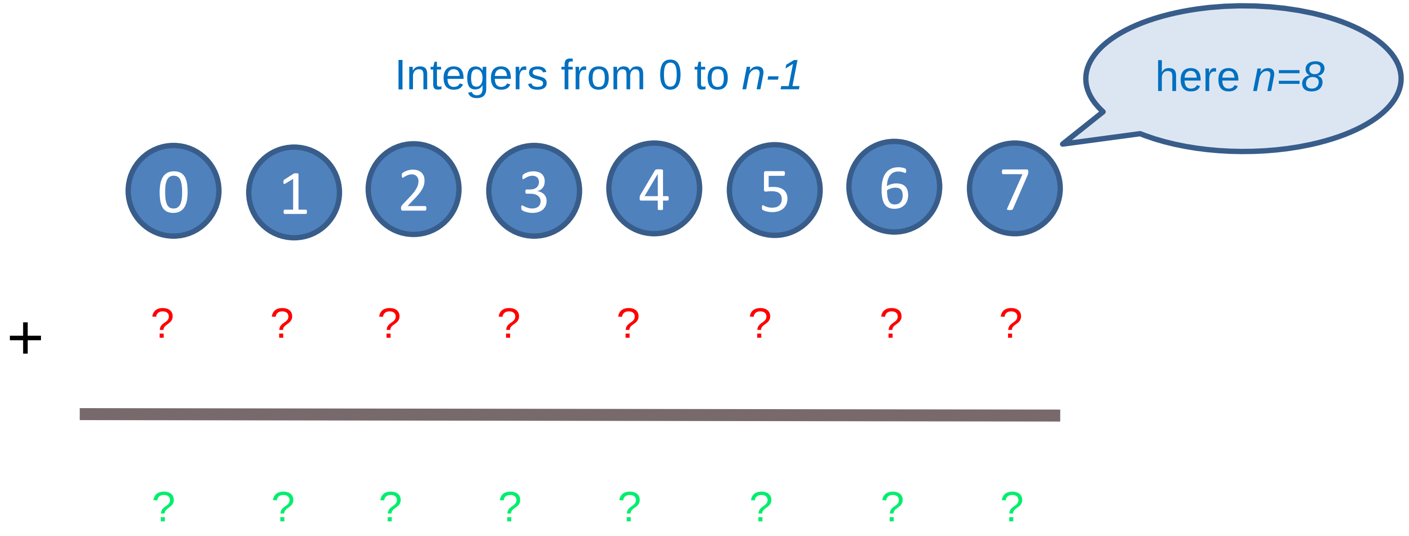
1	6	7	8	9	10	11	12	13
1	7	8	9	10	11	12	13	14
1	8	9	10	11	12	13	14	15
1	9	10	11	12	13	14	15	16
	1	1	1	1	1	1	1	1

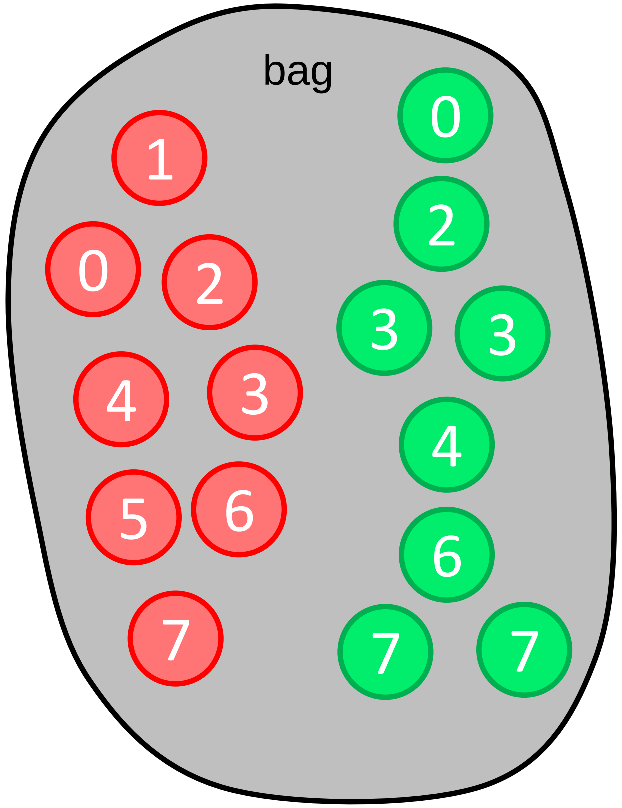
③ Alon's Combinatorial Open Problem

Initial puzzle

Integers from 0 to *n-1*



Integers from 0 to *n-1*



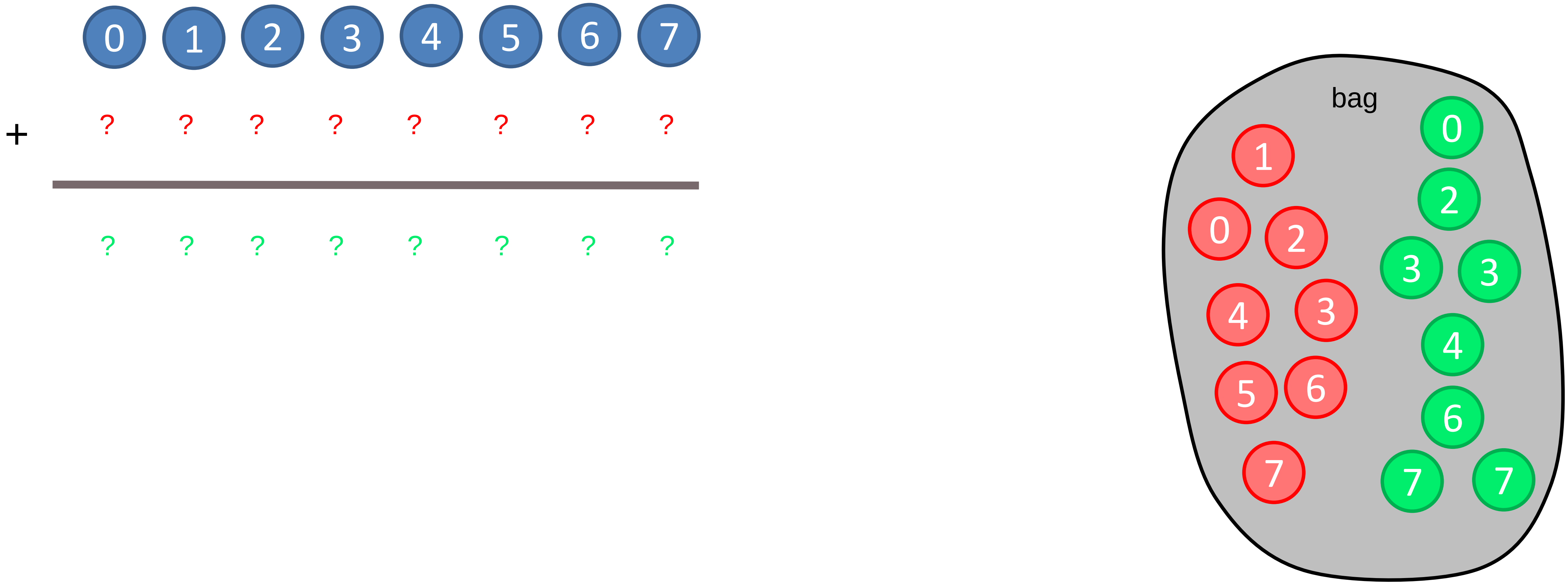
Input = Target sums:

n integers with sum 2(n-1)

③

Alon’s Combinatorial Open Problem

Initial puzzle

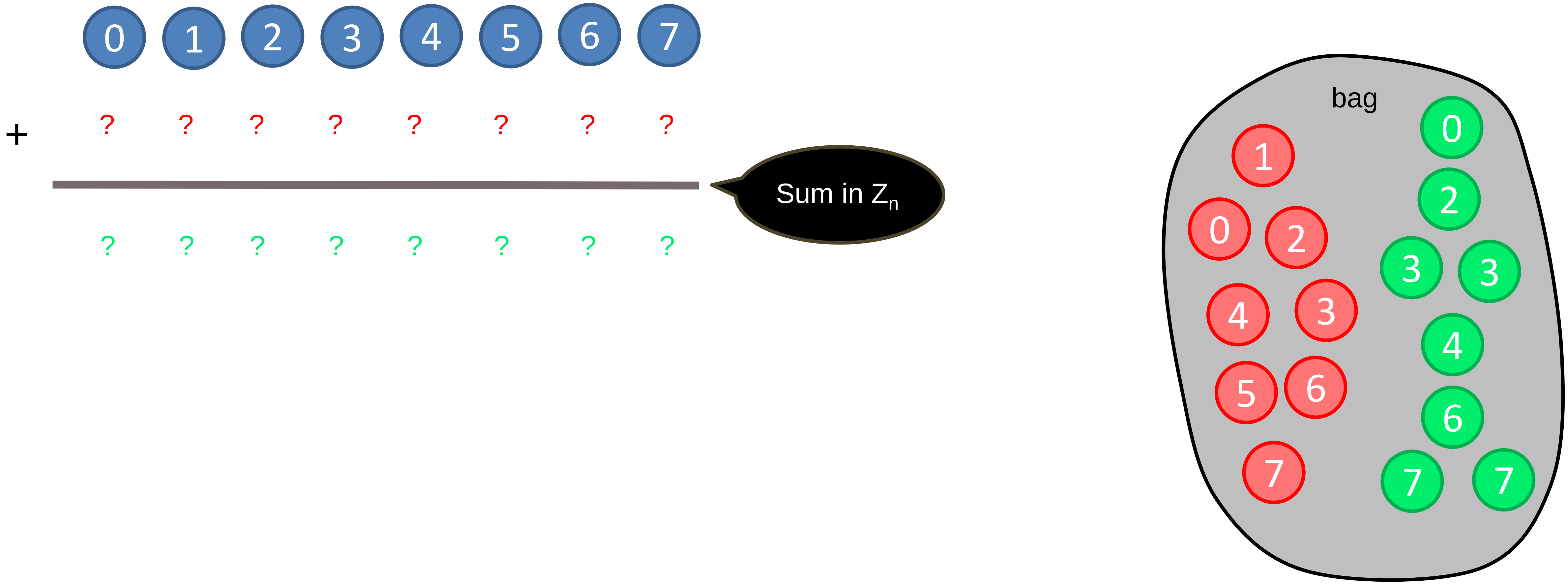


Input = Target sums:

n integers with sum 2(n-1)

③ Alon’s Combinatorial Open Problem

Same puzzle but in Z_n

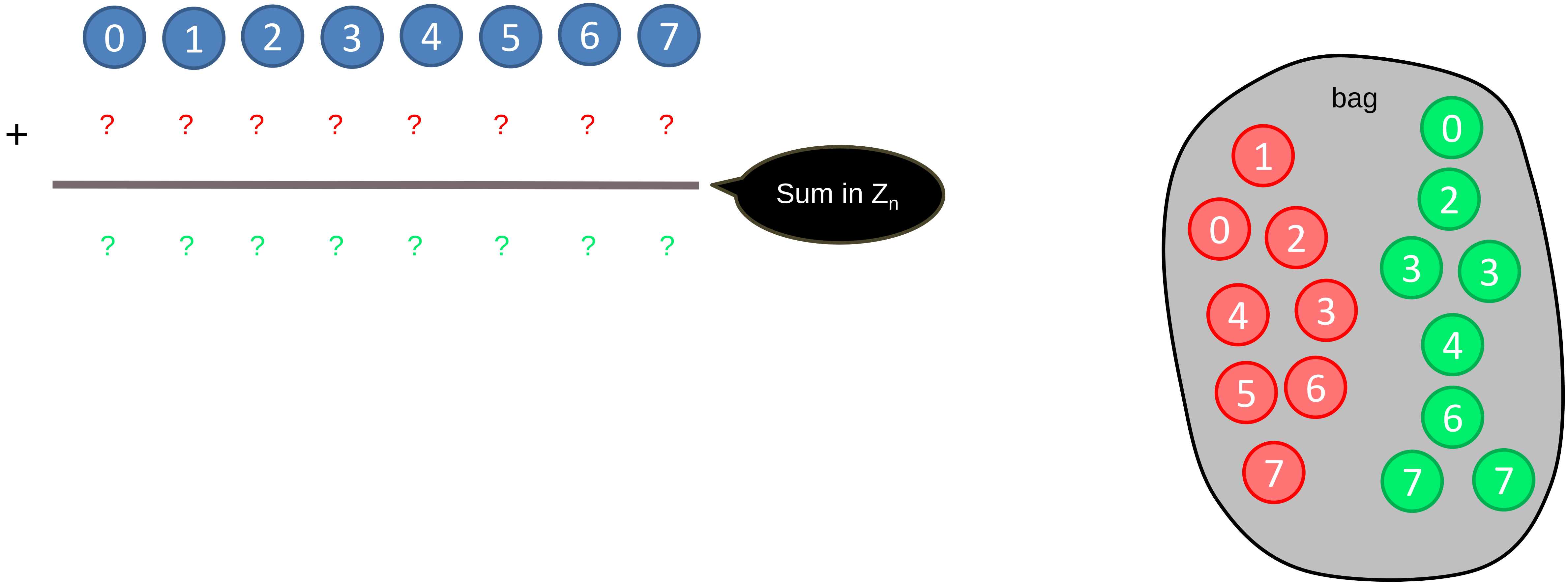


Input = Target sums:

n integers with a null sum

③ Alon’s Combinatorial Open Problem

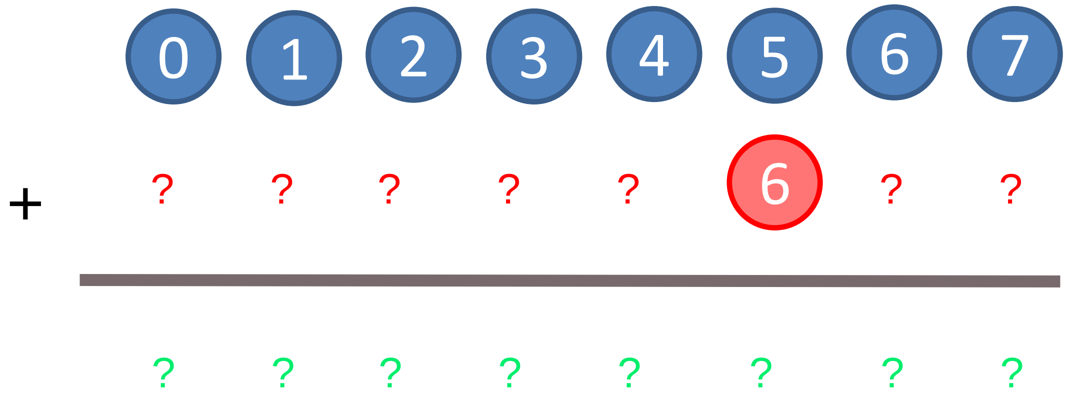
Same puzzle but in Z_n

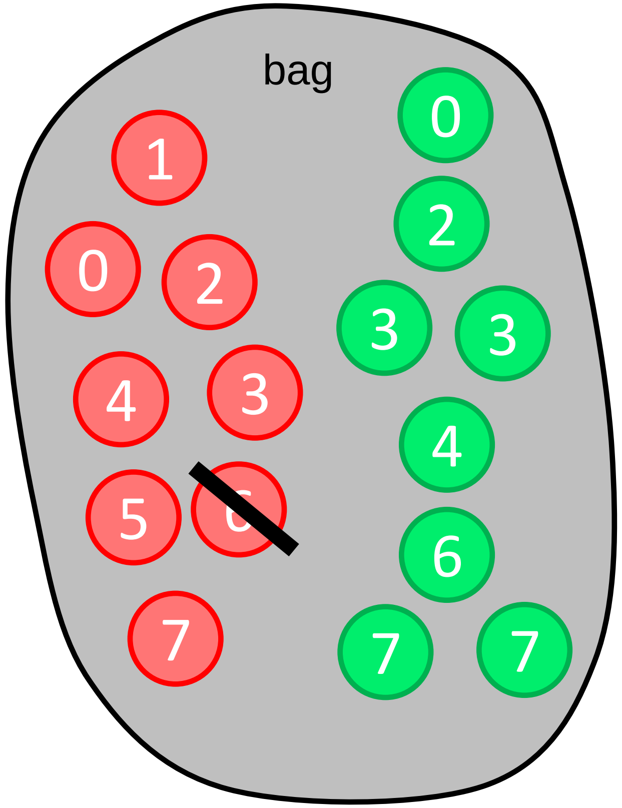


③ Alon's Combinatorial Open Problem

Same puzzle but in Z_n

Integers from 0 to *n-1*

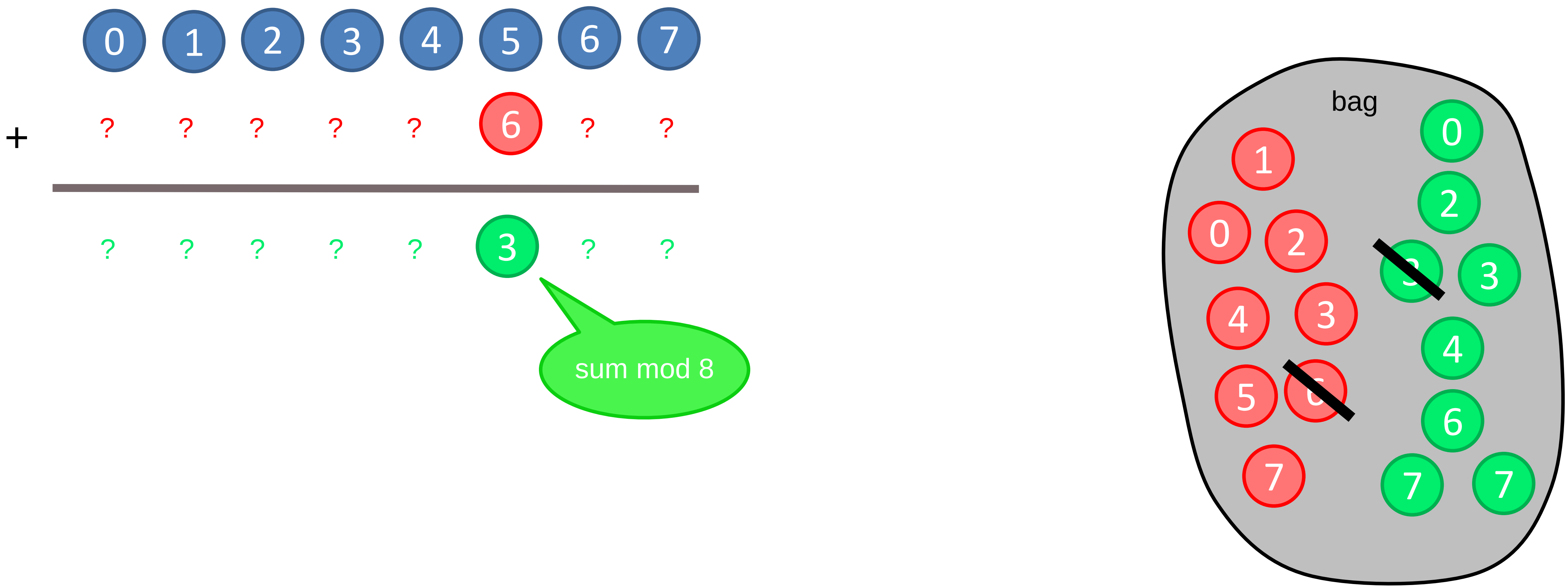




③

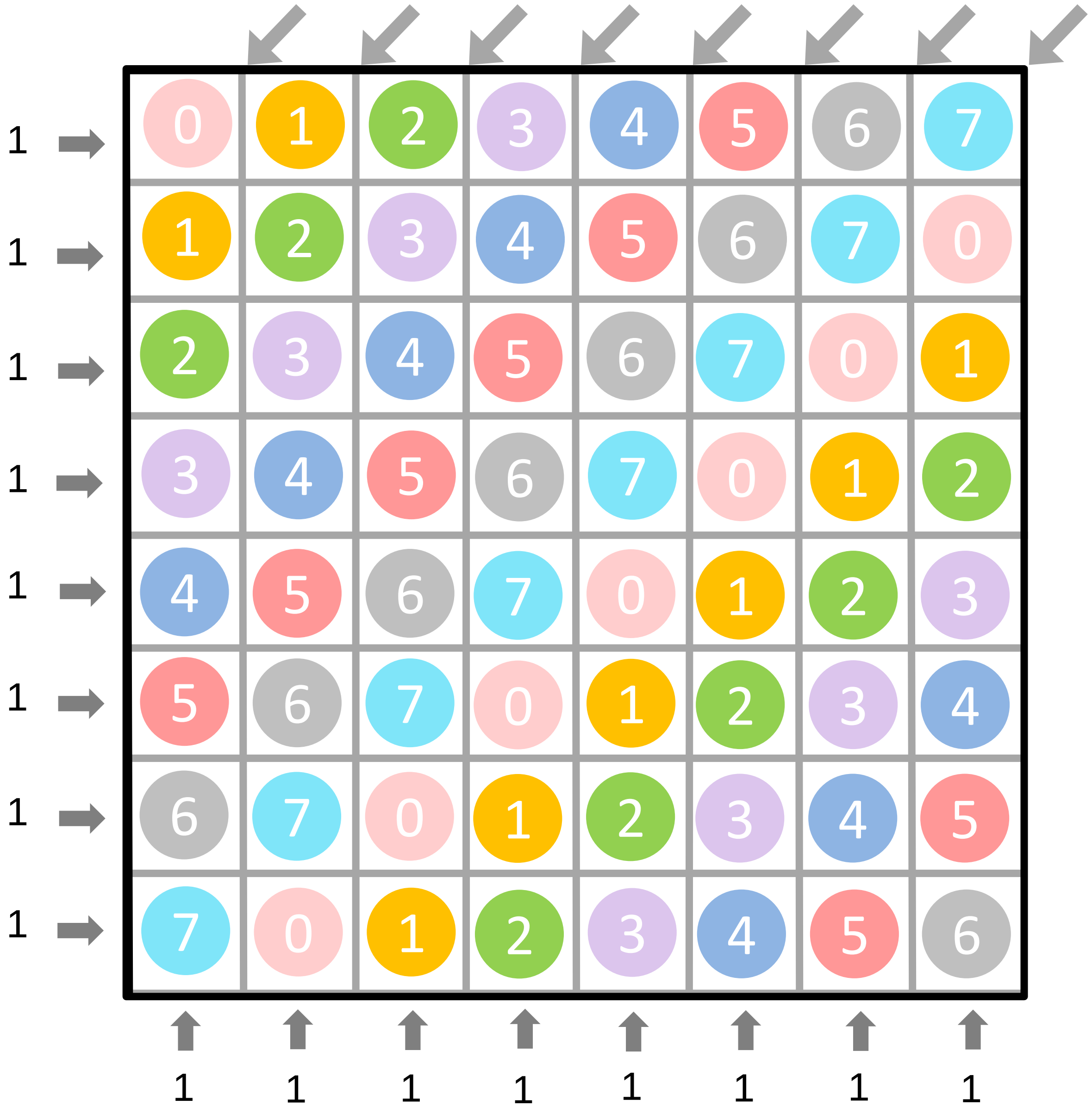
Alon's Combinatorial Open Problem

Same puzzle but in Z_n



③ Alon's Combinatorial Open Problem

Toric grid



1	0	1	2	3	4	5	6	7
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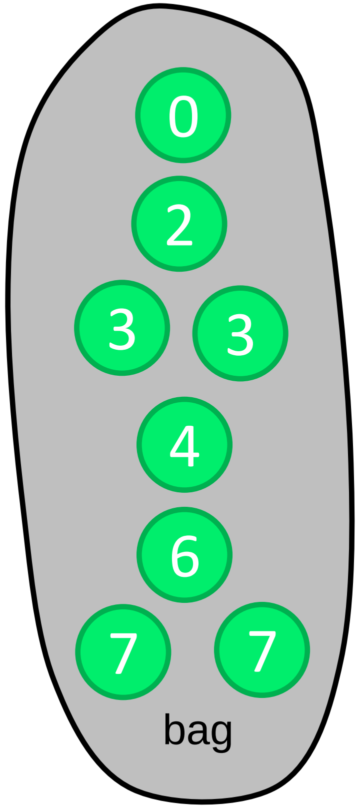
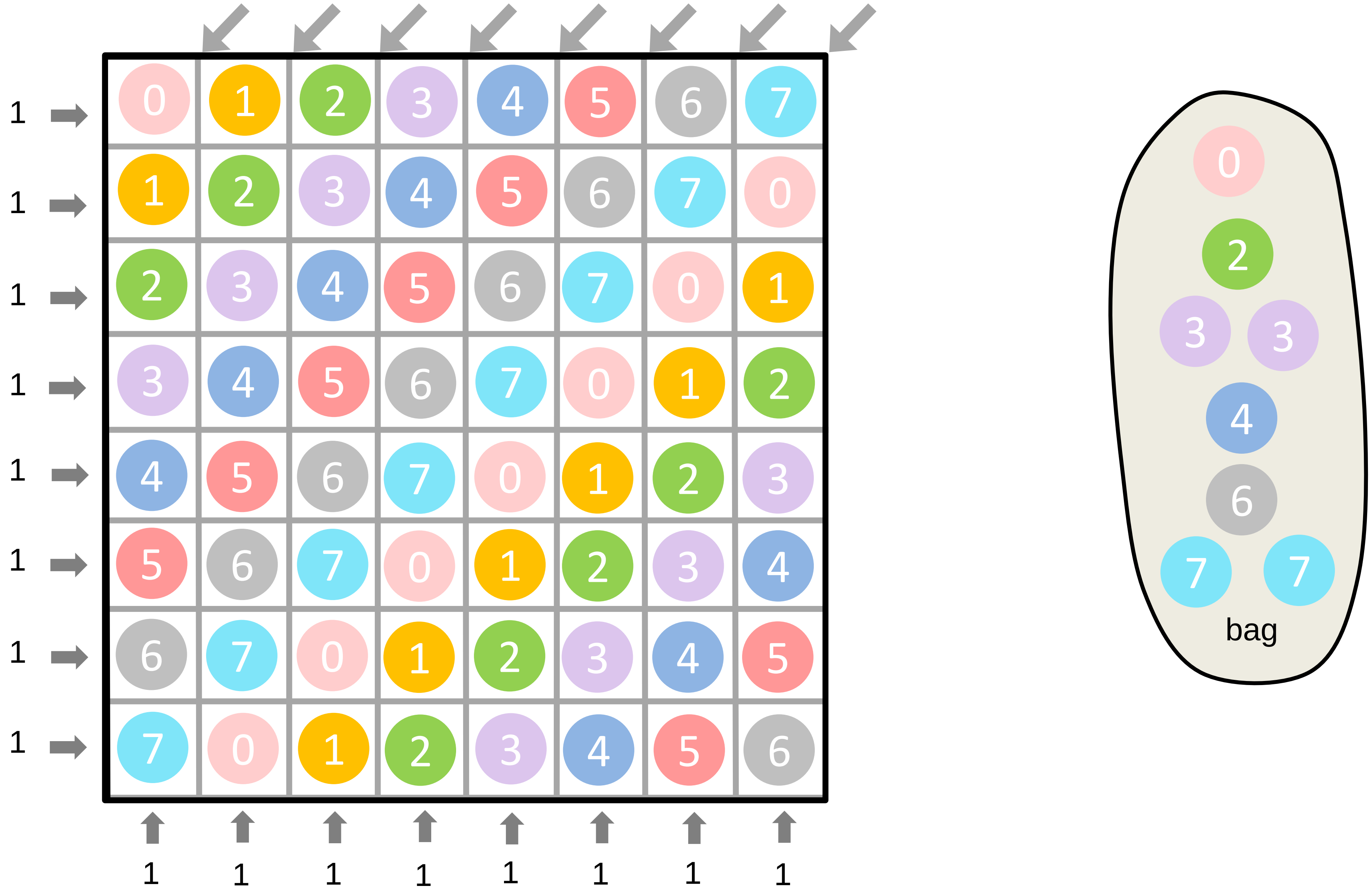
1	2	3	4	5	6	7	0	1
1	3	4	5	6	7	0	1	2
1	4	5	6	7	0	1	2	3
1	5	6	7	0	1	2	3	4
1	6	7	0	1	2	3	4	5
1	7	0	1	2	3	4	5	6
	1	1	1	1	1	1	1	1



③

Alon’s Combinatorial Open Problem

Toric grid



1	0	1	2	3	4	5	6	7
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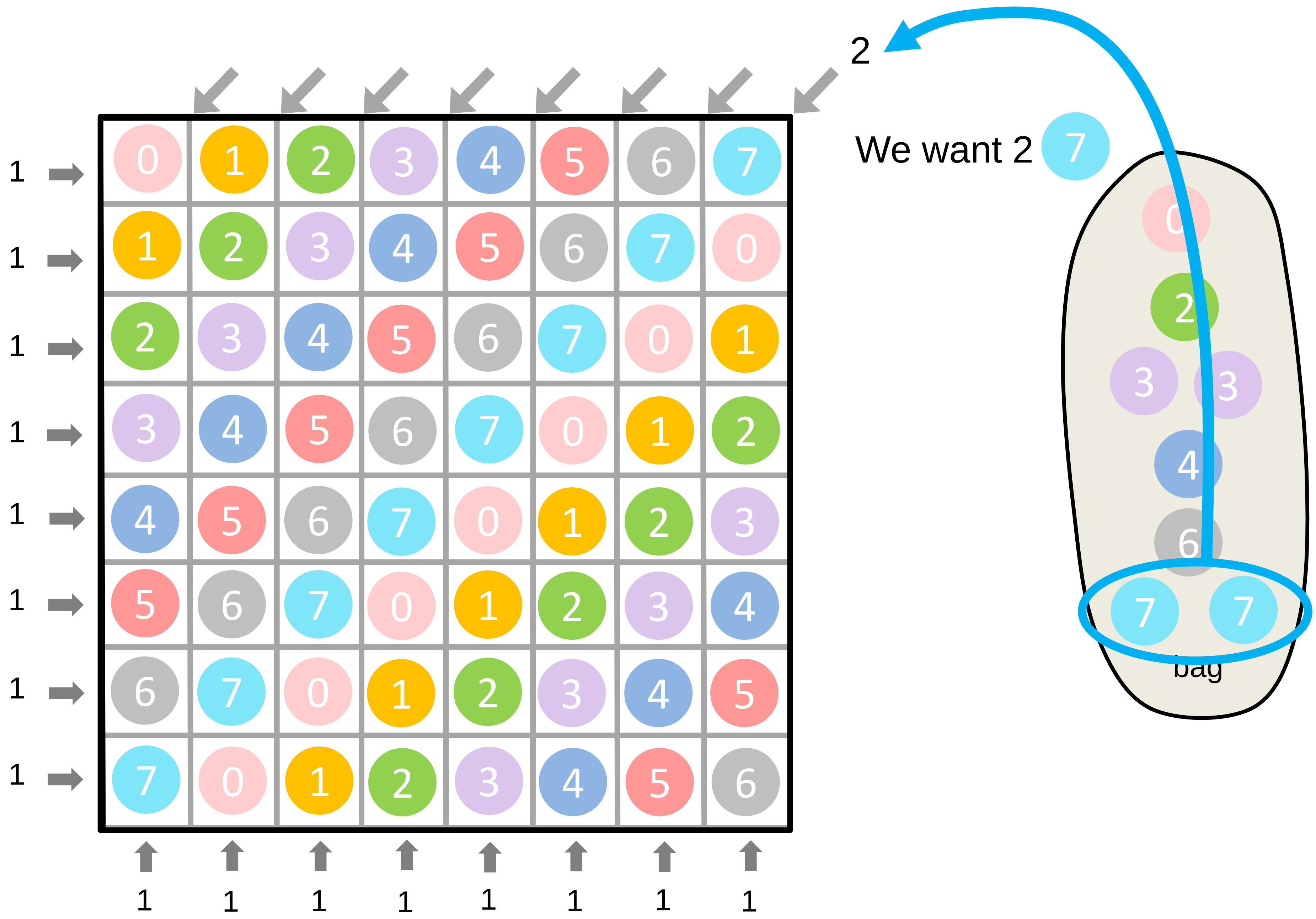
1	2	3	4	5	6	7	0	1
1	3	4	5	6	7	0	1	2
1	4	5	6	7	0	1	2	3
1	5	6	7	0	1	2	3	4
1	6	7	0	1	2	3	4	5
1	7	0	1	2	3	4	5	6
	1	1	1	1	1	1	1	1

Alon's Combinatorial Open Problem



1	0	1	2	3	4	5	6	7
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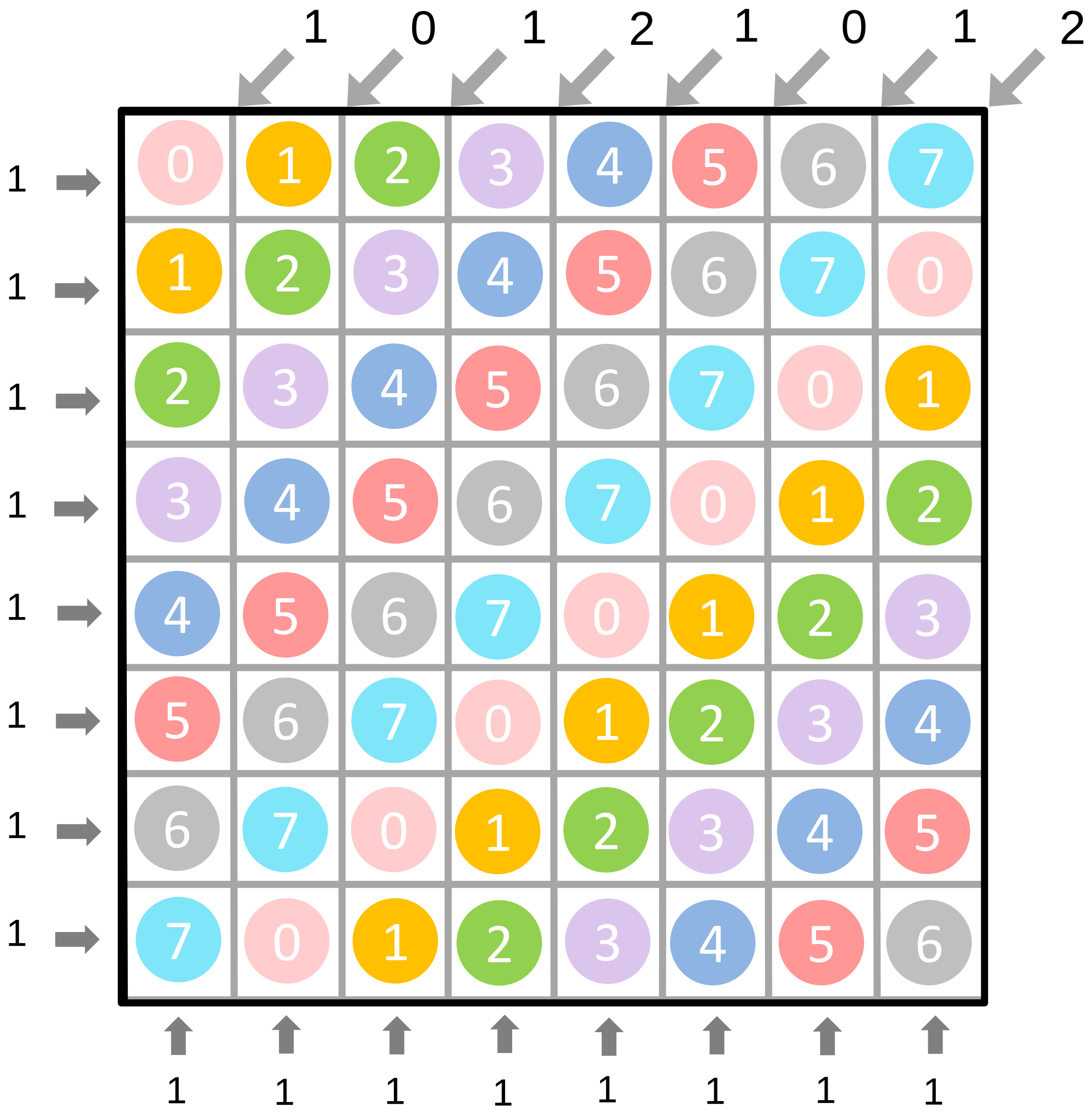
1	2	3	4	5	6	7	0	1
1	3	4	5	6	7	0	1	2
1	4	5	6	7	0	1	2	3
1	5	6	7	0	1	2	3	4
1	6	7	0	1	2	3	4	5
1	7	0	1	2	3	4	5	6
	1	1	1	1	1	1	1	1

We want 2 7

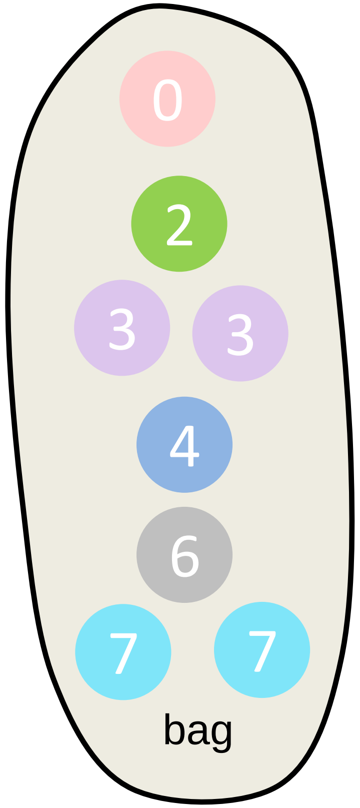
③

Alon’s Combinatorial Open Problem

Toric grid

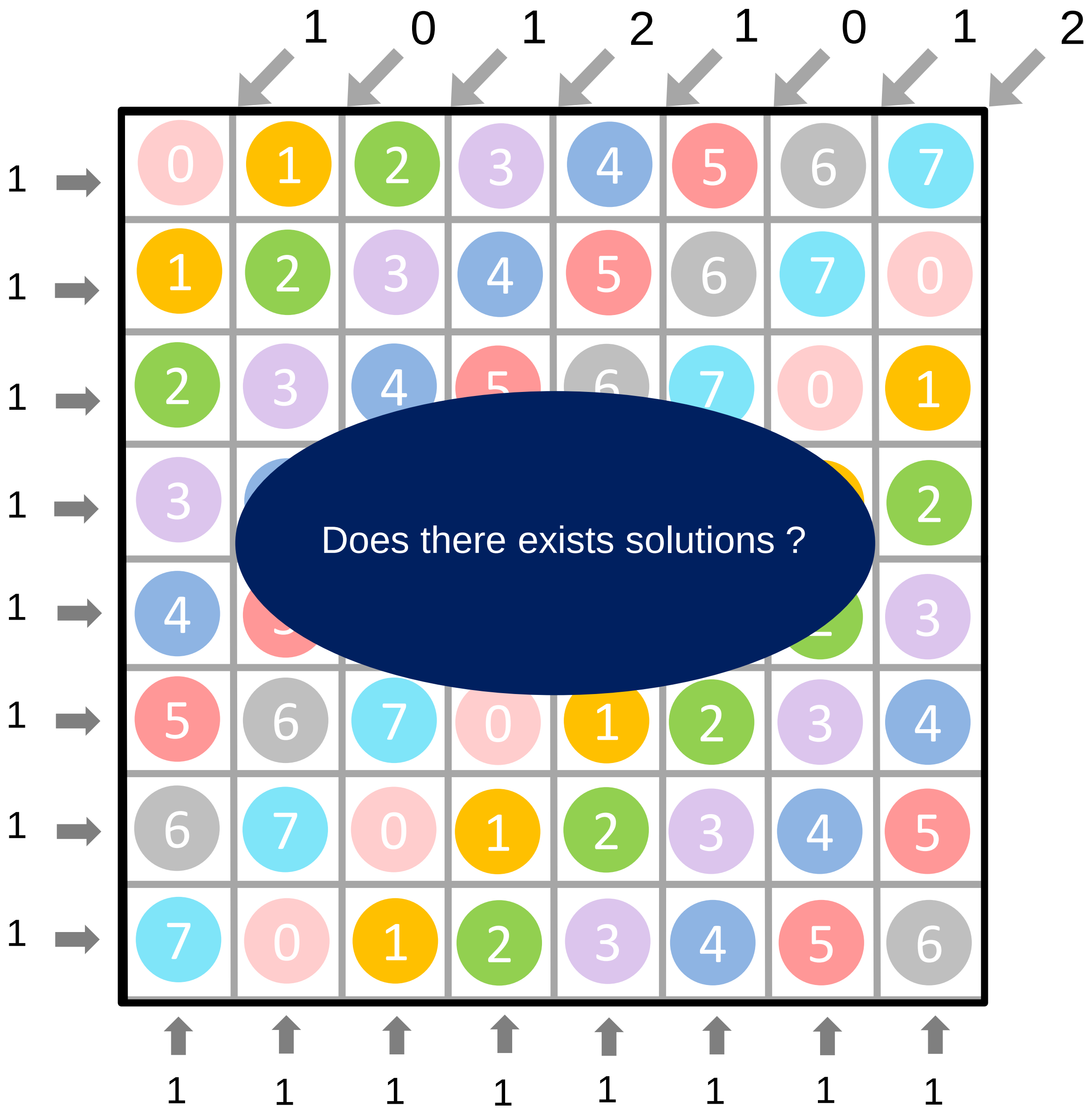


0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

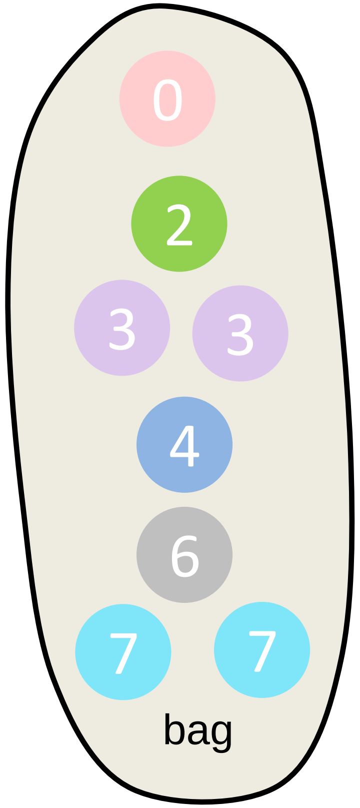


③ Alon’s Combinatorial Open Problem

1 dollar question



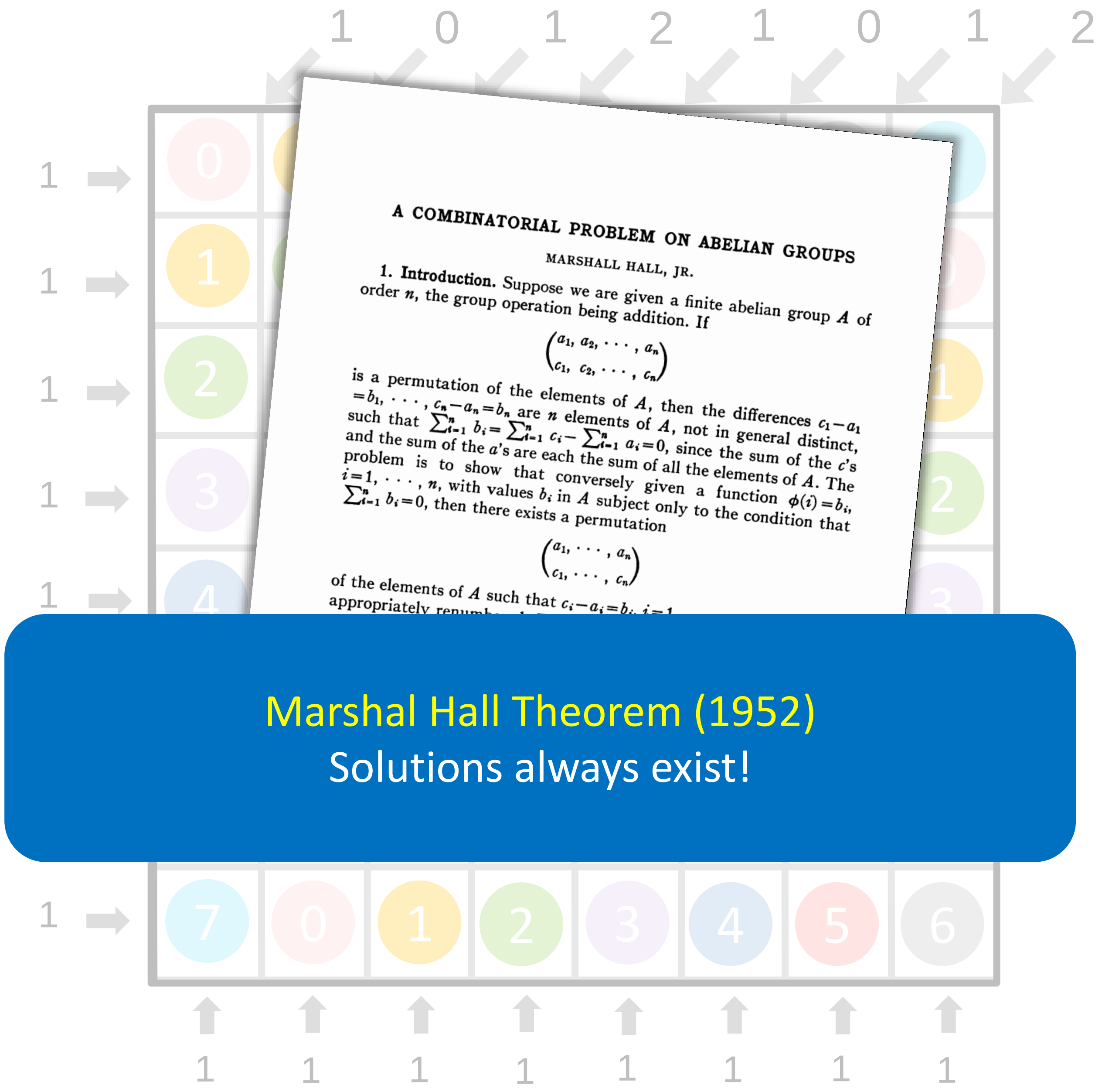
Does there exists solutions ?



Sum of the bag=0

③ Alon’s Combinatorial Open Problem

0 dollar answer = YES



A COMBINATORIAL PROBLEM ON ABELIAN GROUPS

MARSHALL HALL, JR.

1. Introduction. Suppose we are given a finite abelian group A of order n, the group operation being addition. If

$$\begin{pmatrix} a_1, & a_2, & \cdots, & a_n \\ c_1, & c_2, & \cdots, & c_n \end{pmatrix}$$

is a permutation of the elements of A, then the differences $c_1-a_1=b_1, \cdots, c_n-a_n=b_n$ are n elements of A, not in general distinct, such that $\sum_{i=1}^{n} b_i=\sum_{i=1}^{n} c_i-\sum_{i=1}^{n} a_i=0$, since the sum of the c's and the sum of the a's are each the sum of all the elements of A. The problem is to show that conversely given a function $\phi(i)=b_i$, $i=1, \cdots, n$, with values b_i in A subject only to the condition that $\sum_{i=1}^{n} b_i=0$, then there exists a permutation

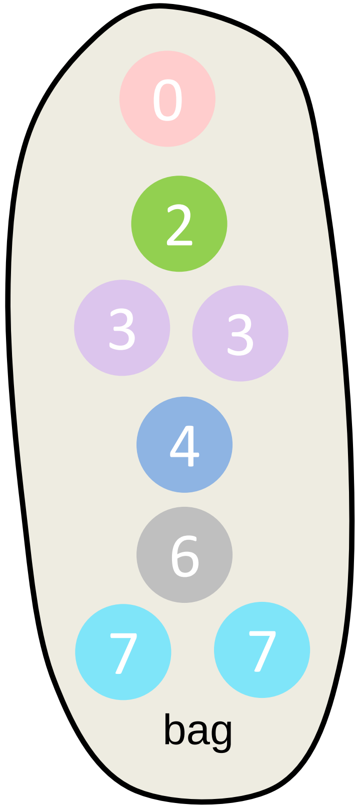
$$\begin{pmatrix} a_1, & \cdots, & a_n \\ c_1, & \cdots, & c_n \end{pmatrix}$$

of the elements of A such that $c_i-a_i=b_i$, $i=1$

appropriately

Marshal Hall Theorem (1952)

Solutions always exist!

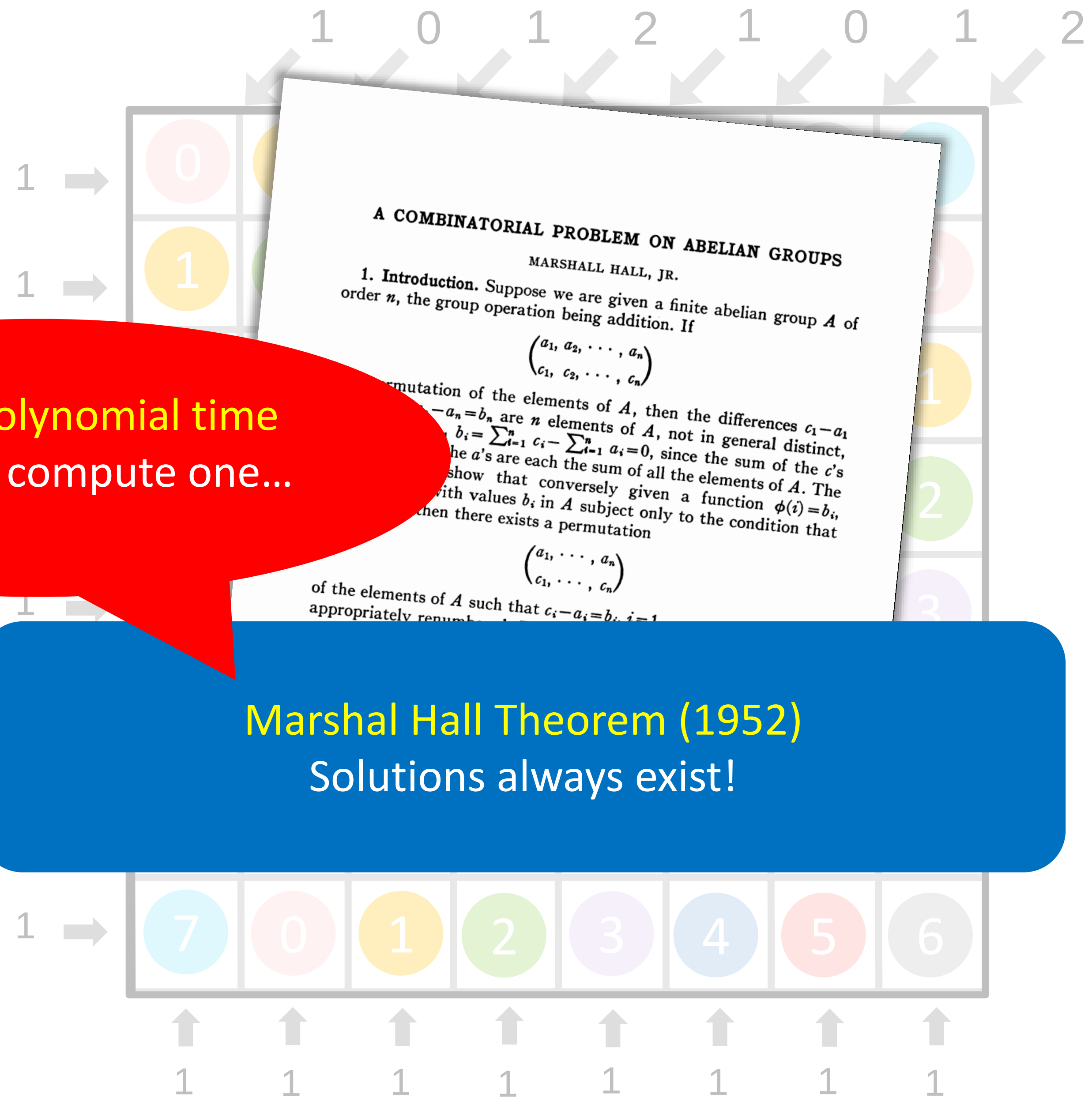


Sum of the bag=0

③

Alon's Combinatorial Open Problem

0 dollar answer = YES



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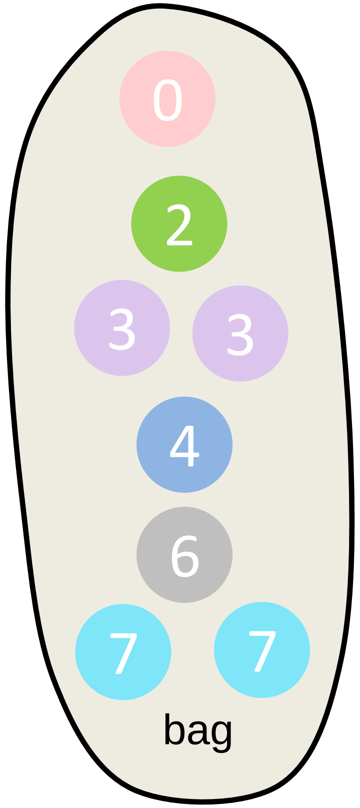
hen there exists a permutation

$$\begin{pmatrix} a_1, \cdots, a_n \\ c_1, \cdots, c_n \end{pmatrix}$$

of the elements of A such that $c_i-a_i=b_i$, $i=1$
appropriately renum

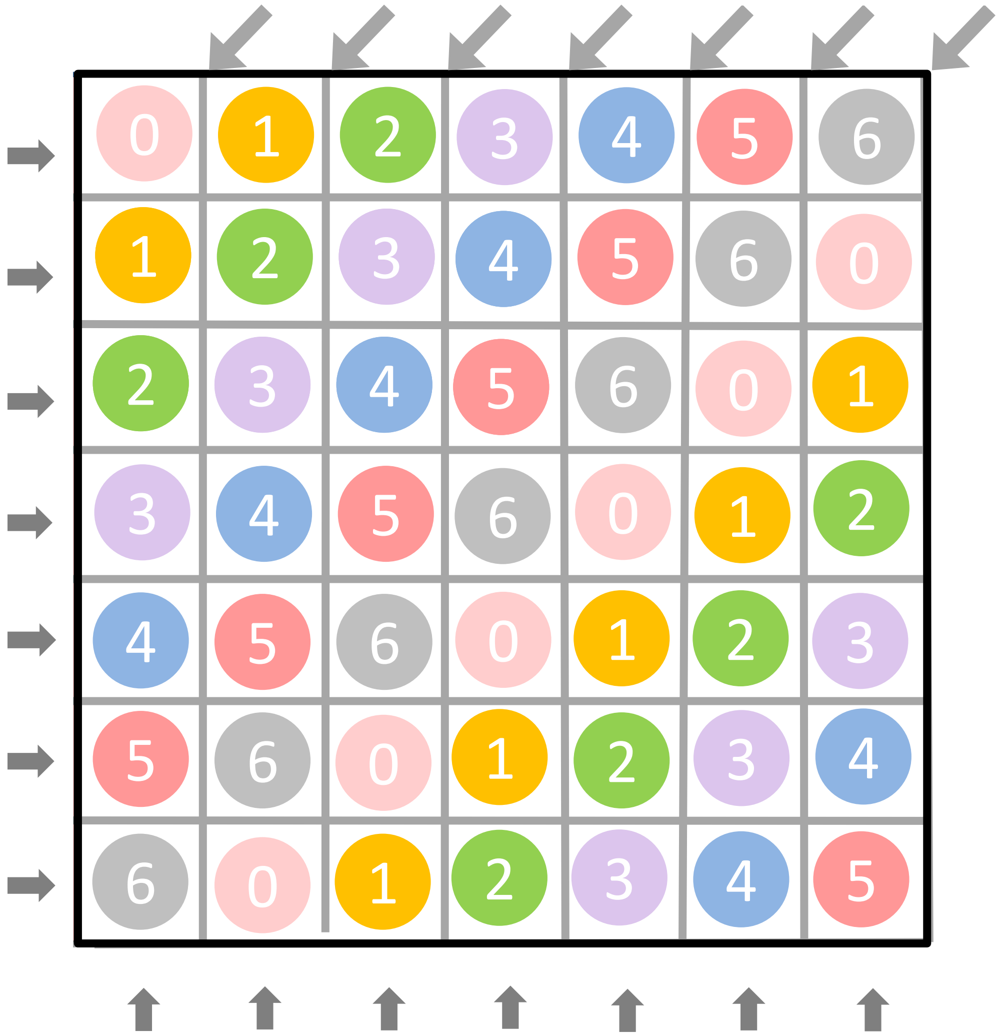
There is a polynomial time algorithm to compute one…

Marshal Hall Theorem (1952)
Solutions always exist!



Sum of the bag=0

③ Alon's Combinatorial Open Problem



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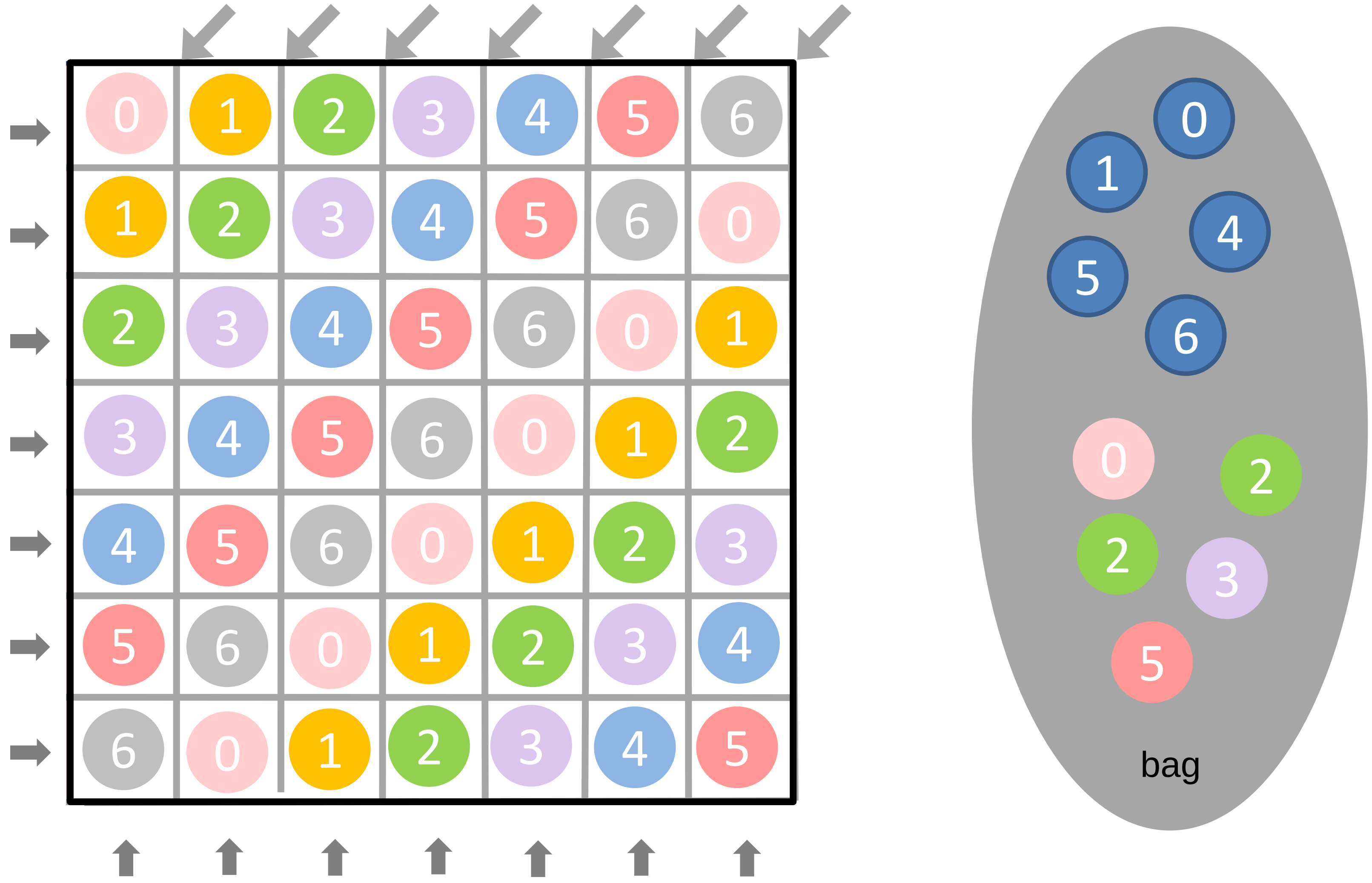
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

n is prime (here 7)

③

Alon’s Combinatorial Open Problem

Input



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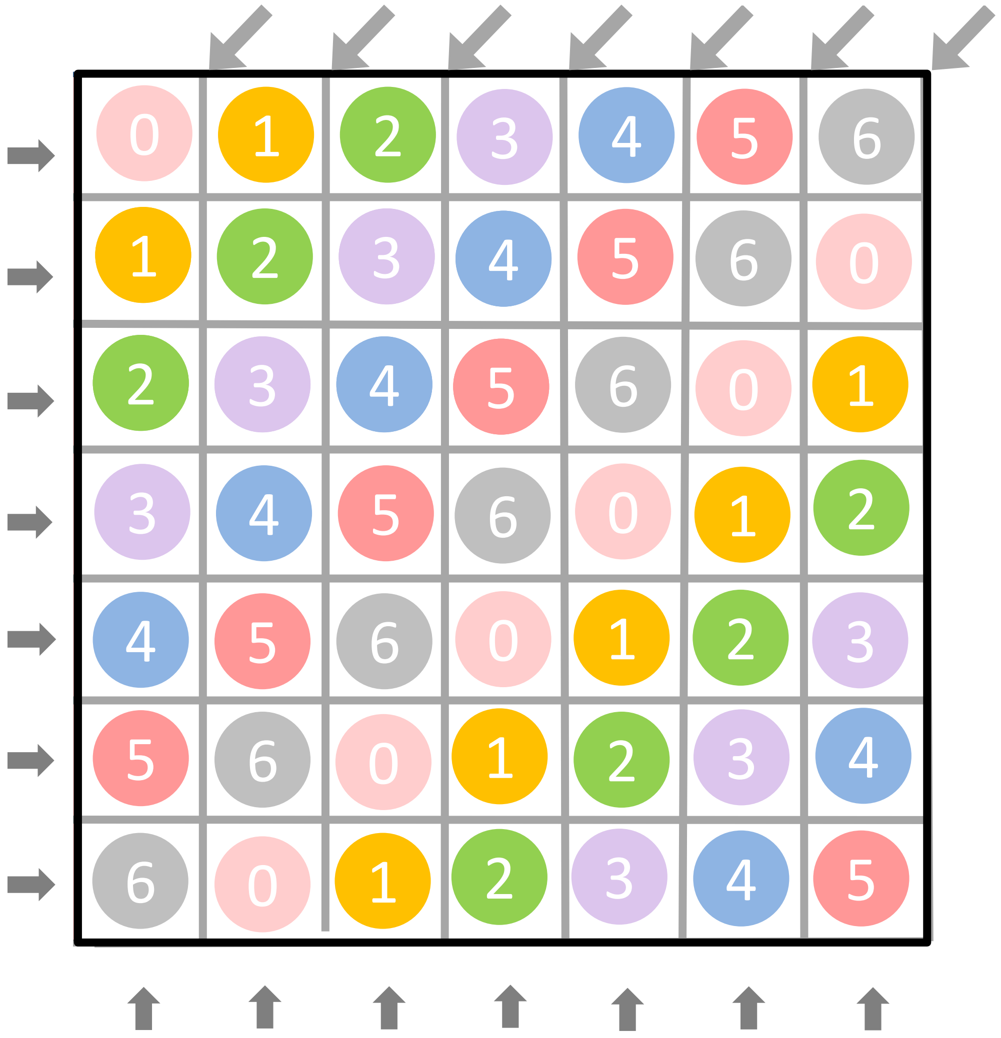
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

③ Alon's Combinatorial Open Problem

Input



0	1	2	3	4	5	6
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2	3	4	5	6	0	1
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4	5	6	0	1	2	3
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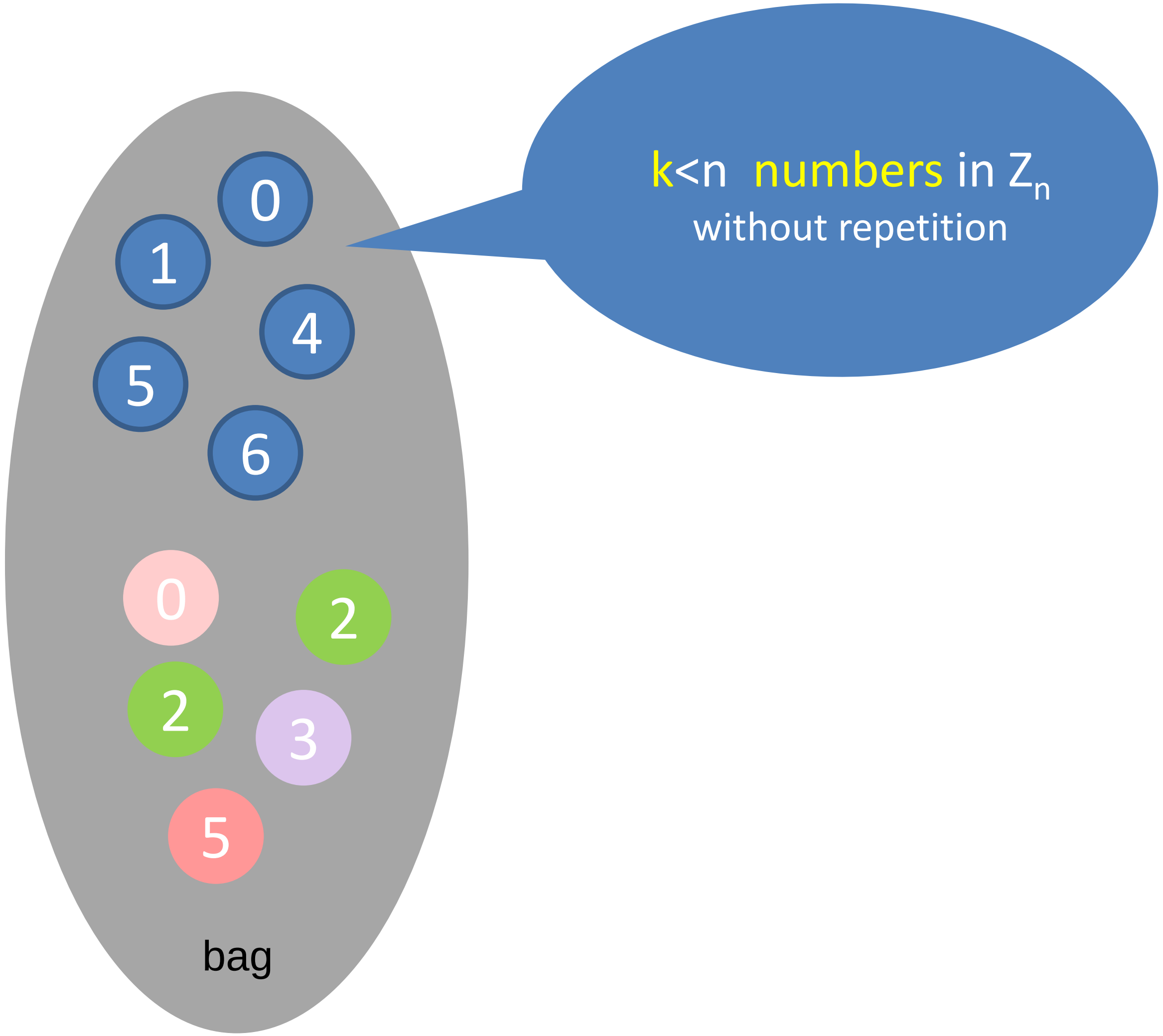
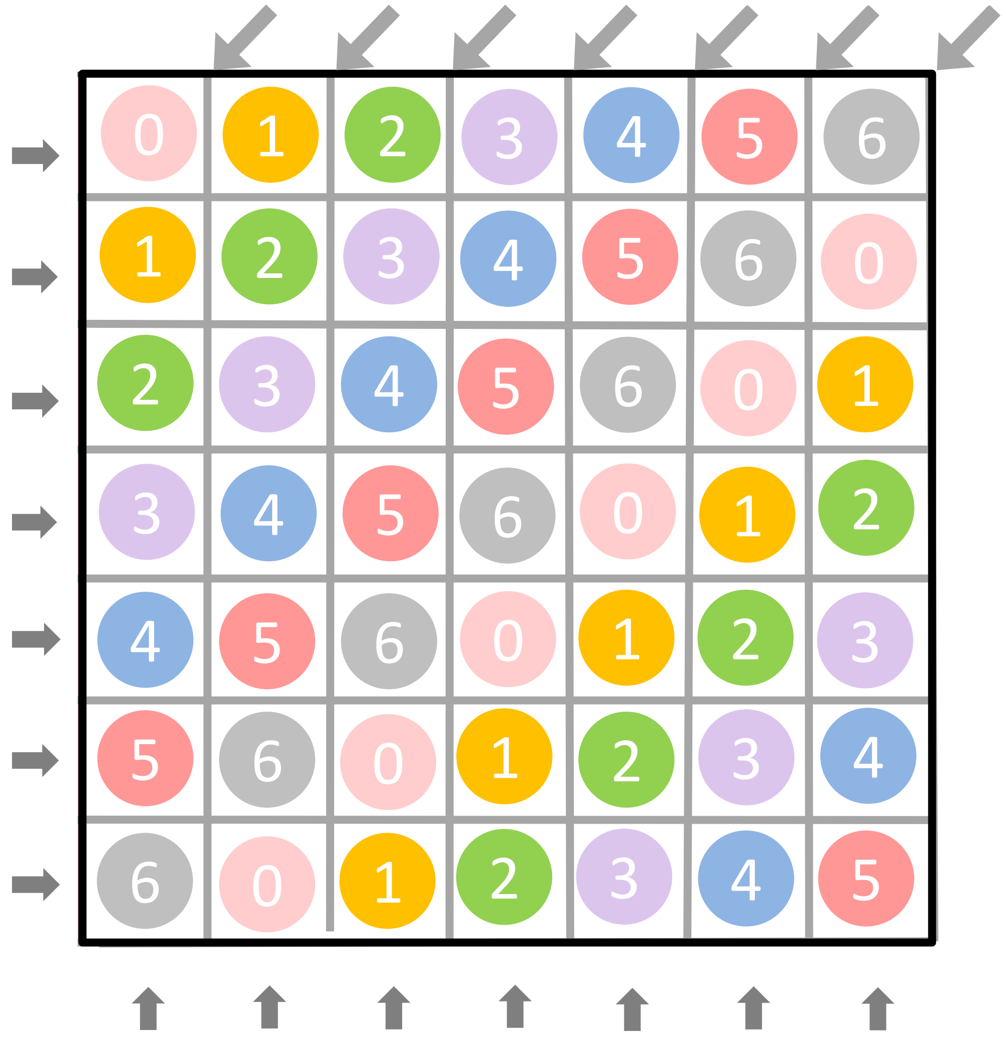
6	0	1	2	3	4	5



k<n numbers in Z_n
without repetition

③ Alon's Combinatorial Open Problem

Input



0	1	2	3	4	5	6
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2	3	4	5	6	0	1
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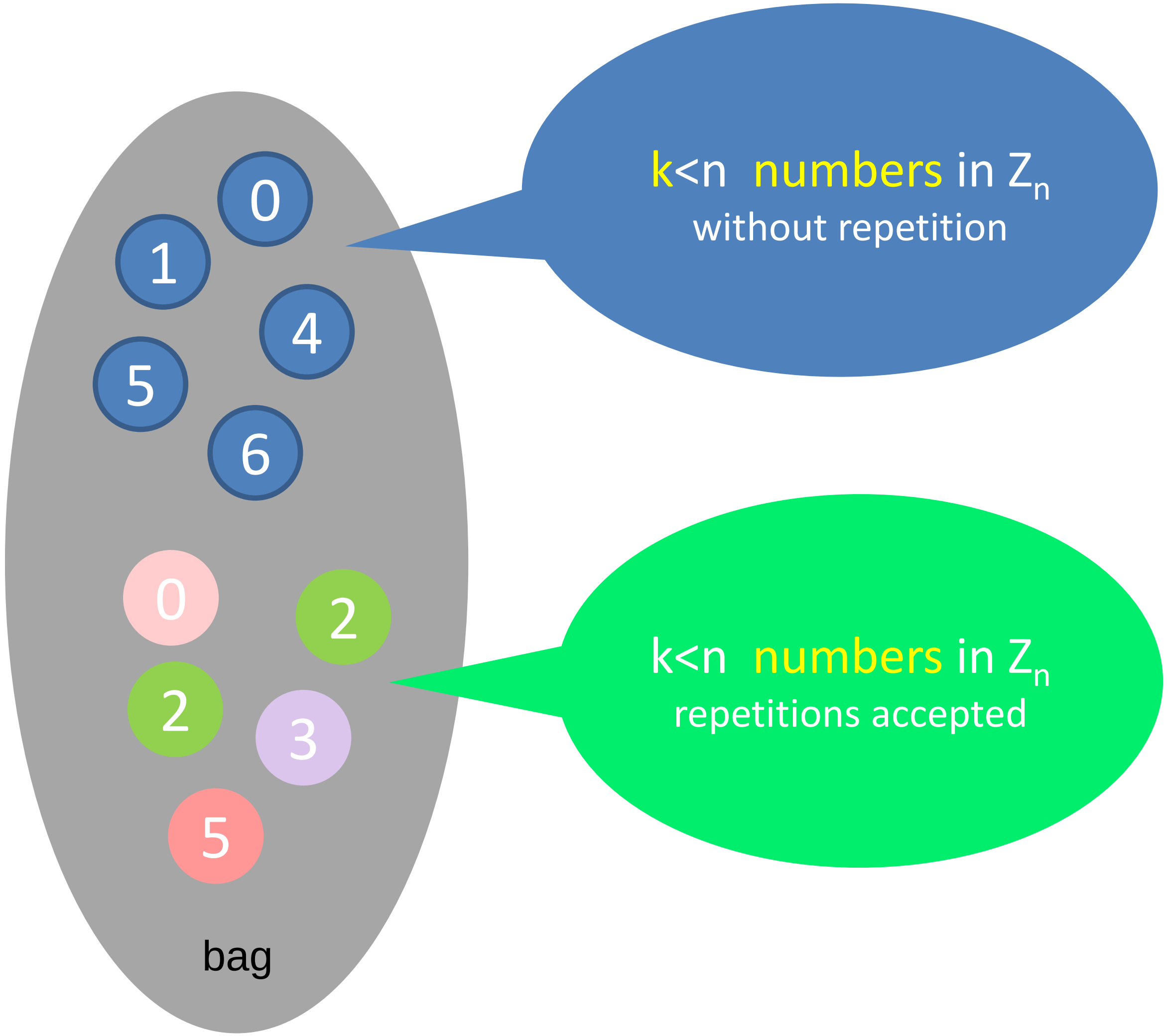
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

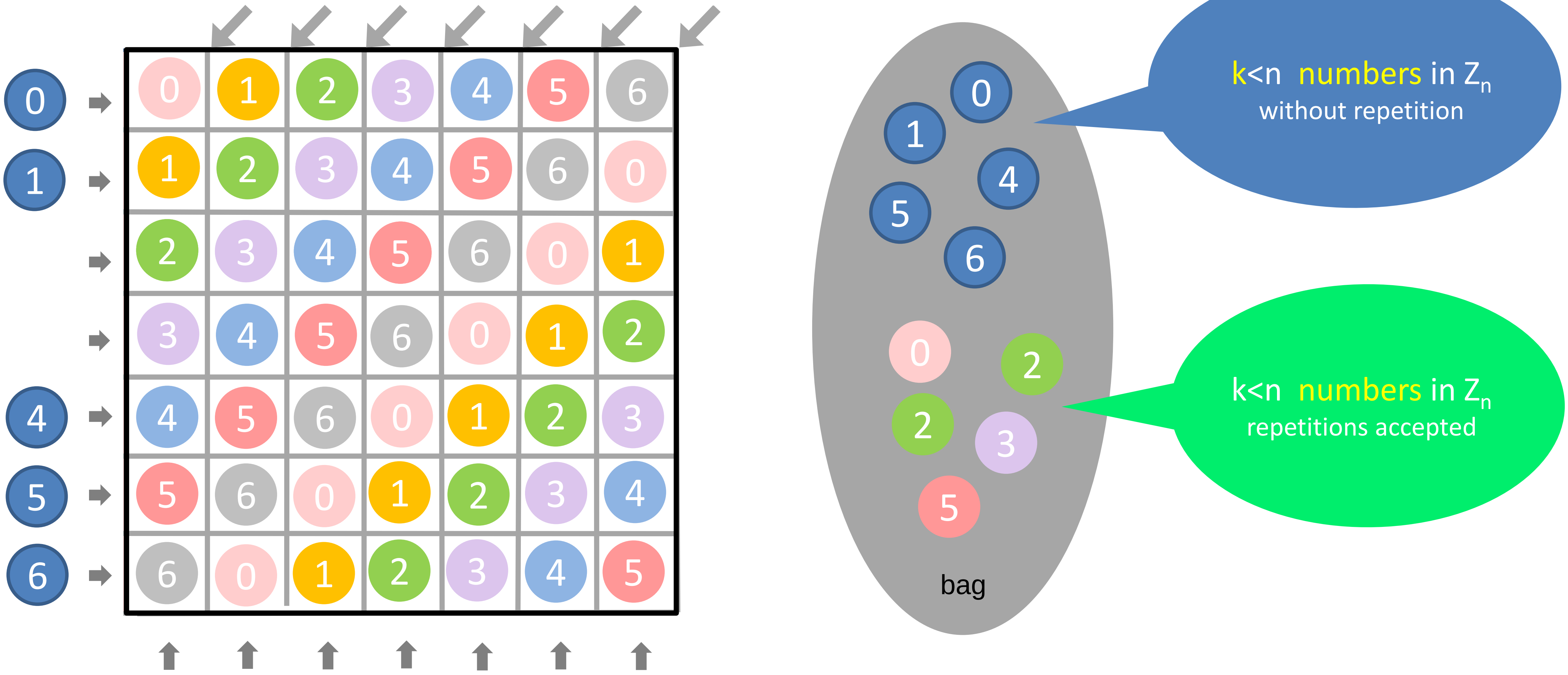


k<n numbers in Z_n
without repetition

k<n numbers in Z_n
repetitions accepted

③ Alon's Combinatorial Open Problem

Input



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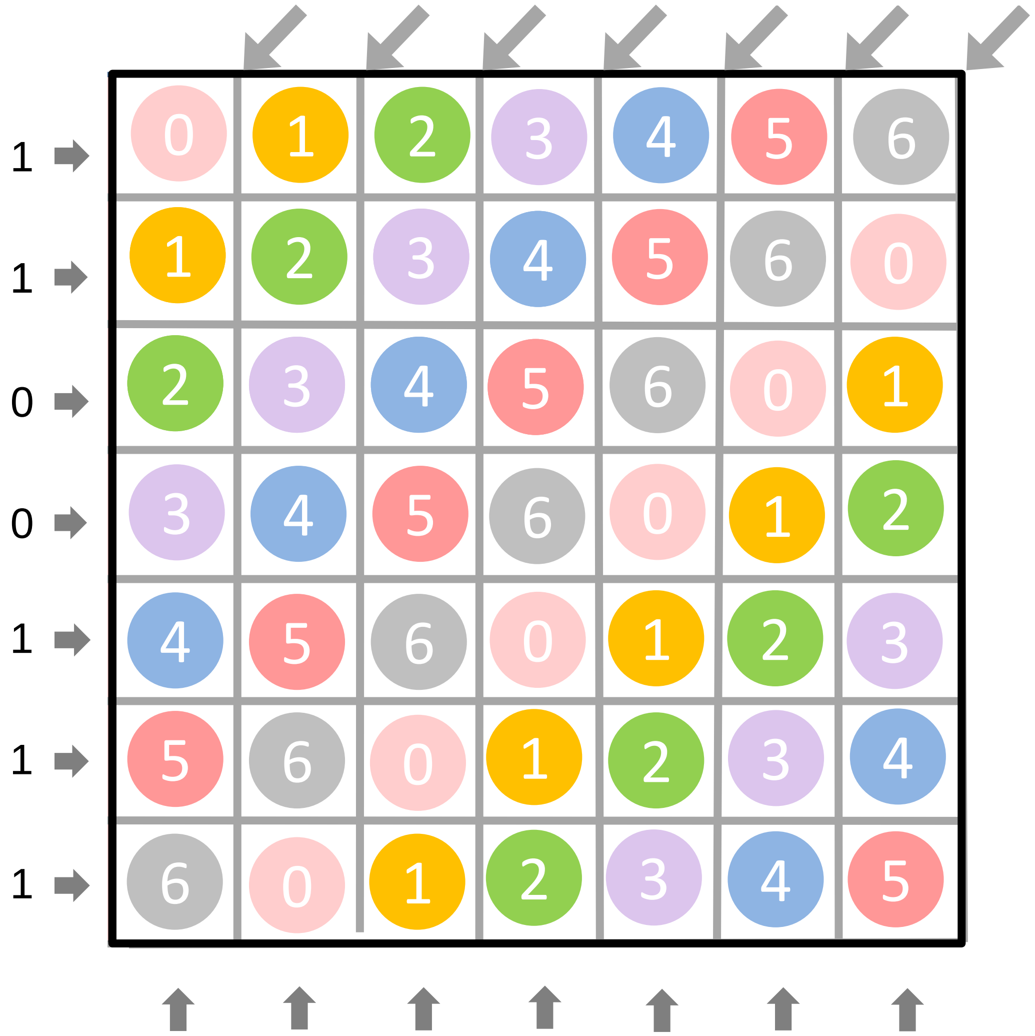
	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

k<n numbers in Z_n
without repetition

k<n numbers in Z_n
repetitions accepted

③ Alon's Combinatorial Open Problem

Input



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0	3	4	5	6	0	1	2
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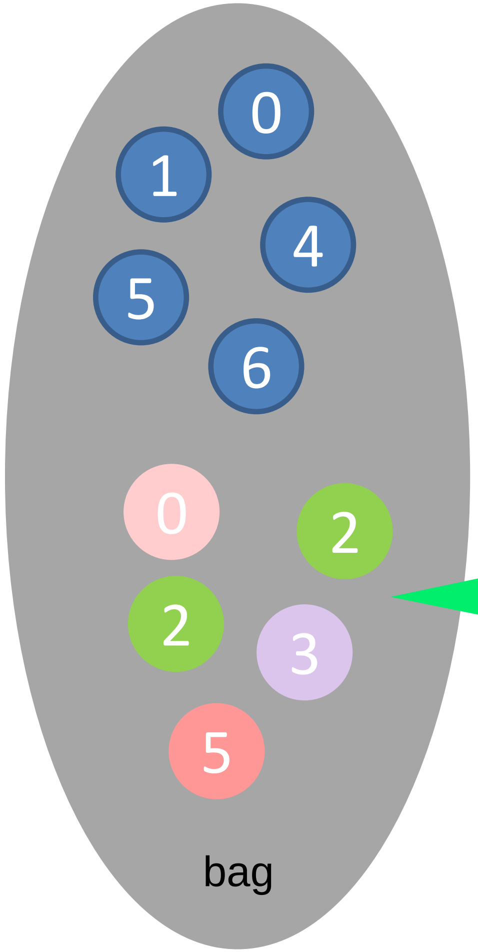
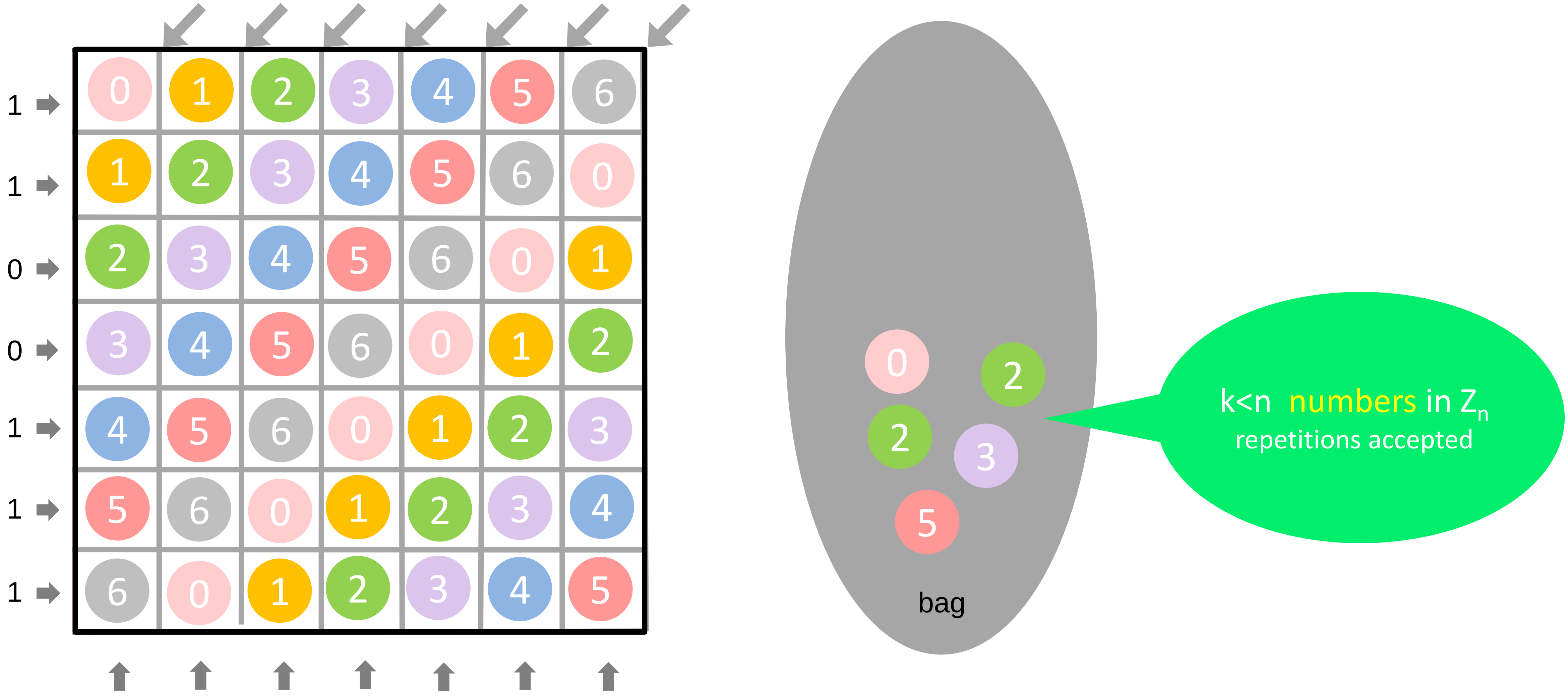
1	6	0	1	2	3	4	5



k<n numbers in Z_n
repetitions accepted

③ Alon’s Combinatorial Open Problem



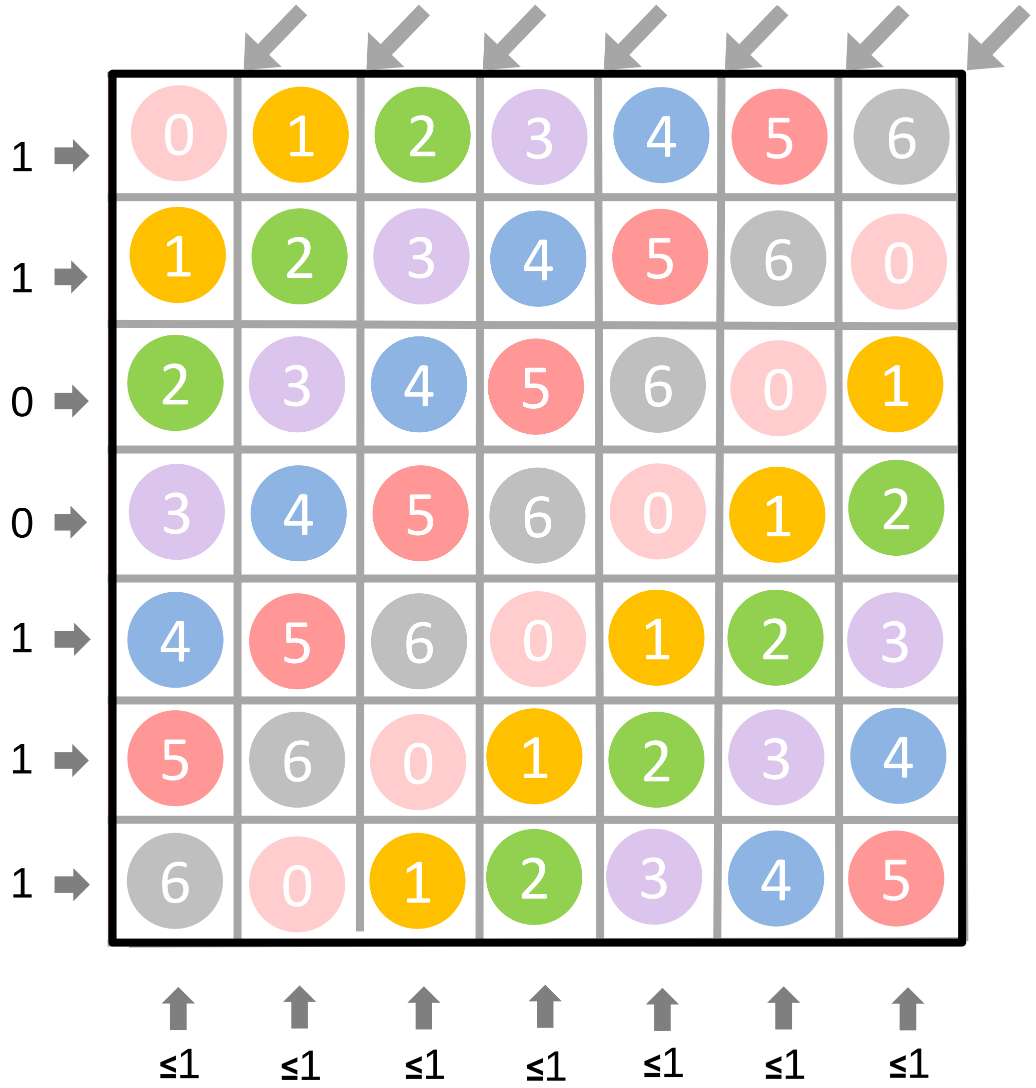
1	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
0	2	3	4	5	6	0	1
0	3	4	5	6	0	1	2
1	4	5	6	0	1	2	3
1	5	6	0	1	2	3	4
1	6	0	1	2	3	4	5

k<n numbers in Z_n

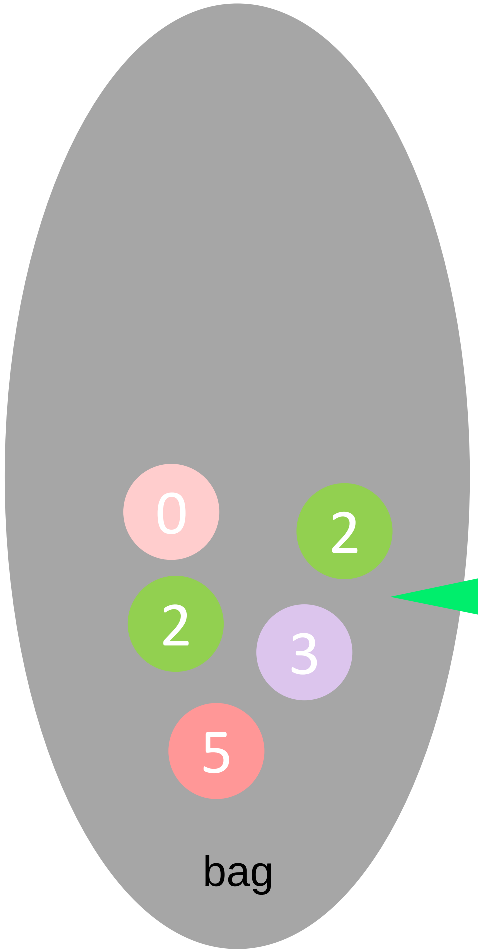
repetitions accepted

③ Alon’s Combinatorial Open Problem

Input



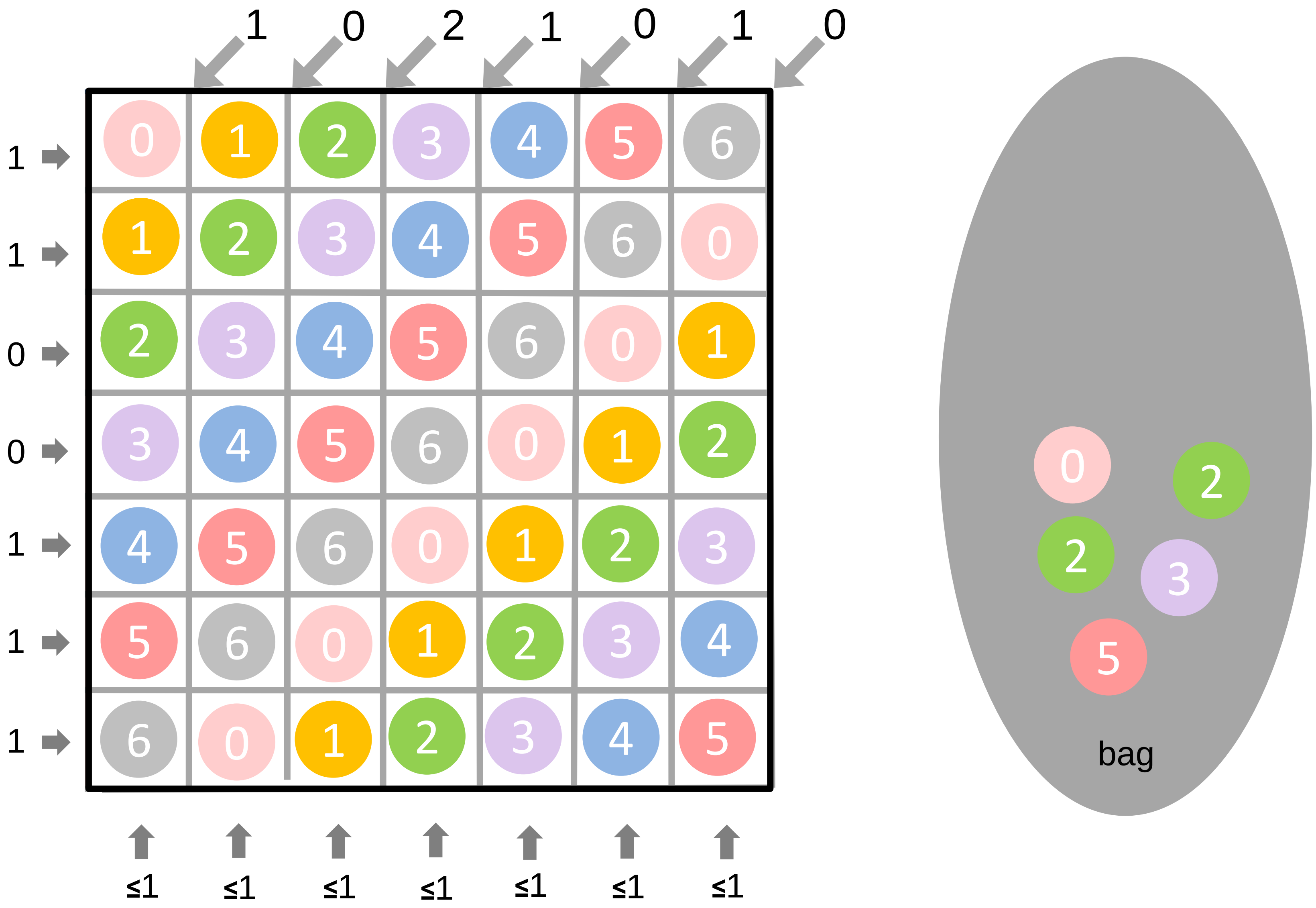
1	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
0	2	3	4	5	6	0	1
0	3	4	5	6	0	1	2
1	4	5	6	0	1	2	3
1	5	6	0	1	2	3	4
1	6	0	1	2	3	4	5
	≤1	≤1	≤1	≤1	≤1	≤1	≤1



k<n numbers in Z_n
repetitions accepted

③ Alon's Combinatorial Open Problem

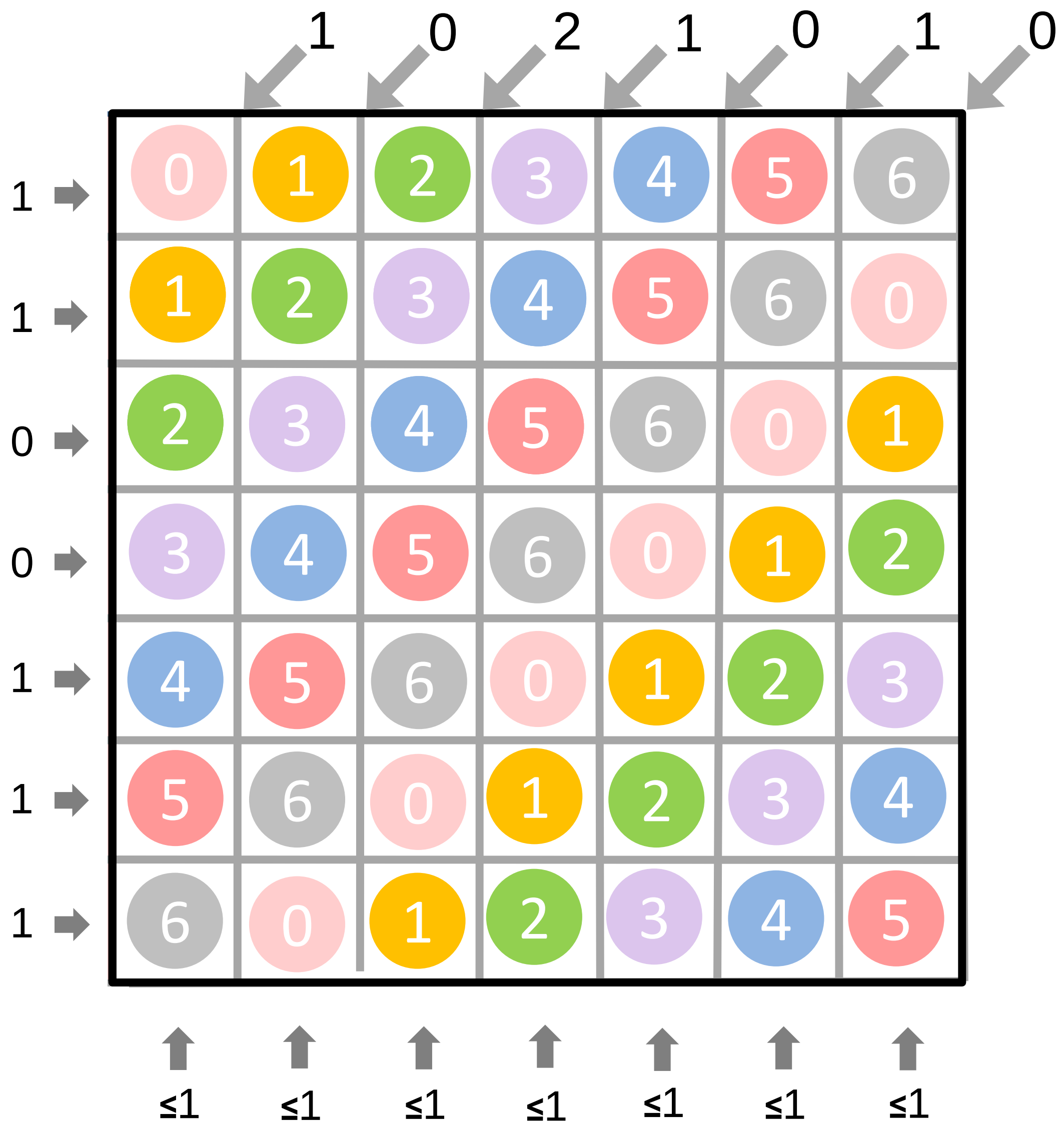
Input



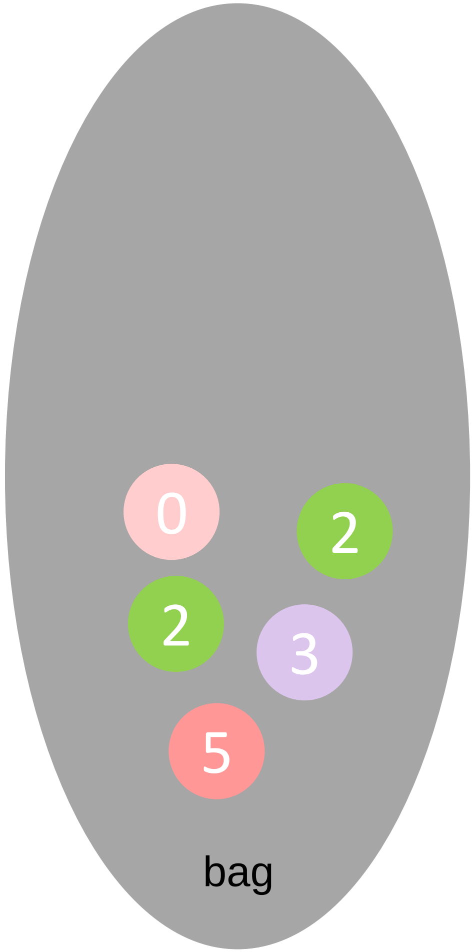
1 →	0	1	2	3	4	5	6
1 →	1	2	3	4	5	6	0
0 →	2	3	4	5	6	0	1
0 →	3	4	5	6	0	1	2
1 →	4	5	6	0	1	2	3
1 →	5	6	0	1	2	3	4
1 →	6	0	1	2	3	4	5
	↑ ≤1	↑ ≤1	↑ ≤1	↑ ≤1	↑ ≤1	↑ ≤1	↑ ≤1

③ Alon's Combinatorial Open Problem

2 dollars question

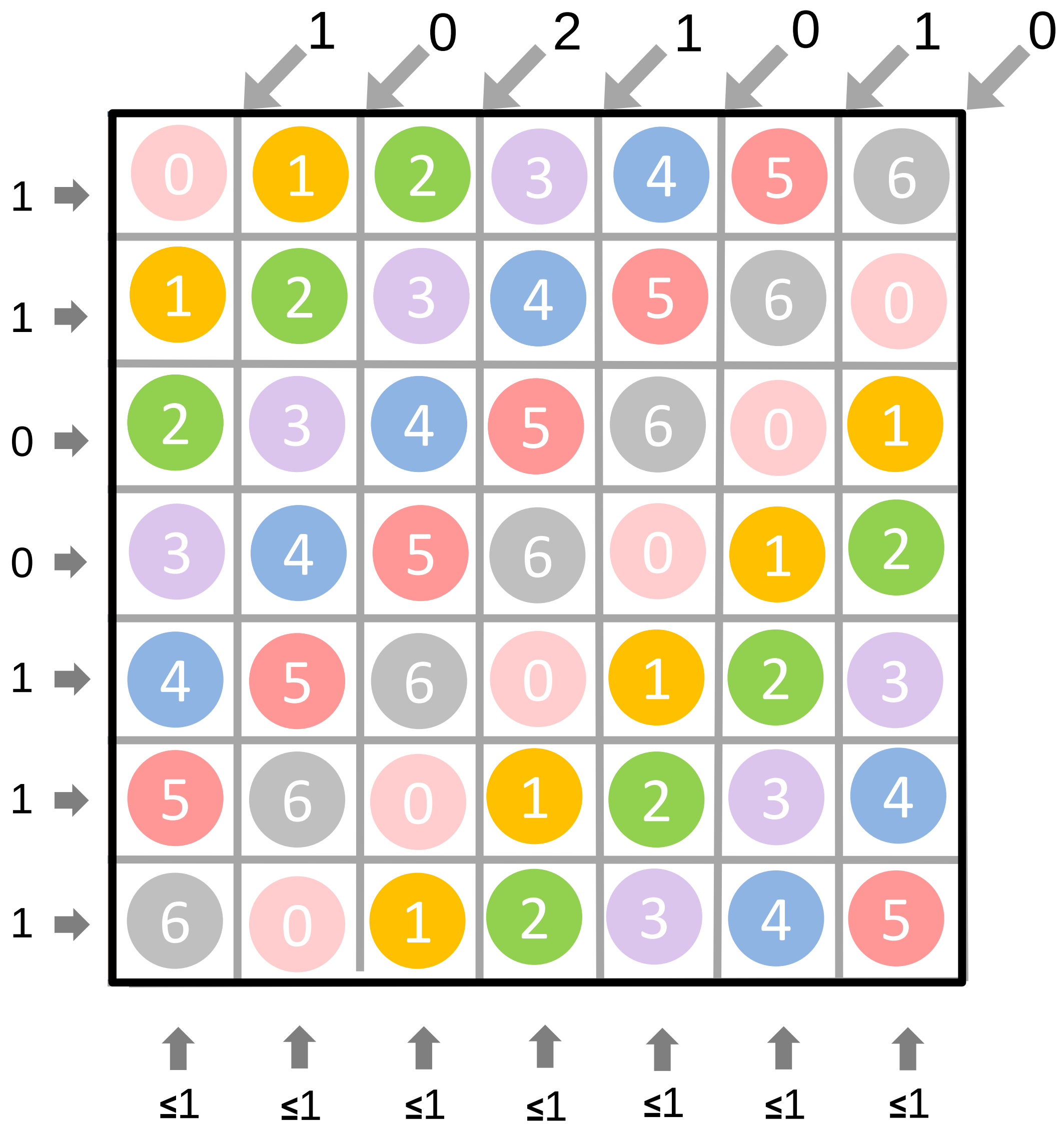


	1	0	2	1	0	1	0
1	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
0	2	3	4	5	6	0	1
0	3	4	5	6	0	1	2
1	4	5	6	0	1	2	3
1	5	6	0	1	2	3	4
1	6	0	1	2	3	4	5
	≤1	≤1	≤1	≤1	≤1	≤1	≤1



③ Alon’s Combinatorial Open Problem

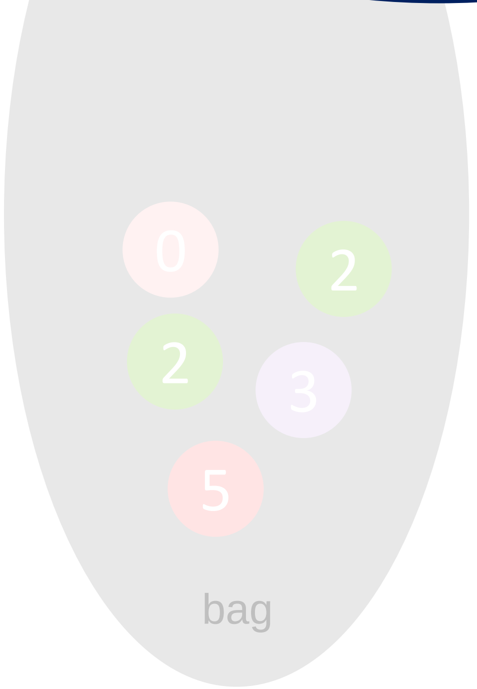
2 dollars question



1 0 2 1 0 1 0

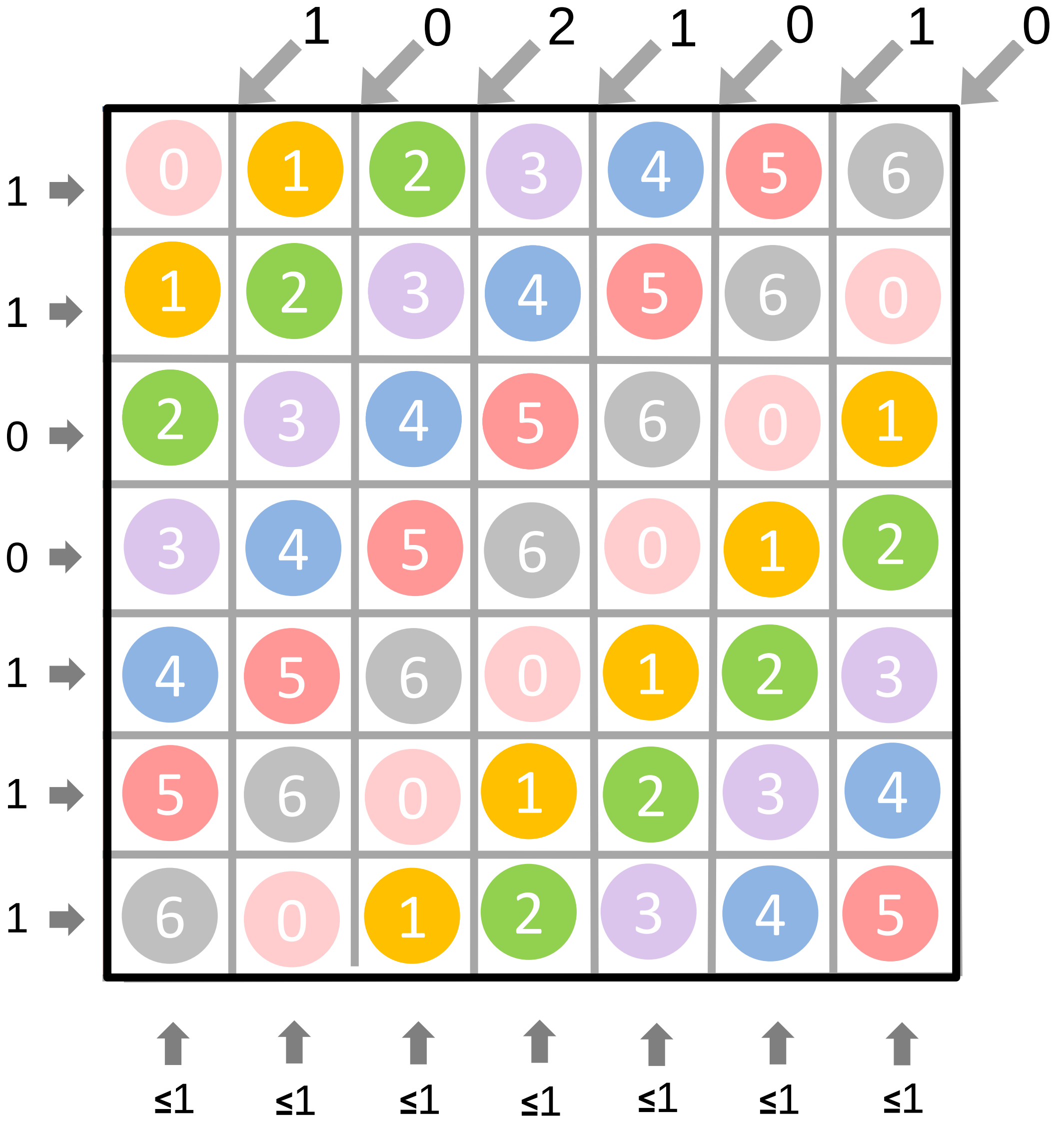
1	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
0	2	3	4	5	6	0	1
0	3	4	5	6	0	1	2
1	4	5	6	0	1	2	3
1	5	6	0	1	2	3	4
1	6	0	1	2	3	4	5
	≤1	≤1	≤1	≤1	≤1	≤1	≤1

Does there exist solutions ?

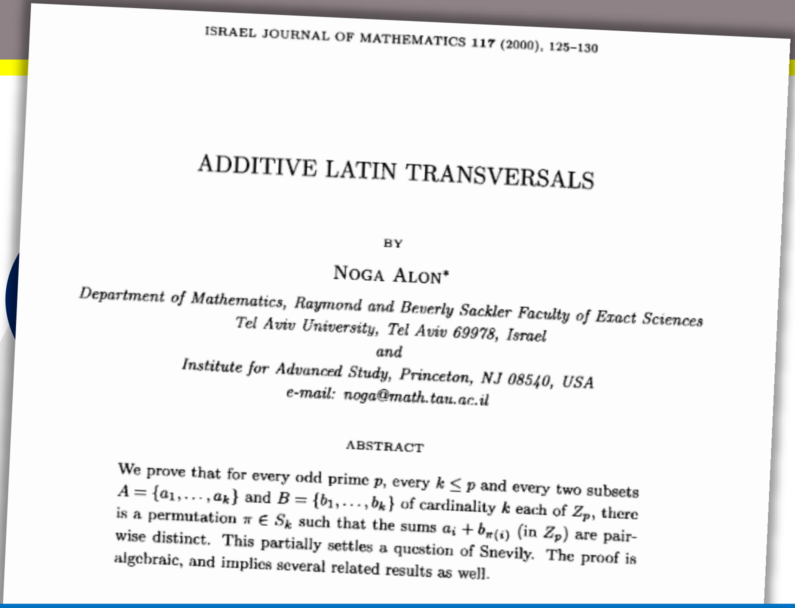


Alon's Combinatorial Open Problem

Again



	1	0	2	1	0	1	0
1	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
0	2	3	4	5	6	0	1
0	3	4	5	6	0	1	2
1	4	5	6	0	1	2	3
1	5	6	0	1	2	3	4
1	6	0	1	2	3	4	5
	≤1	≤1	≤1	≤1	≤1	≤1	≤1



ISRAEL JOURNAL OF MATHEMATICS 117 (2000), 125–130

ADDITIVE LATIN TRANSVERSALS

BY

NOGA ALON*

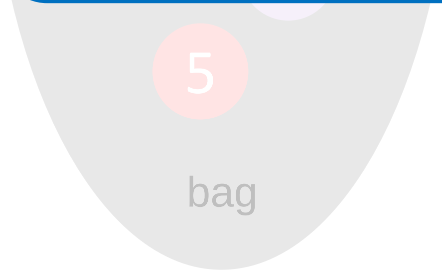
Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv 69978, Israel
and
Institute for Advanced Study, Princeton, NJ 08540, USA
e-mail: noga@math.tau.ac.il

ABSTRACT

We prove that for every odd prime p, every $k \leq p$ and every two subsets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ of cardinality k each of Z_p, there is a permutation $\pi \in S_k$ such that the sums $a_i + b_{\pi(i)}$ (in Z_p) are pairwise distinct. This partially settles a question of Snevily. The proof is algebraic, and implies several related results as well.

Alon's Theorem (2000)

Solutions always exist!

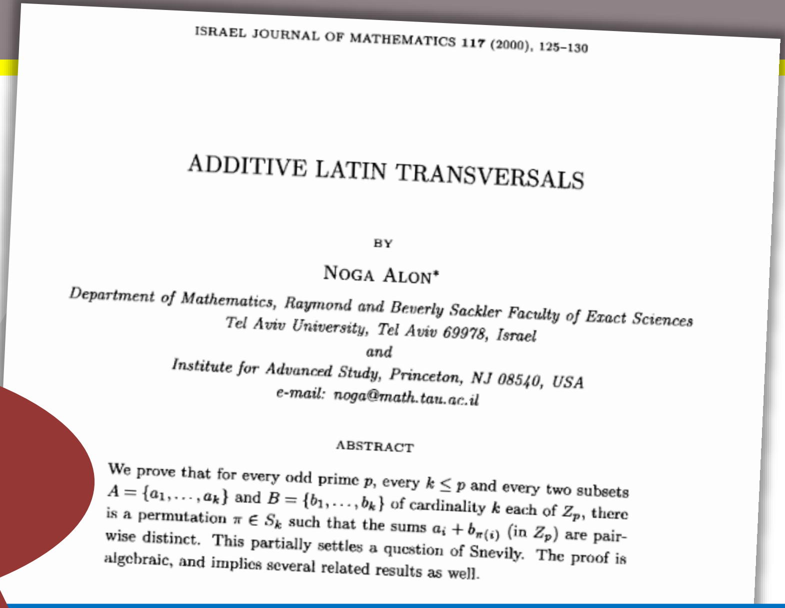


③ Alon's Combinatorial Open Problem

Again

	1	0	2	1	0	1	0
1	0	1	2	3	4	5	6
1	1	2	3	4	5		
0	2	3	4	5	6		
0	3	4	5	6	0	1	2
1	4	5	6	0	1	2	3
1	5	6	0	1	2	3	4
1	6	0	1	2	3	4	5
	≤1	≤1	≤1	≤1	≤1	≤1	≤1

Proved by Nullstellensatz



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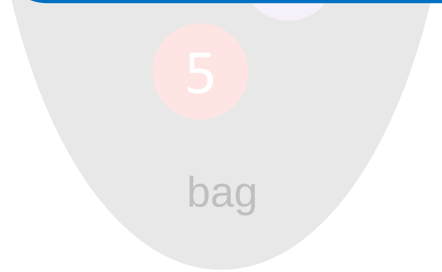
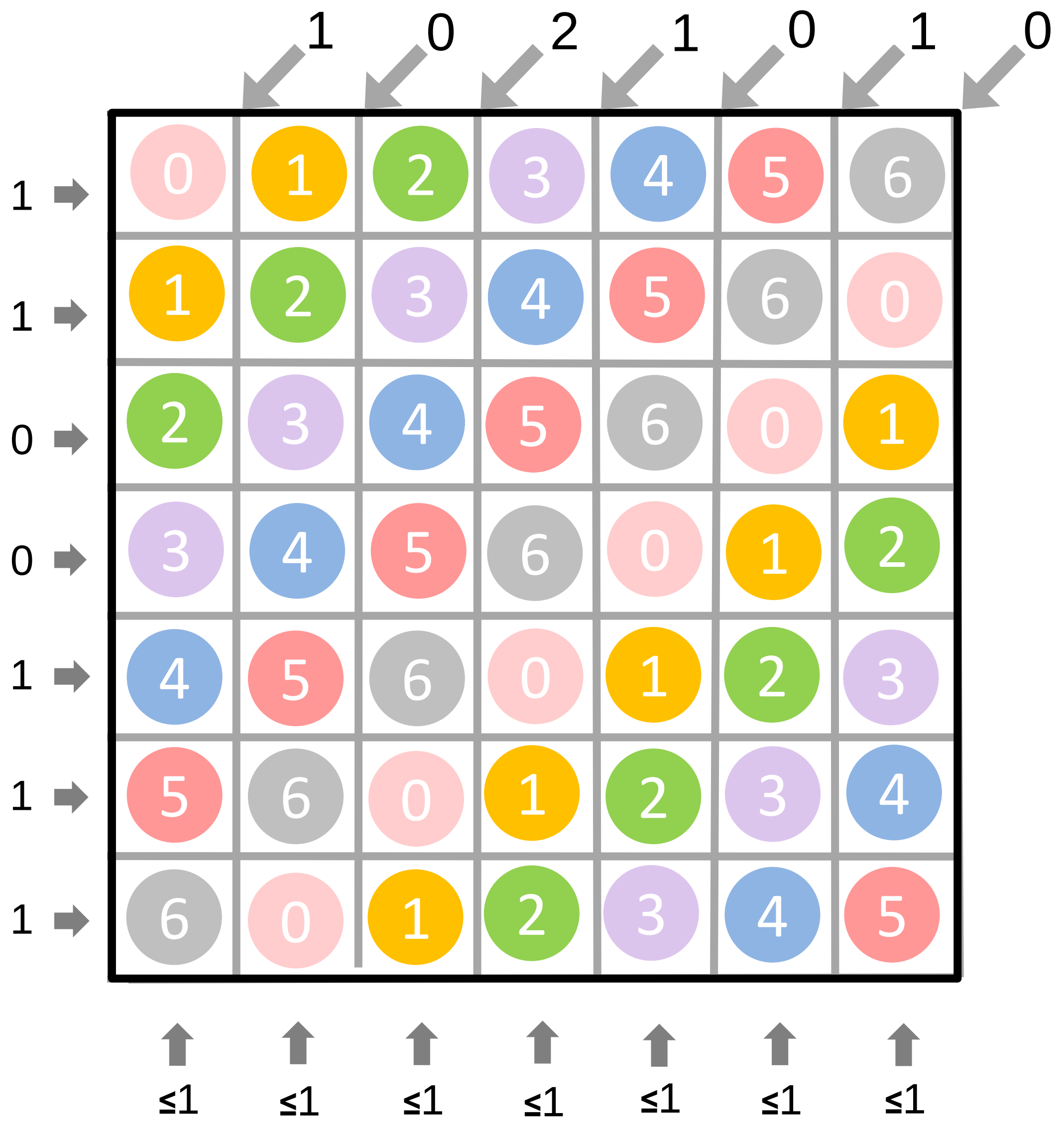


③ Alon’s Combinatorial Open Problem



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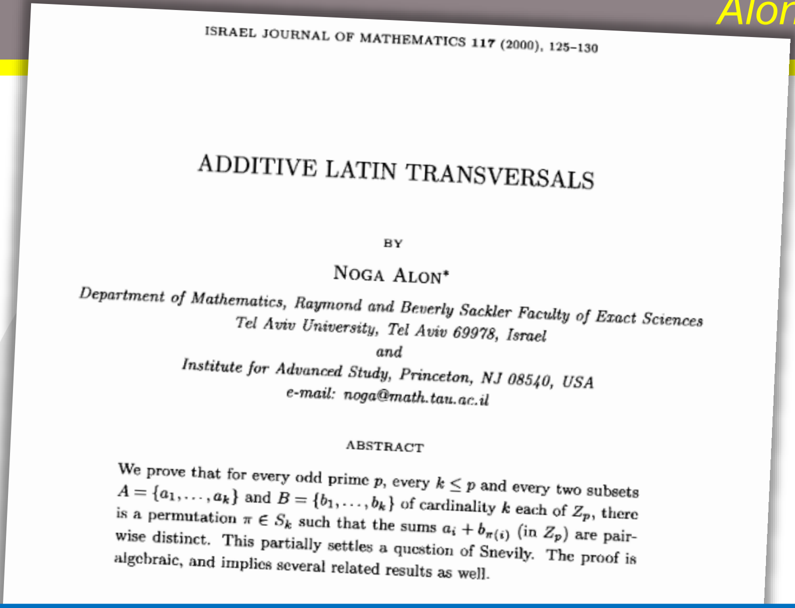
0	2	3	4	5	6	0	1
0	3	4	5	6	0	1	2
1	4	5	6	0	1	2	3
1	5	6	0	1	2	3	4
1	6	0	1	2	3	4	5
	≤1	≤1	≤1	≤1	≤1	≤1	≤1



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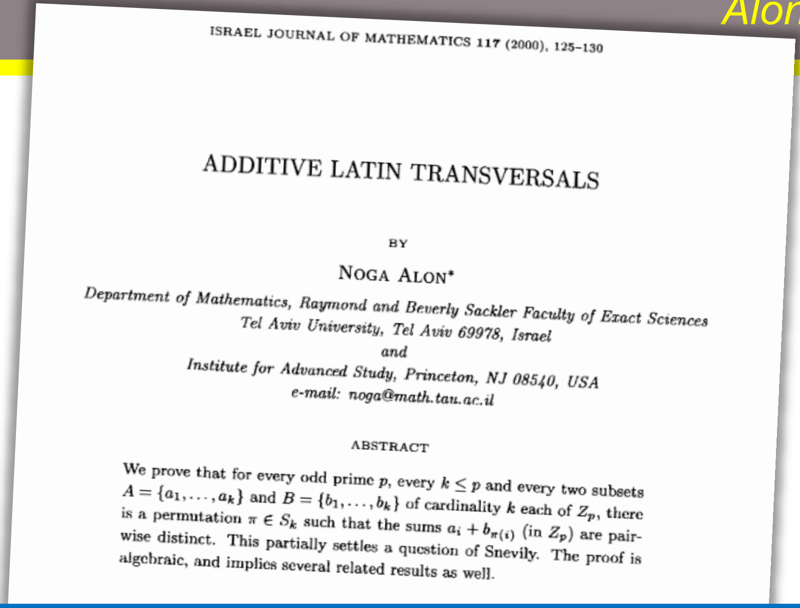
ABSTRACT

We prove that for every odd prime p, every $k \leq p$ and every two subsets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ of cardinality k each of Z_p, there is a permutation $\pi \in S_k$ such that the sums $a_i + b_{\pi(i)}$ (in Z_p) are pairwise distinct. This partially settles a question of Snevily. The proof is algebraic, and implies several related results as well.

Alon’s Theorem (1999)
Solutions always exist!

No polynomial time algorithm is known…

③ Alon's Combinatorial Open Problem



ISRAEL JOURNAL OF MATHEMATICS **117** (2000), 125–130

ADDITIVE LATIN TRANSVERSALS

BY

NOGA ALON*
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Tel Aviv University, Tel Aviv 69978, Israel
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e-mail: noga@math.tau.ac.il

ABSTRACT

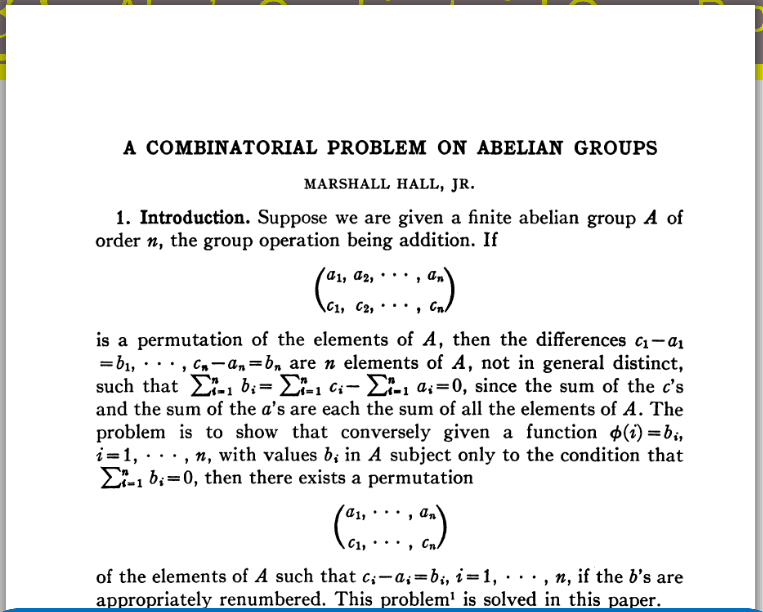
We prove that for every odd prime p, every $k \le p$ and every two subsets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ of cardinality k each of Z_p, there is a permutation $\pi \in S_k$ such that the sums $a_i + b_{\pi(i)}$ (in Z_p) are pairwise distinct. This partially settles a question of Snevily. The proof is algebraic, and implies several related results as well.

Alon's Theorem (1999)
Solutions always exist!

Find one…

No polynomial time
algorithm is known…

③ oblem



A COMBINATORIAL PROBLEM ON ABELIAN GROUPS

MARSHALL HALL, JR.

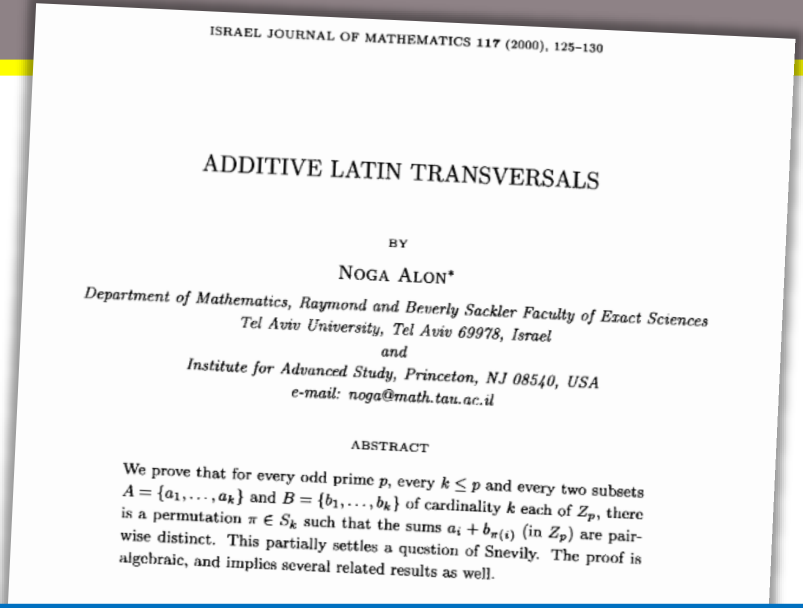
1. Introduction. Suppose we are given a finite abelian group A of order n, the group operation being addition. If

$$\begin{pmatrix} a_1, & a_2, & \cdots, & a_n \\ c_1, & c_2, & \cdots, & c_n \end{pmatrix}$$

is a permutation of the elements of A, then the differences $c_1-a_1=b_1, \cdots, c_n-a_n=b_n$ are n elements of A, not in general distinct, such that $\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} c_i - \sum_{i=1}^{n} a_i = 0$, since the sum of the c's and the sum of the a's are each the sum of all the elements of A. The problem is to show that conversely given a function $\phi(i)=b_i$, $i=1, \cdots, n$, with values b_i in A subject only to the condition that $\sum_{i=1}^{n} b_i=0$, then there exists a permutation

$$\begin{pmatrix} a_1, & \cdots, & a_n \\ c_1, & \cdots, & c_n \end{pmatrix}$$

of the elements of A such that $c_i-a_i=b_i$, $i=1, \cdots, n$, if the b's are appropriately renumbered. This problem[1] is solved in this paper.



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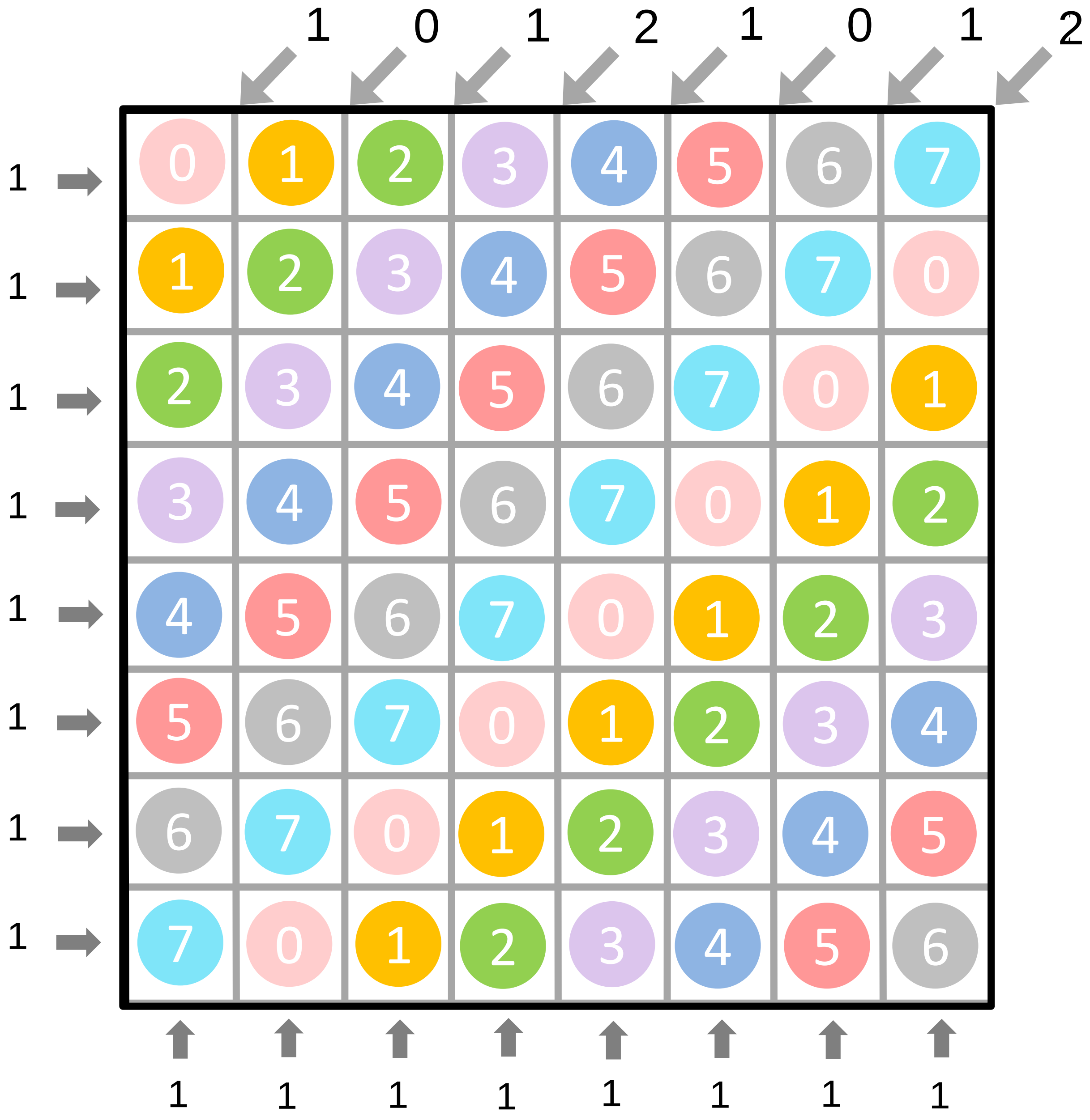
Hall's Theorem (1952)
Solutions always exist!

Alon's Theorem (1999)
Solutions always exist!

There is a
polynomial time algorithm

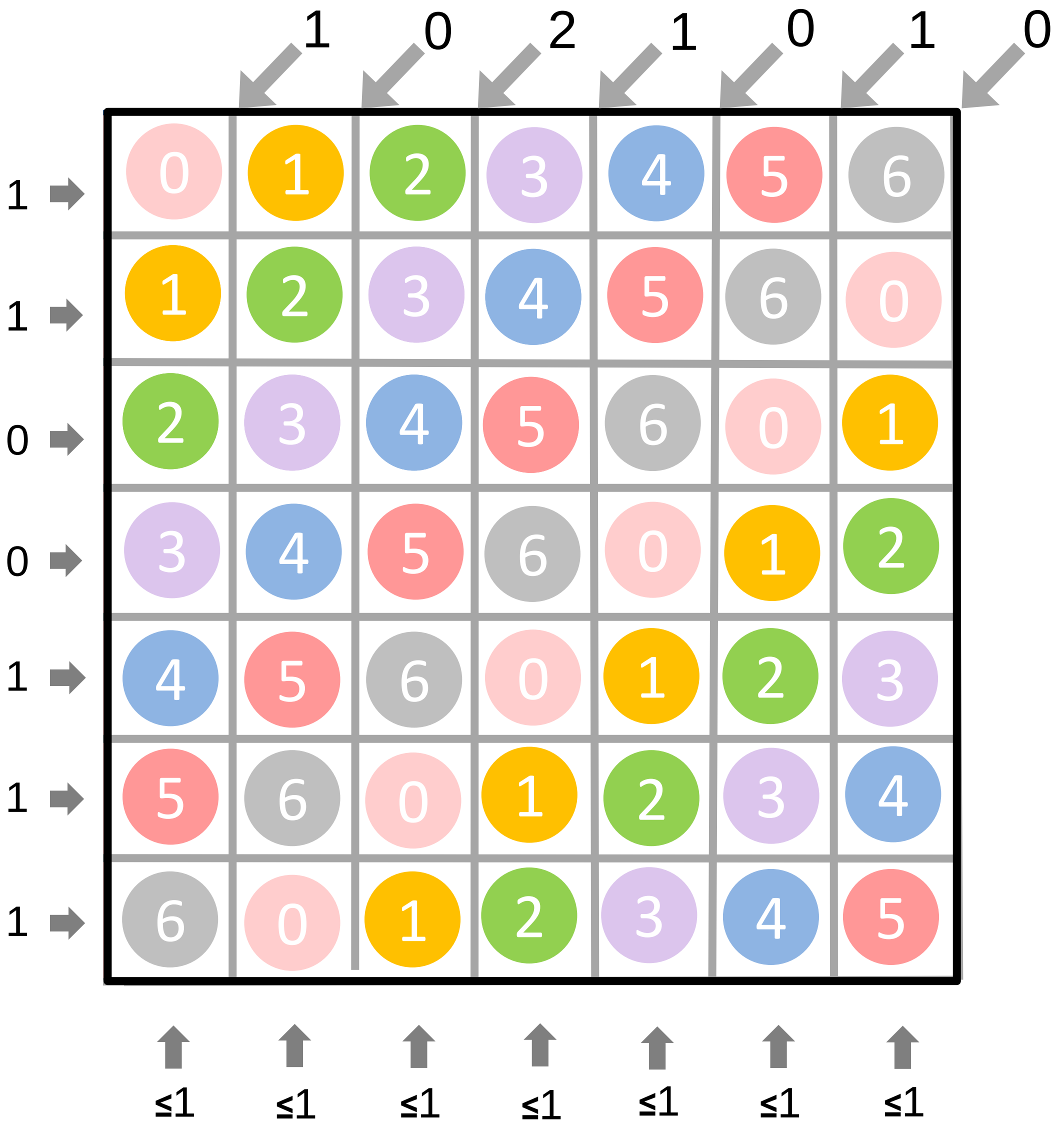
Find one…

No polynomial time
algorithm is known…



0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

A polynomial time algorithm is known



0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

A polynomial time algorithm is unknown

Conclusion

Peter Schwander, physicist
at AT&T Bell labs
(in the 90s)

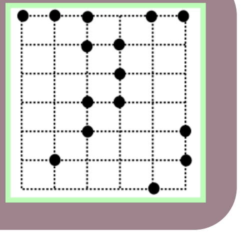
Question

Larry Shepp, CT expert,
AT&T Bell labs
(in the 90s)

Discrete Tomography deals with the reconstruction
of a binary function
on a discrete domain

$f : \mathit{Lattice} \rightarrow \{0 , 1\}$

namely a lattice set



Conclusion

End ?

Three-dimensional atomic imaging of crystalline nanoparticles,
Sandra Van Aert, Kees J. Batenburg et al… in Nature 2011

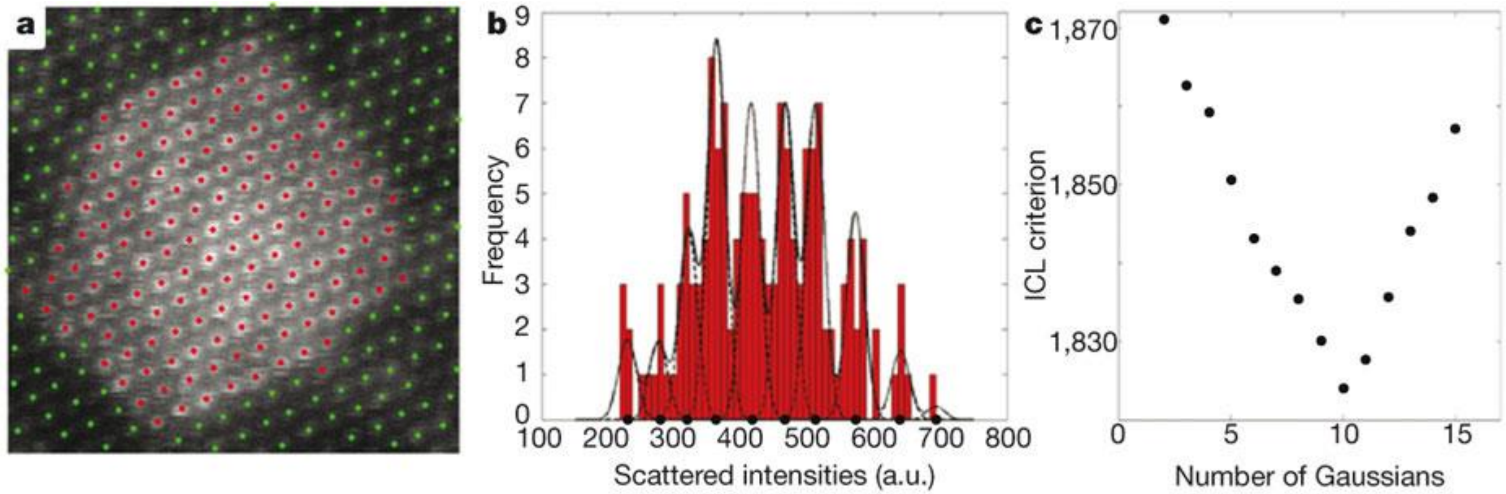


Image of an Ag nanoparticle by Electron microscopy.

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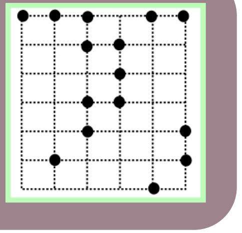

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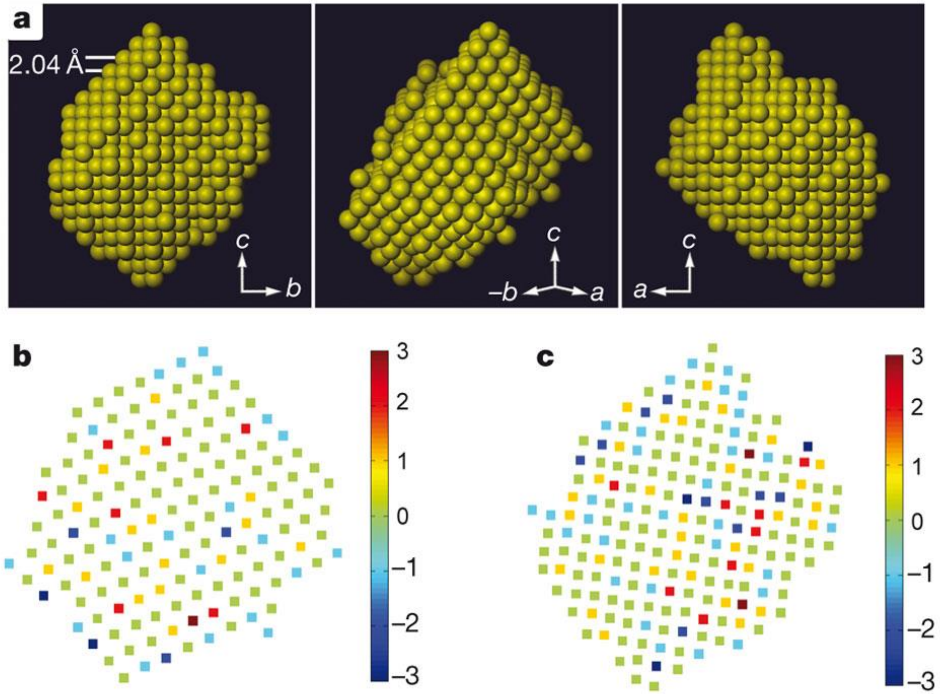

3D reconstruction of an Ag nanoparticle by Discrete Tomography

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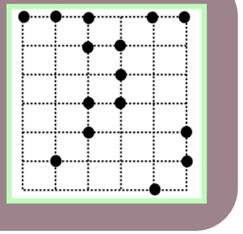

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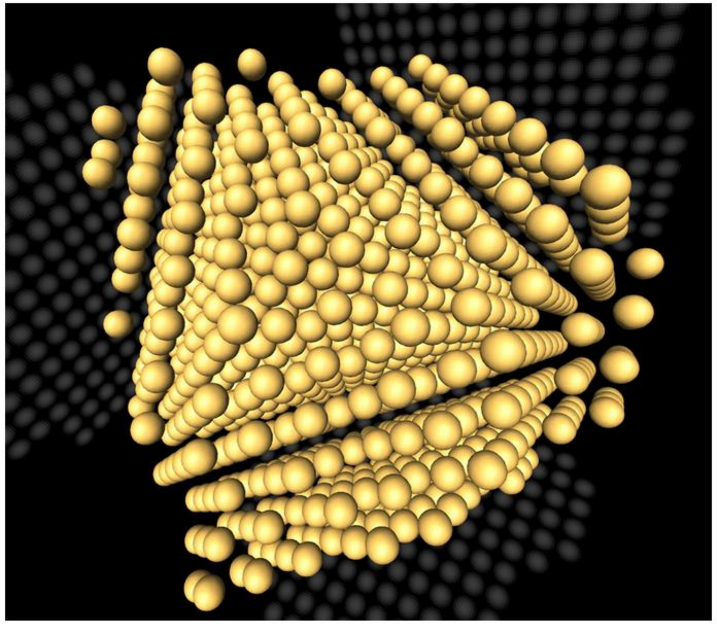
Three-dimensional atomic imaging of crystalline nanoparticles,
Sandra Van Aert, Kees J. Batenburg et al… in Nature 2011



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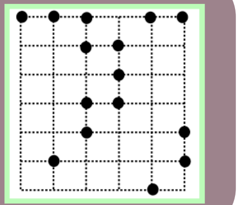

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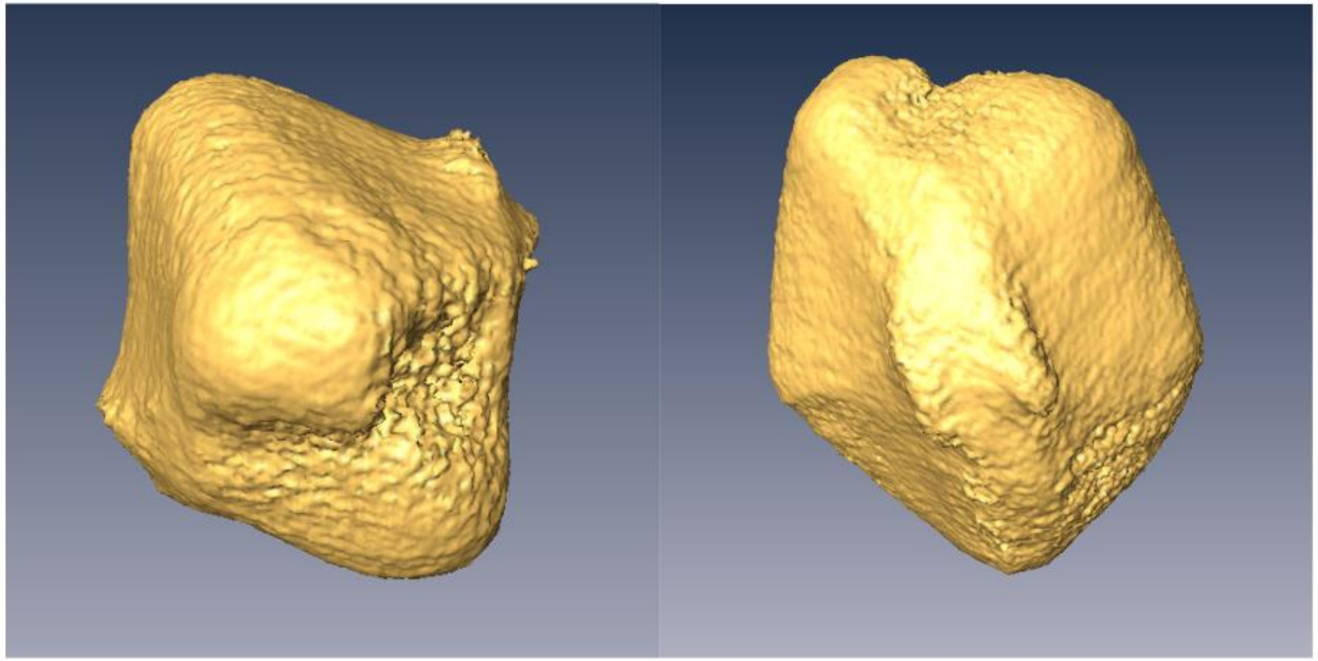
3D Imaging of Nanomaterials by Discrete Tomography,
Kees J. Batenburg et al… in Ultramicroscopy 2009



3D reconstruction of a gold nanoparticle.

Discrete Tomography deals with the reconstruction
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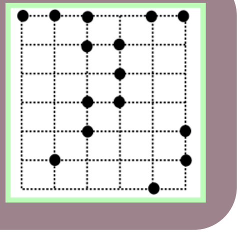
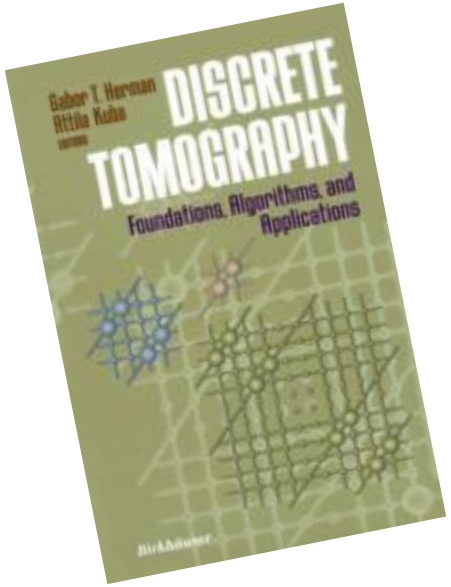
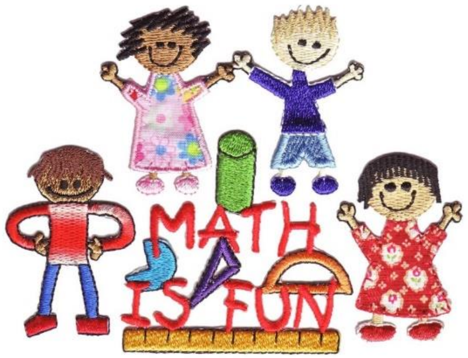
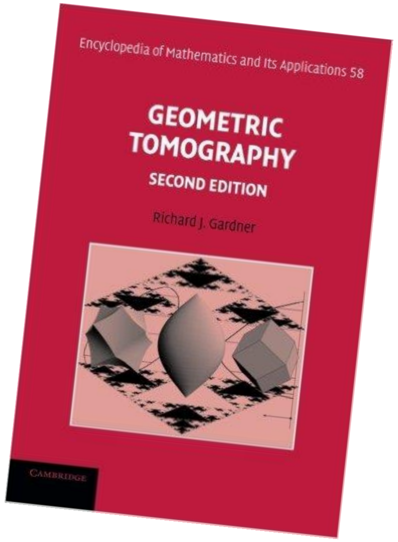


Books



A. Kuba and G. Herman’s book.





Richard Gardner’s book.