# Locally Turn Bounded Curves and Their Application to Topology Preservation of Shapes

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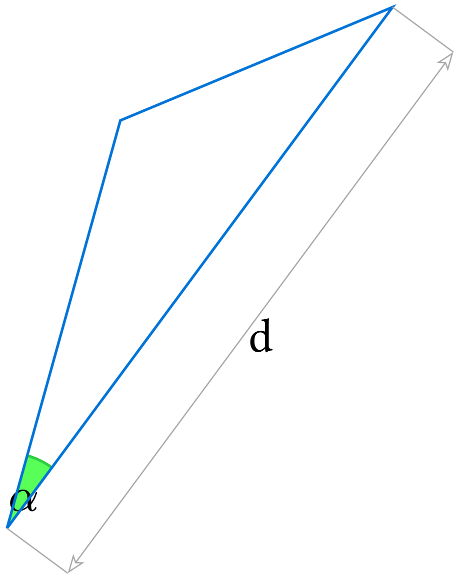
# Digital Geometry

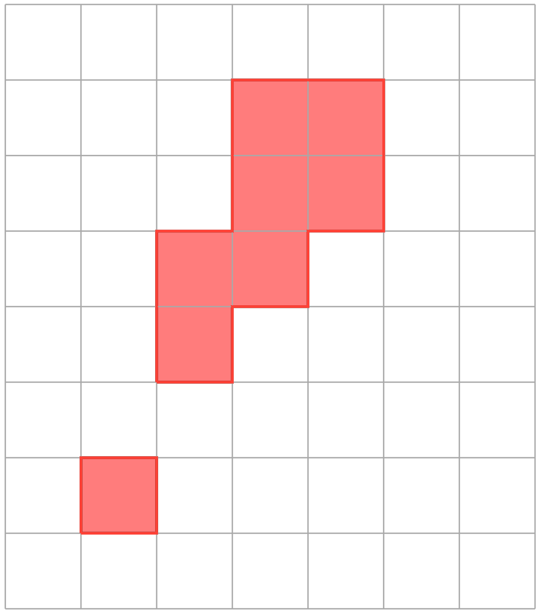
Digital Geometry: Geometry on discrete grid

(pixel/voxel representation of a shape)

- acquisition on numerical devices
- representation efficient for several algorithms
- integer based computations

# Digitization

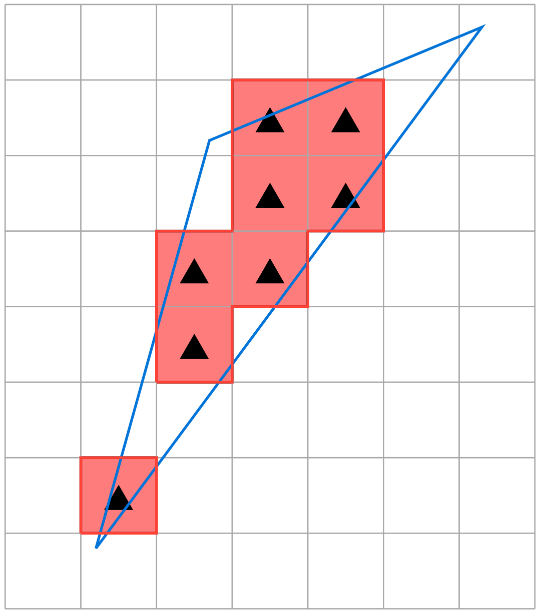


Length ?

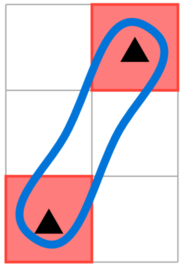
Topology ?

# Gauss Digitization

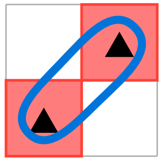


Gauss digitization of $S$: $S \cap h\mathbb{Z}^2$

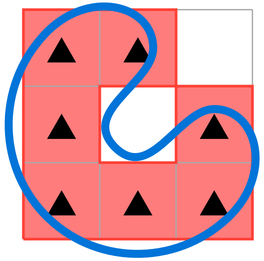
# Digital Topology



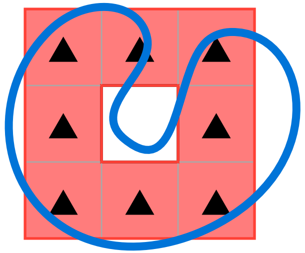
Not 8-connected



Not 4-connected

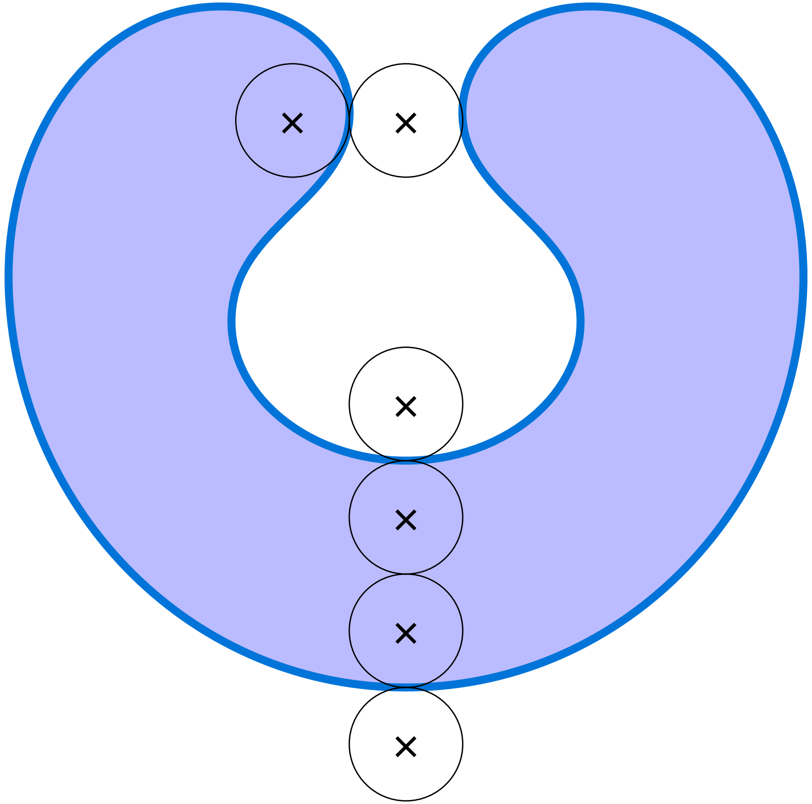


Not well-composed



Not simply connected

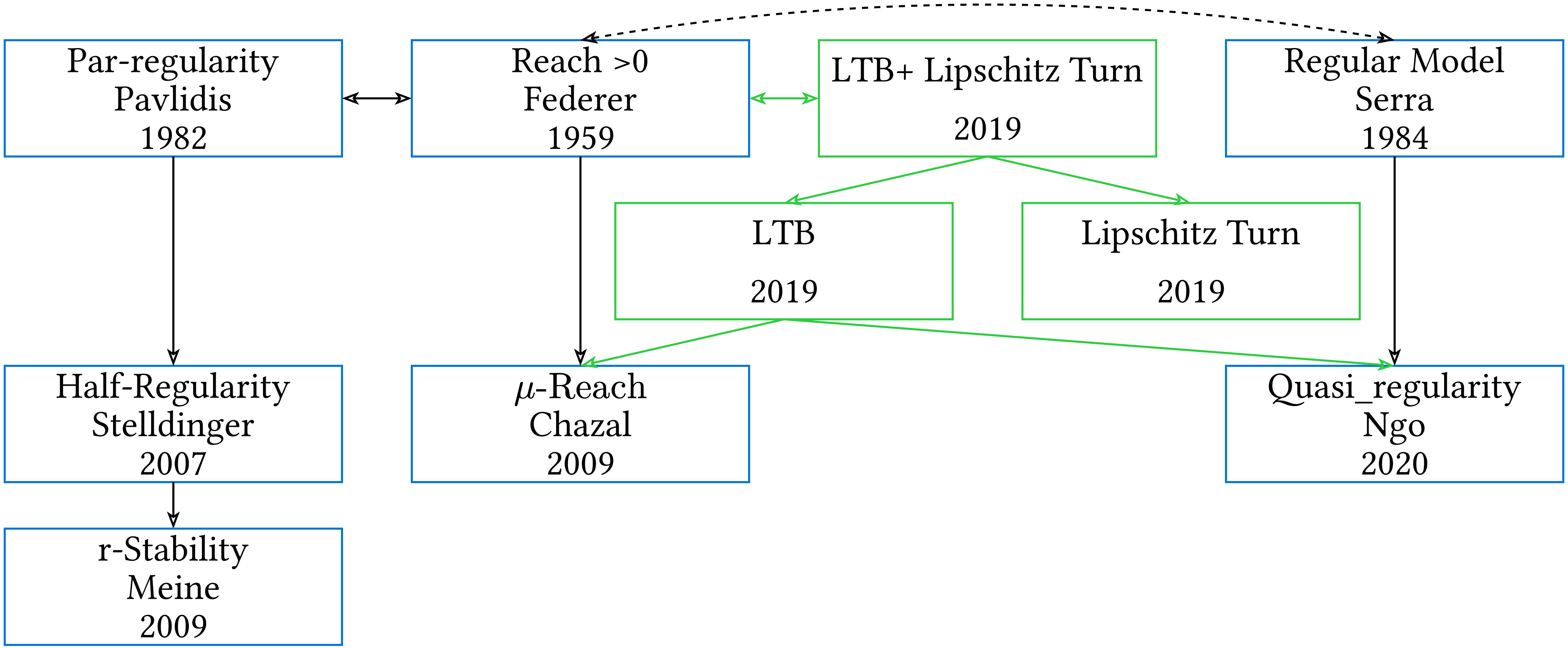
# Par-regularity *T. Pavlidis*



# Results on Par-Regular Shapes

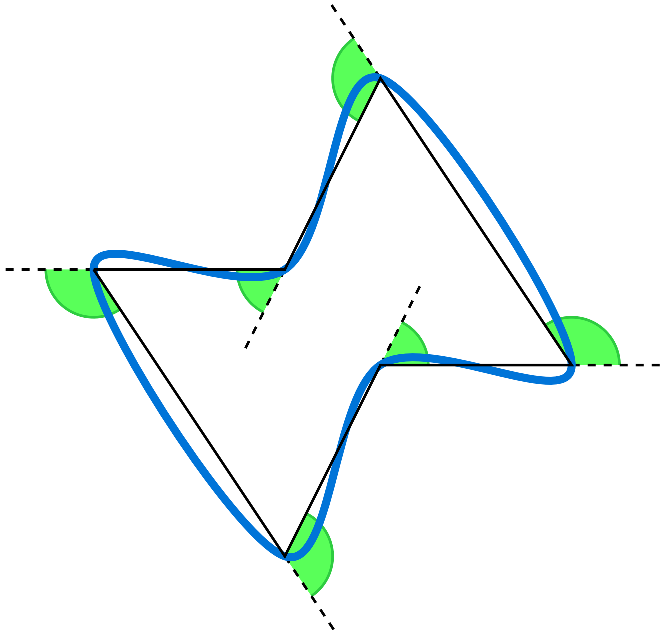
- The Gauss/surfacic/intersection Digitization of a par($r$)-regular shape is well-composed on a grid with a step $h < \sqrt{2}r$. *L. Latecki, U. Eckhardt, and A. Rosenfeld*
- Multigrid convergence of integral estimator based on a normal estimation. *J.-O. Lachaud and B. Thibert*

# Generalizations of Par-regularity



# Turn of a curve

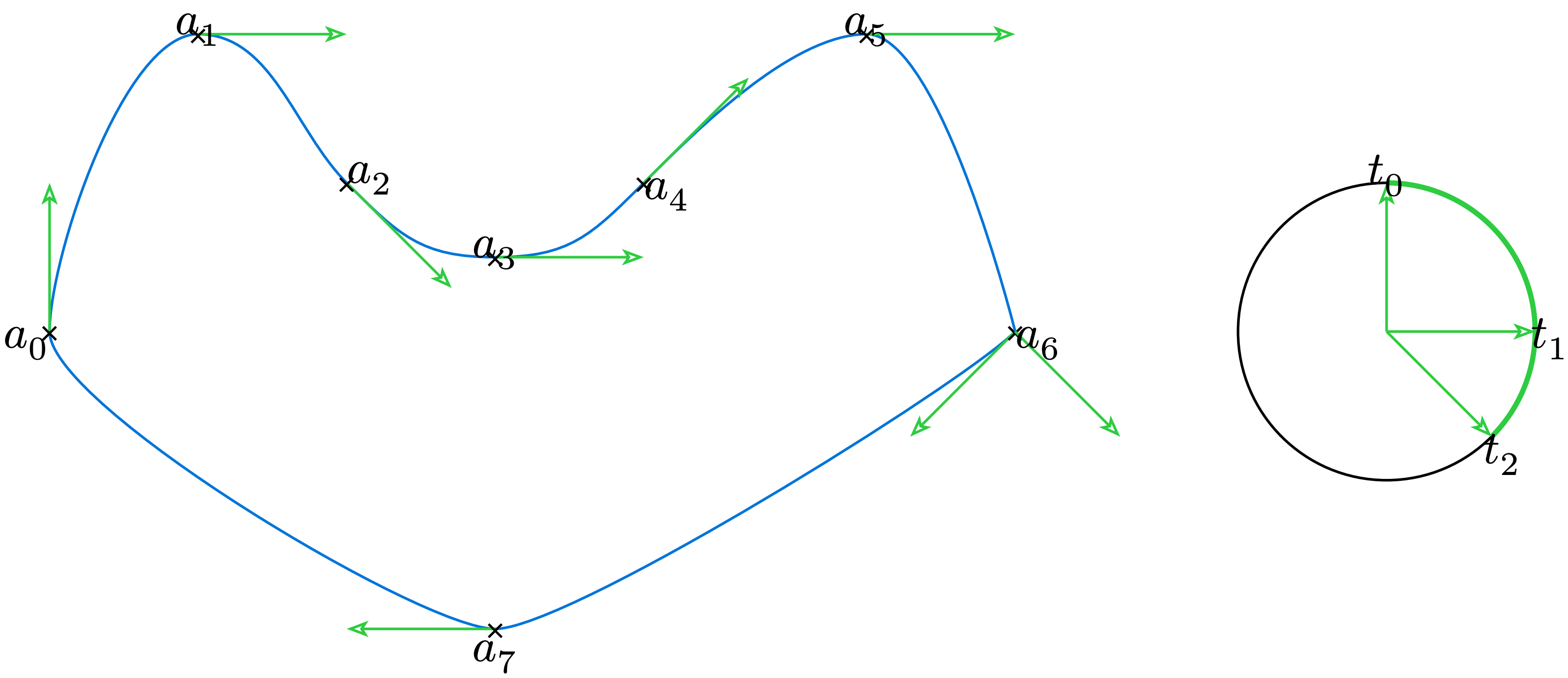
# Definition *J. W. Milnor*



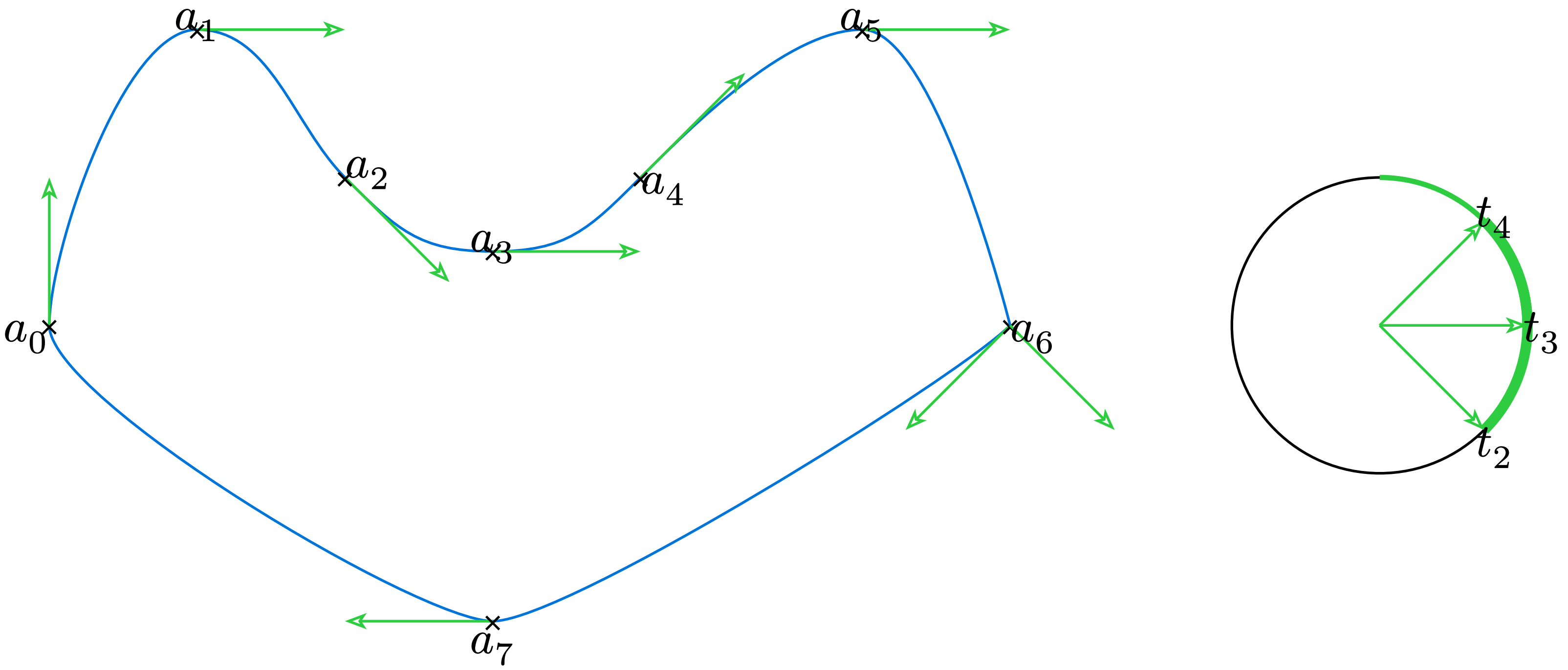
$$\kappa(P) = \sum_{i \in \mathbb{Z}/N\mathbb{Z}} \angle(a_{i-1}a_i, a_i a_{i+1})$$

$$\kappa(\mathcal{C}) = \sup_{P \text{ inscribed in } \mathcal{C}} \kappa(P)$$

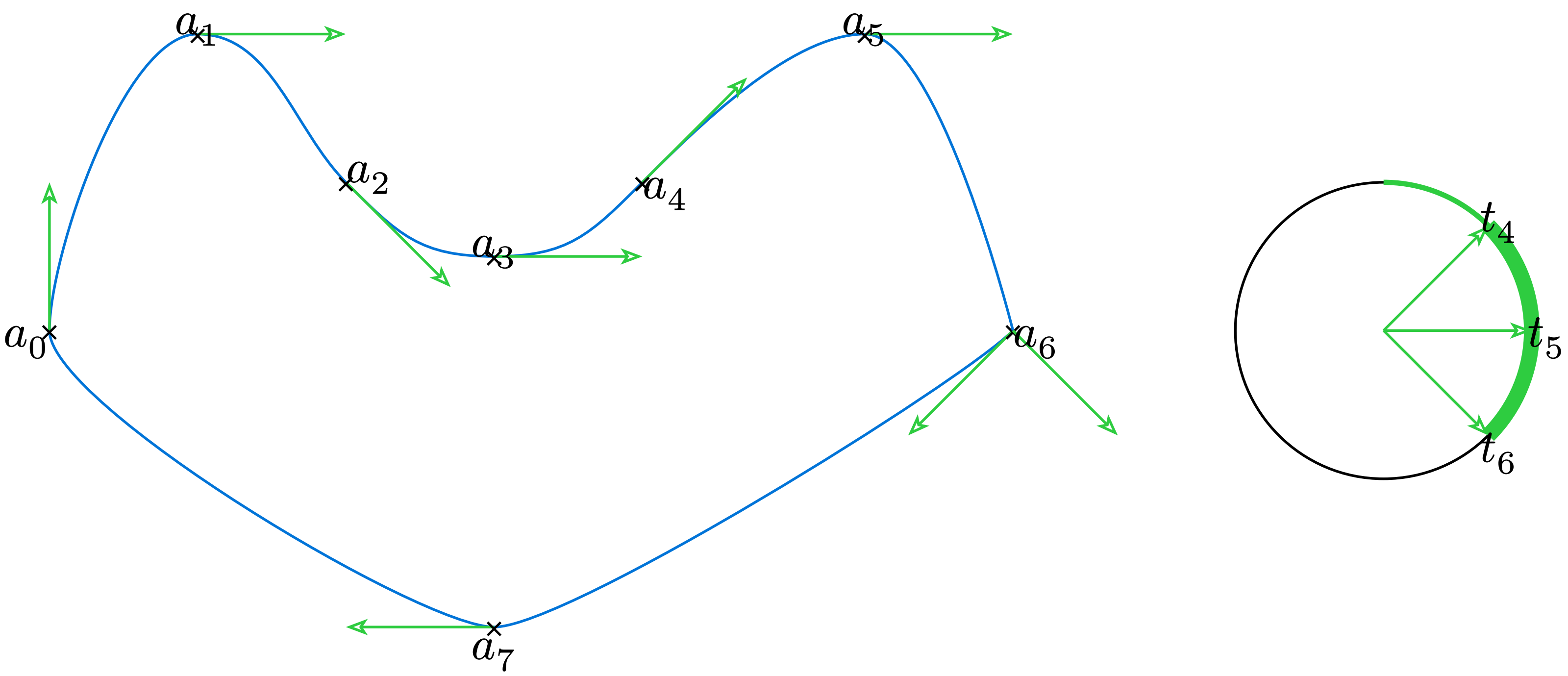
# Tangent Indicatrix



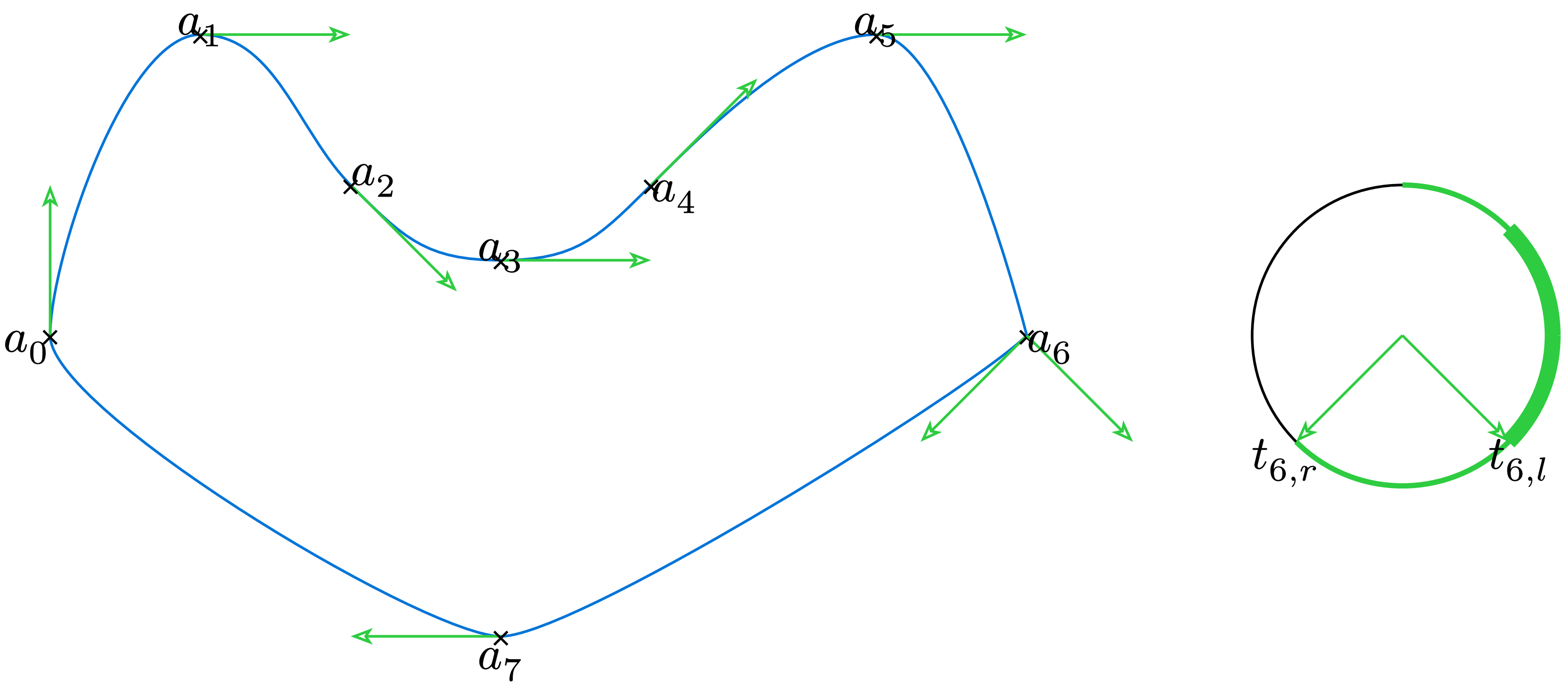
# Tangent Indicatrix



# Tangent Indicatrix



# Tangent Indicatrix



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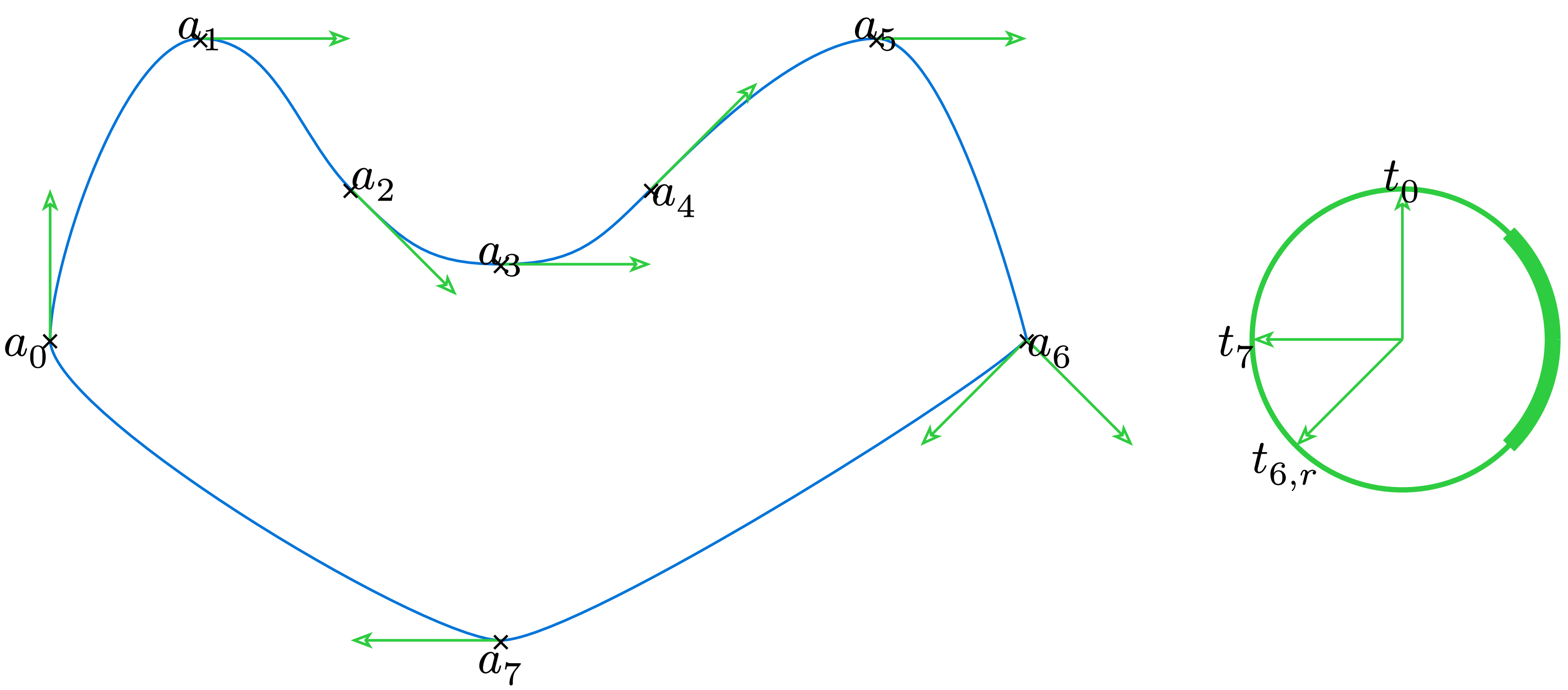

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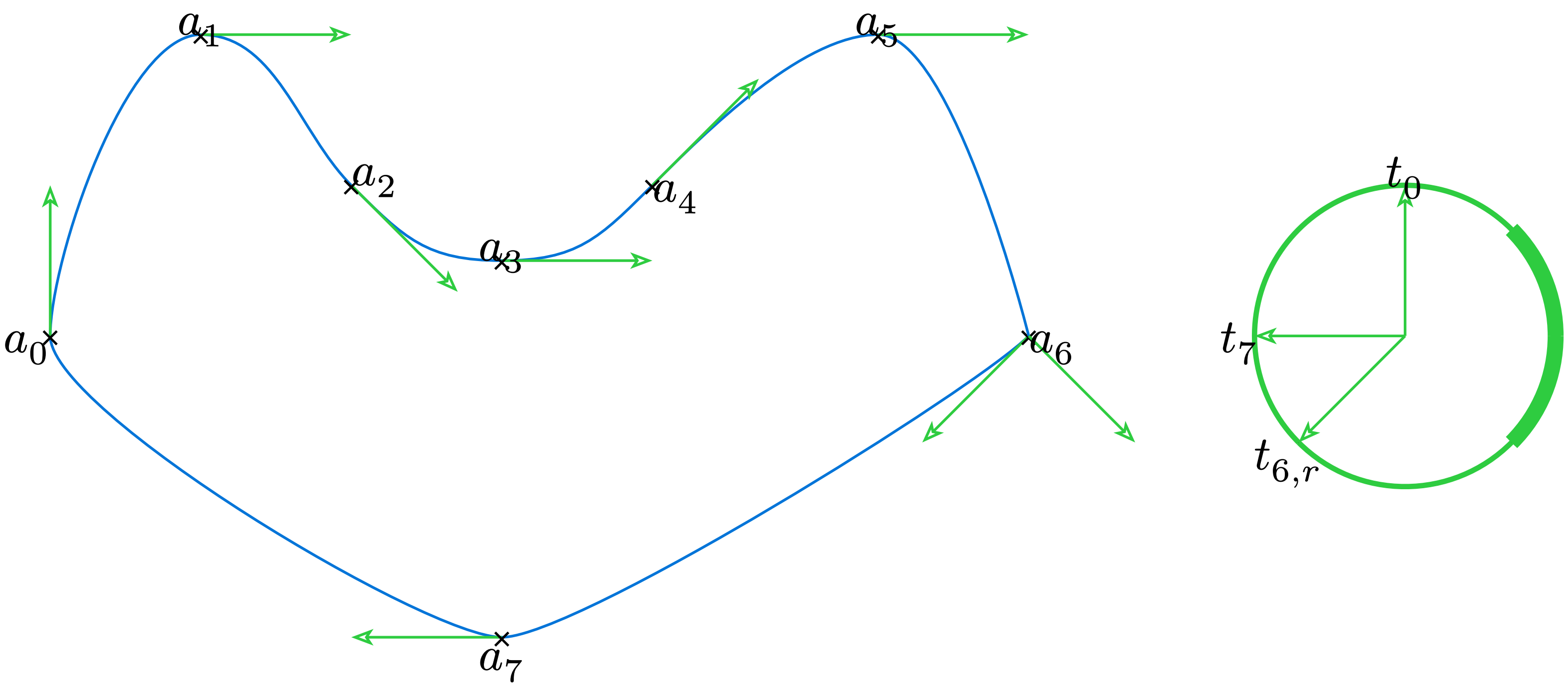


# Important Properties

- For twice differentiable curves $\mathcal{C}$ :

$$\kappa(\mathcal{C}) = \int_0^{\text{Length}(\mathcal{C})} |k(s)| \, ds$$

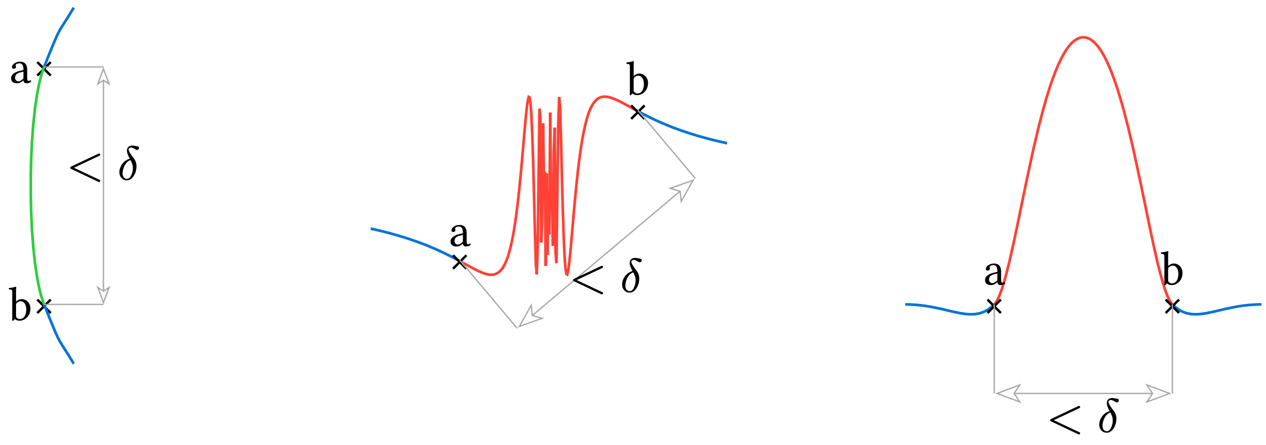
- Fenchel's Theorem: For any Jordan curve $\mathcal{C}$, $\kappa(\mathcal{C}) \geq 2\pi$.
  $\mathcal{C}$ has a convex interior iff $\kappa(\mathcal{C}) = 2\pi$.

# Locally Turn-Bounded curves

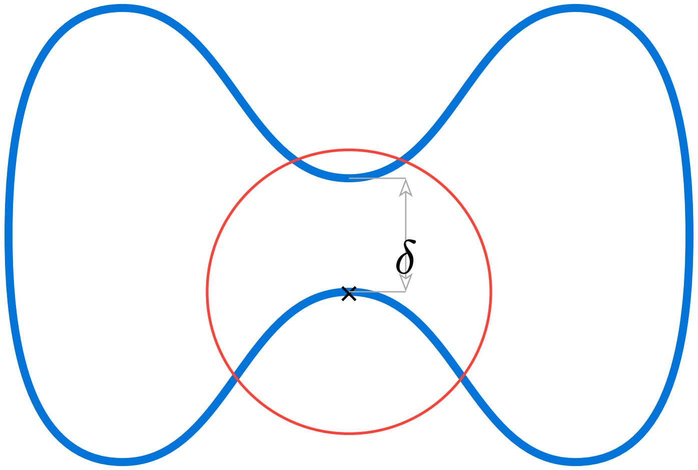
# Definition of LTB curves

*E. Le Quentrec, L. Mazo, E. Baudrier, and M. Tajine*

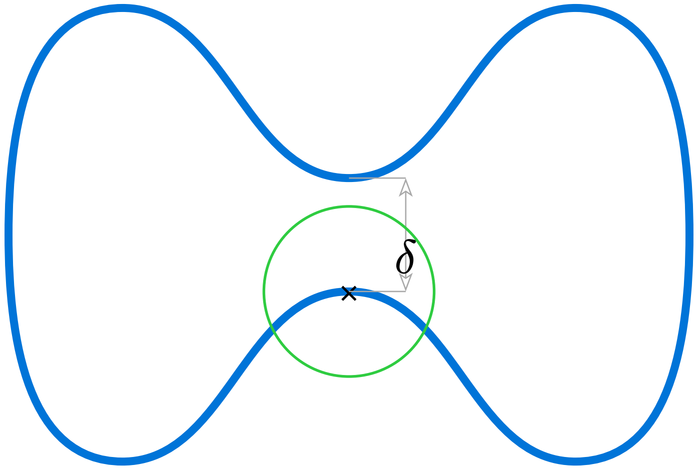
- A Jordan curve $\mathcal{C}$ is Locally Turn-Bounded by $(\theta, \delta)$ if for any pair of points $(a, b)$ of $\mathcal{C}$ such that $d(a, b) < \delta$, there exists an arc $\mathcal{C}_a^b$ of $\mathcal{C}$ joining $a$ and $b$ and such that $\kappa(\mathcal{C}_a^b) \leq \theta$.



# Local connectivity



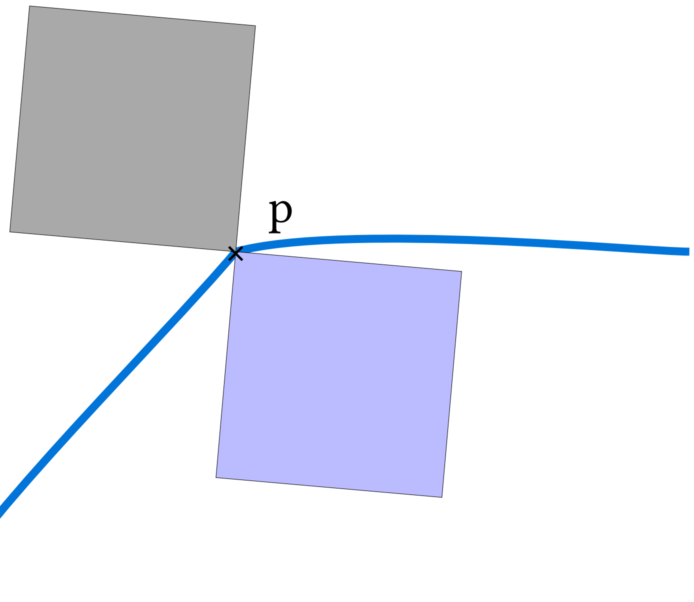
# Local connectivity




$\mathcal{C}$ $\left(\frac{\pi}{2}, \delta\right)$-LTB curve , $a \in \mathcal{C}$,

$\Rightarrow B(a, \varepsilon) \cap \mathcal{C}$ is path-connected for $\varepsilon \leq \delta$.

# Square-regularity




$\mathcal{C}$ $\left(\frac{\pi}{2}, \delta\right)$-LTB, $p \in \mathcal{C}$, there exists a square of edge-size $\frac{\delta}{2}$ and containing $p$ included in the closure of the interior component of $\mathcal{C}$.

# Curves with Lipschitz Turn

$\mathcal{C}$ has a $\frac{1}{r}$-Lipschitz turn if for any arc $\mathcal{A}$ of $\mathcal{C}$,

$$\kappa(\mathcal{A}) \leq \frac{1}{r} \operatorname{Length}(\mathcal{A})$$

# Relationships with other notions

- Any par($r$)-regular curve is $\left(\theta, 2r\sin\left(\frac{\theta}{2}\right)\right)$-LTB for $\theta \in (0, \pi)$ and has a $\frac{1}{r}$-Lipschitz turn
- Any $\left(\frac{\pi}{2}, \delta\right)$-LTB curve having a $\frac{1}{r}$-Lipschitz turn is par($r_1$)-regular for any $r_1 < \min\left(\frac{\delta}{2}, r\right)$
- LTB curves are quasi-regular
- For $\theta \in [0, \pi]$, $\delta > 0$ and $\mathcal{C}$ a $(\theta, \delta)$-LTB curve,

$$\delta \leq 2r_{\mu(\mathcal{C})}$$

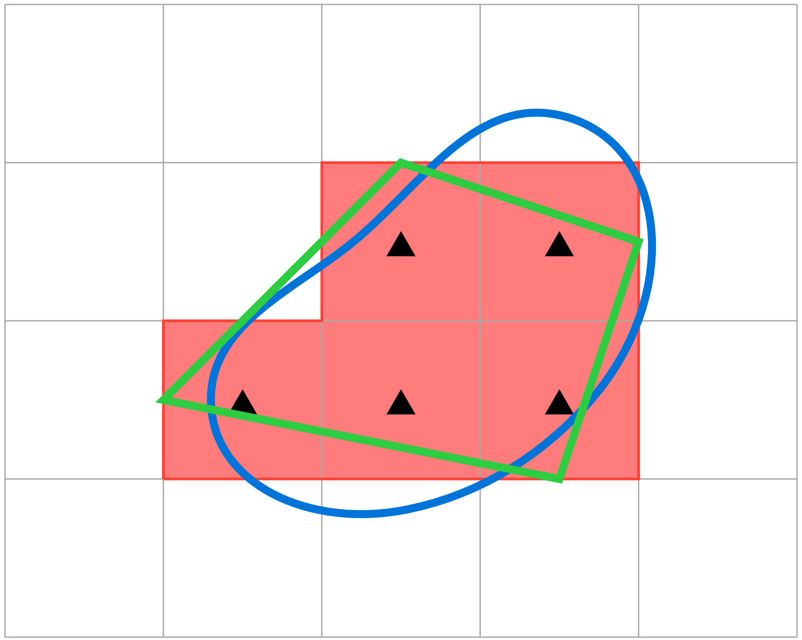
with $\mu = \cos\frac{\theta}{2}$

# Digitization of a LTB curve

$\mathcal{C}$ $\left(\frac{\pi}{2}, \delta\right)$-LTB curve

- Any pixel of the Gauss Digitization of $\mathcal{C}$ is one-quarter covered by the shape bounded by $\mathcal{C}$.
- Any pixel which is three-quarter covered by the shape bounded by $\mathcal{C}$ belongs to the Gauss Digitization of $\mathcal{C}$.
- The Gauss Digitization of $\mathcal{C}$ is well-composed, 4-connected and simply connected if $h < \min\left(\frac{\sqrt{2}}{2}\delta, \frac{1}{2}\operatorname{diam}(\mathcal{C})\right)$

# Length estimation



# Links between turn and length

- [*A. D. Alexandrov and Y. G. Reshetnyak*] : Let $\mathcal{C}$ an arc of ends $a$ and $b$ s.t. $\kappa(\mathcal{C}) \leq \frac{\pi}{2}$,

$$\text{Length}(\mathcal{C}) \leq \frac{\|b-a\|}{\cos\left(\frac{\kappa(\mathcal{C})}{2}\right)}$$

- [*E. Le Quentrec*]: Let $\mathcal{C}$ a par($r$)-regular curve with $r > 0$. Let $\mathcal{C}_a^b$ be a subarc delimlited by two point $a$ and $b$ such that $\|b-a\| < 2r$.

$$\text{Length}\left(\mathcal{C}_a^b\right) \leq 2r \arcsin\left(\frac{\|b-a\|}{2r}\right)$$

# Future work

- Extension to surfaces in 3D (PhD started 1st October: Lysandre Macke)

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