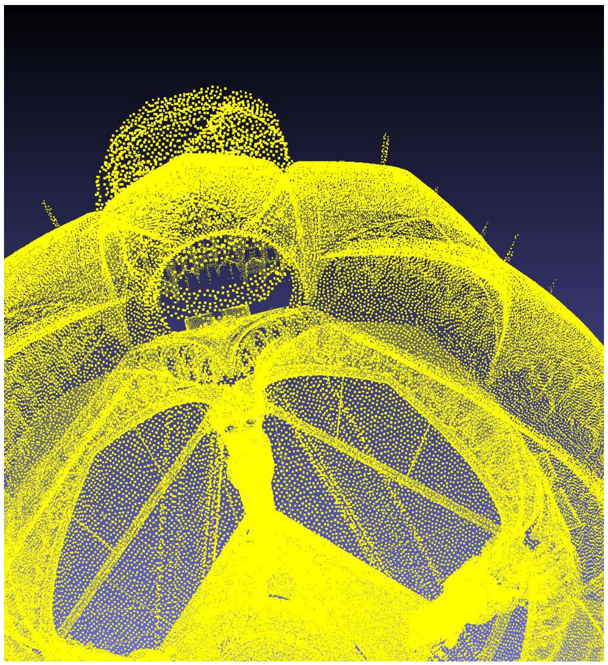
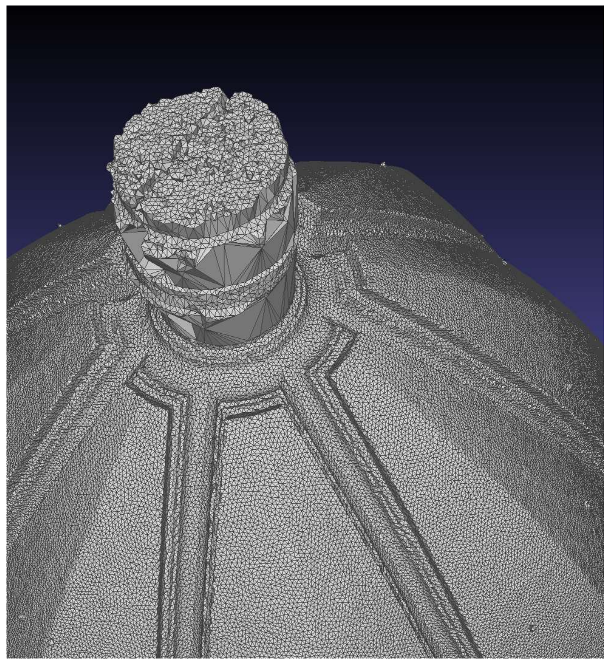
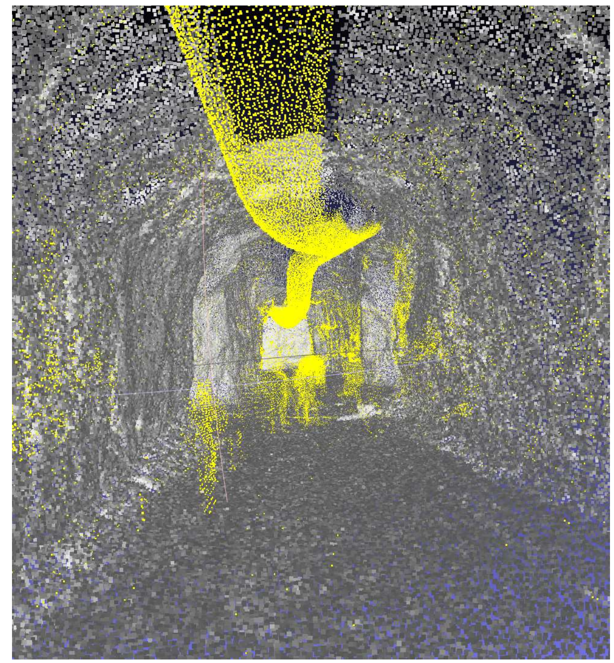
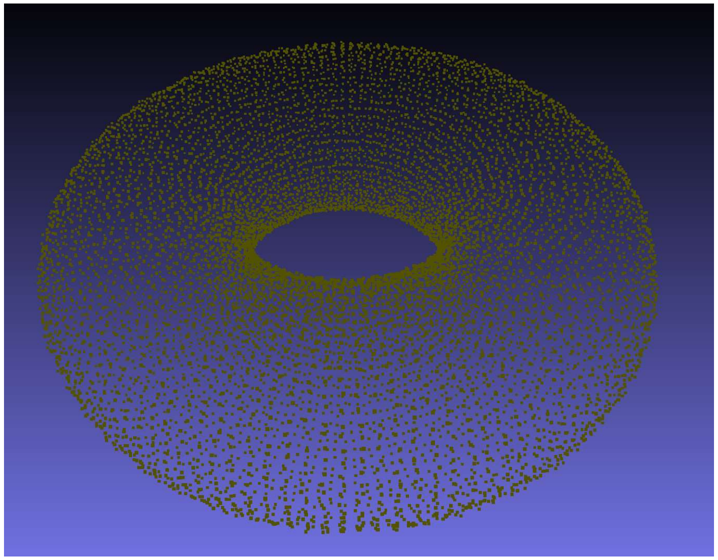
Homological approaches to manifold reconstruction

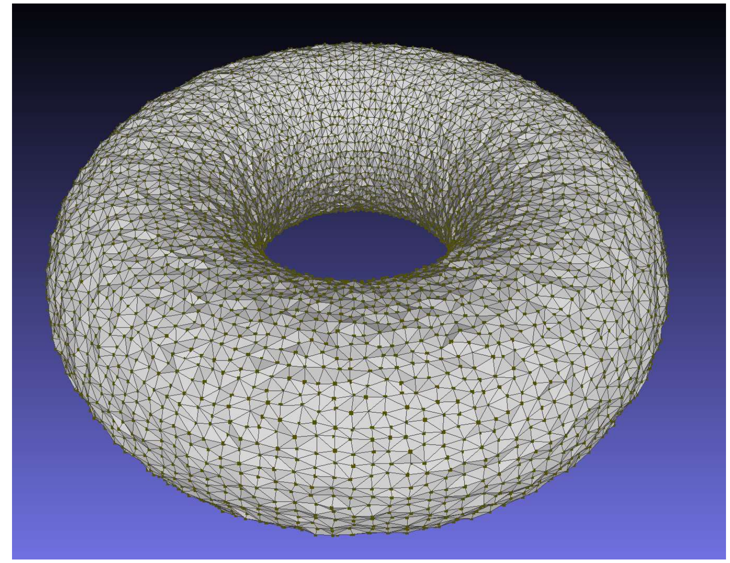
André Lieutier
Aix en Provence

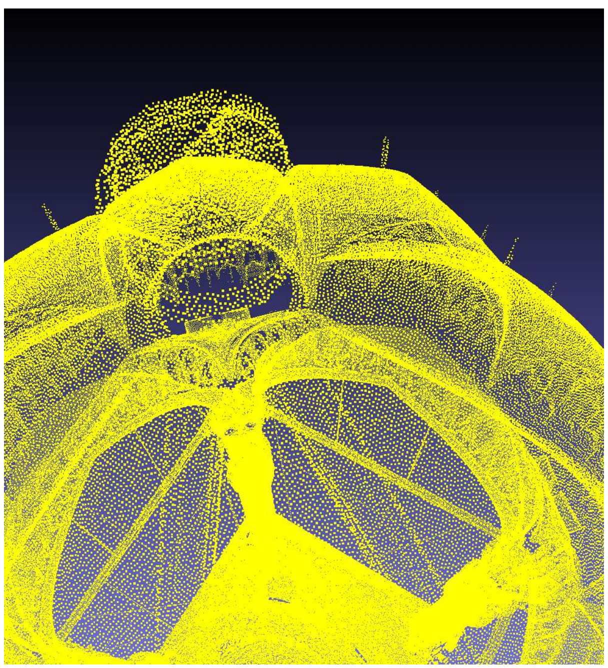
  

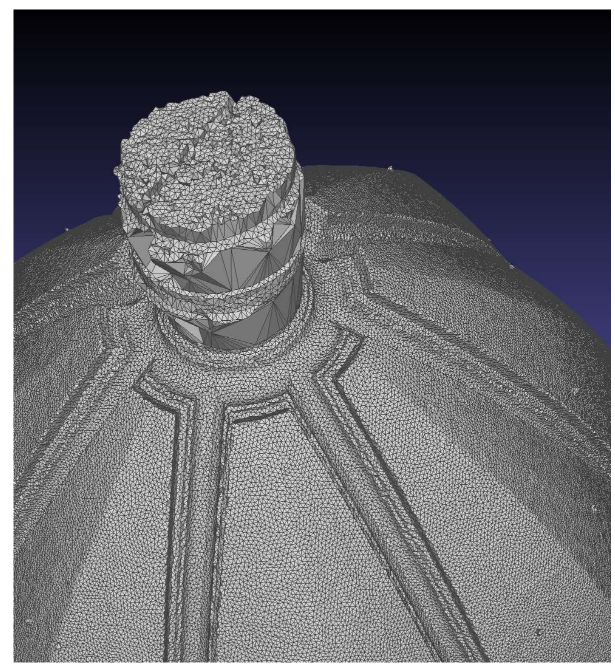
Géométrie et Informatique
Geometry and Computing
CIRM
21-25 October 2024

Manifold triangulation

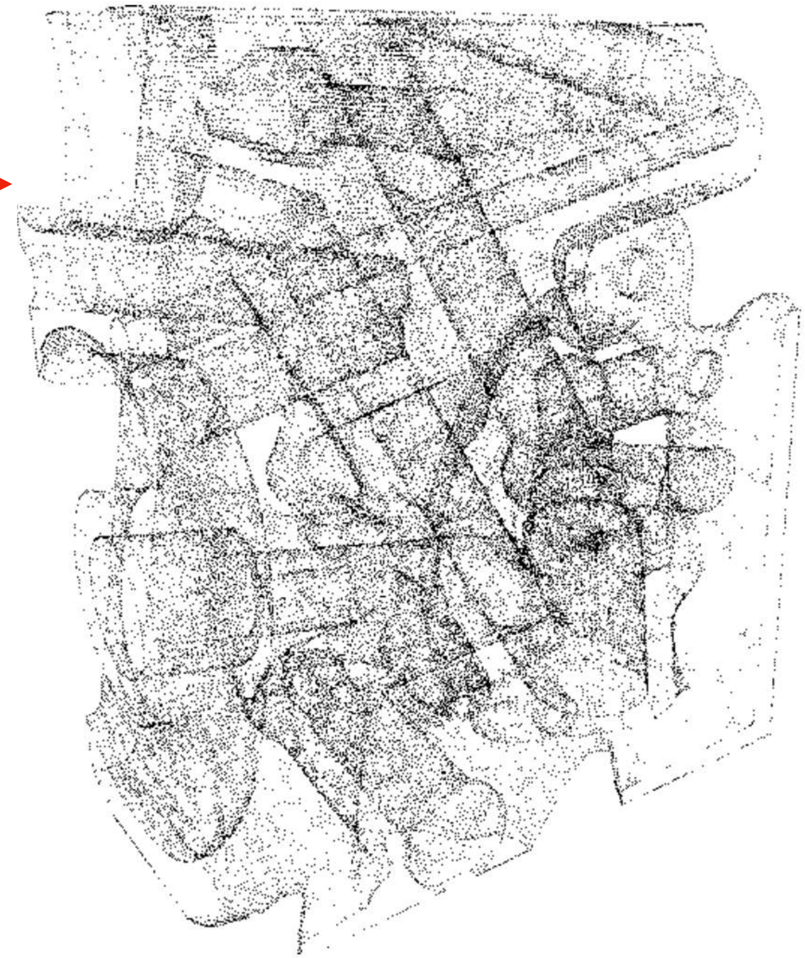






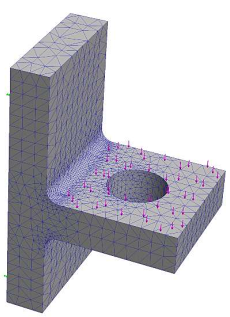


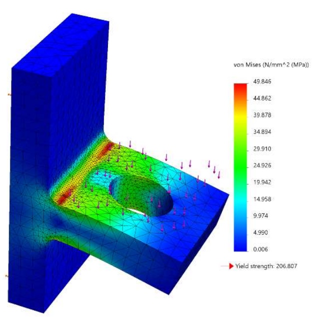
Topological faithful reconstruction and topological inference : motivation

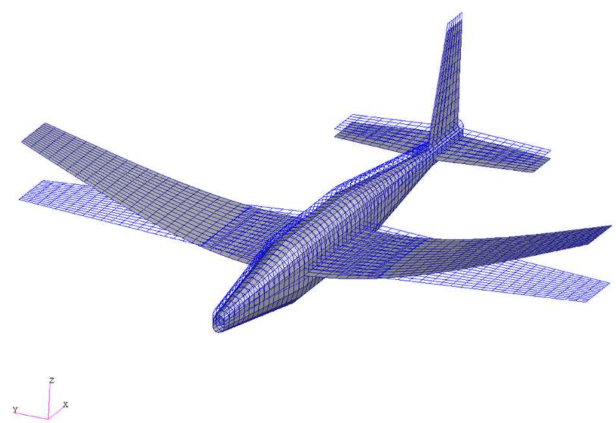


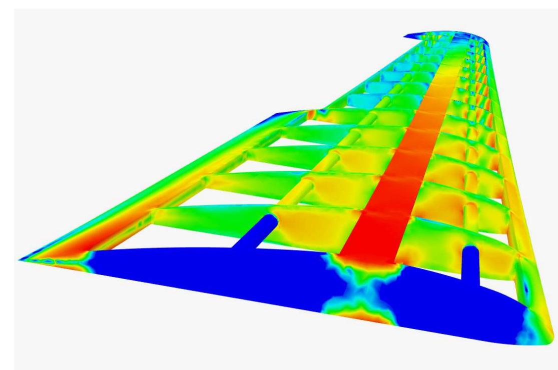
Reconstruction beyond visual realism:
understanding the **topology**

Topological faithful reconstruction and topological inference : motivation









Reconstruction beyond visual realism:
understanding the **topology**

Topological faithful reconstruction and topological inference : motivation

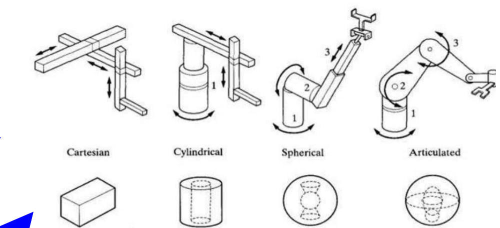
MANIFOLD LEARNING (TDA)

input
:

Data= movie/pictures

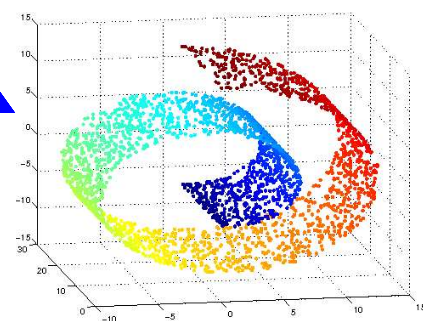
Topological inference
(topology learning)

output:



Robot configuration space

Manifold
reconstruction

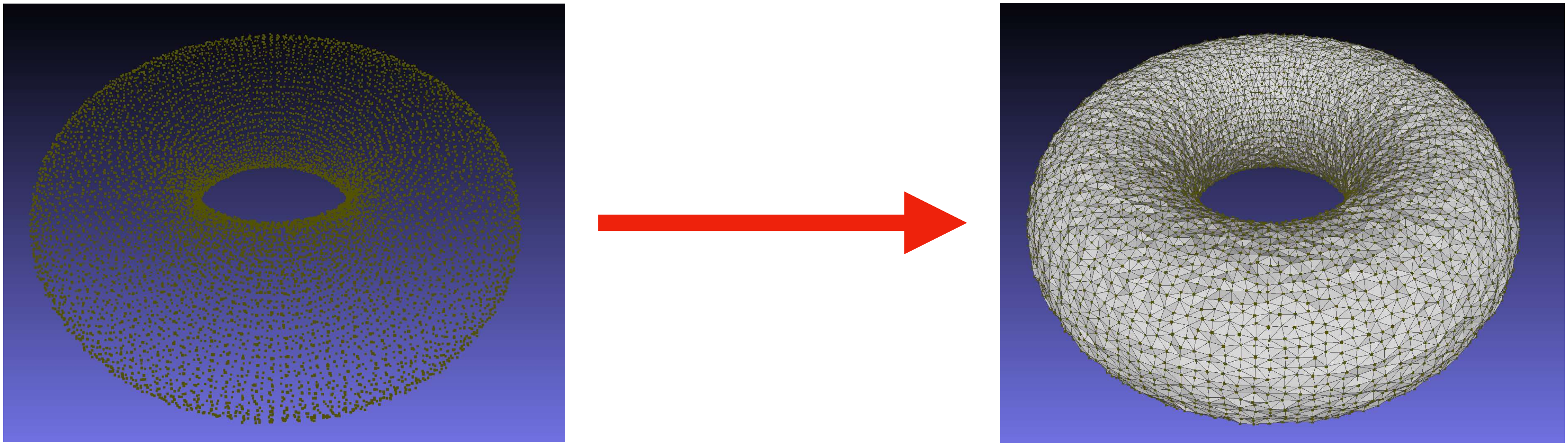


For example as
simplicial complex
= triangulation

topology
computation

What does it mean to recover the topology ?

(of subsets of euclidean space)



Given a set P of point sampling a submanifold $\mathscr{M}$, output a simplicial complex M with vertices in P, which is homeomorphic to the manifold, in other words M is a **triangulation** of $\mathscr{M}$.

Moreover we require the triangulation to be **geometrically close**:

The map $\phi : \mathscr{M} \to M$ that realizes the homeomorphism should satisfy:

$$\sup_x \|\phi(x) - x\| \text{ is small}$$

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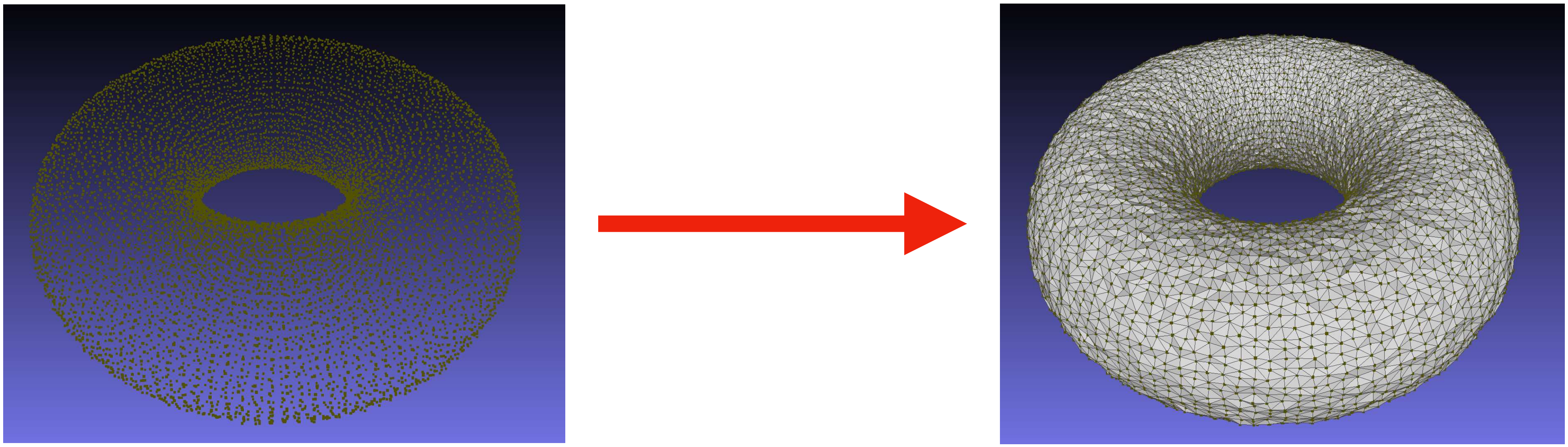

While many approaches have been explored, we focus here on **variational methods**: the triangulation M should be obtained as **the minimum of some functional under constraints**.

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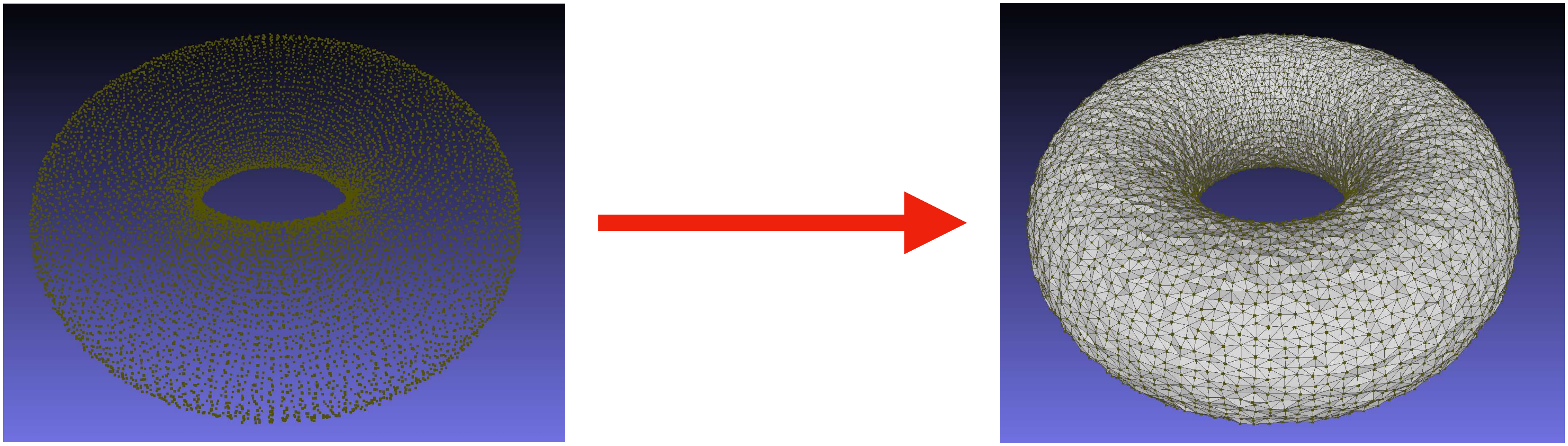
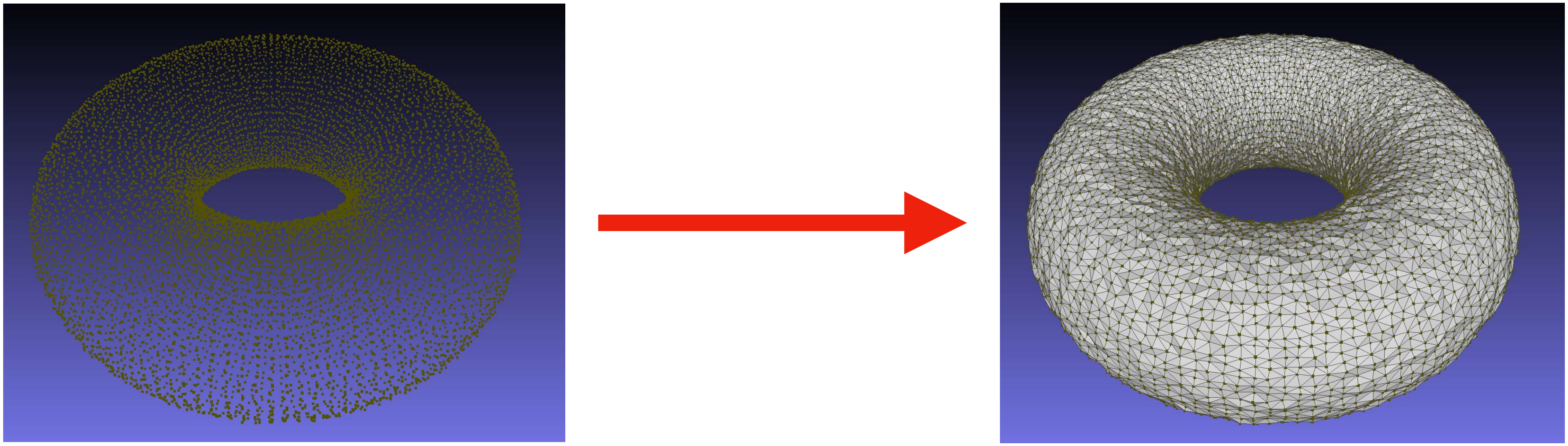


While many approaches have been explored, we focus here on **variational methods**: the triangulation M should be obtained as **the minimum of some functional under constraints**.

More precisely, as the support of a **simplicial chain which is minimal within some homology class**.

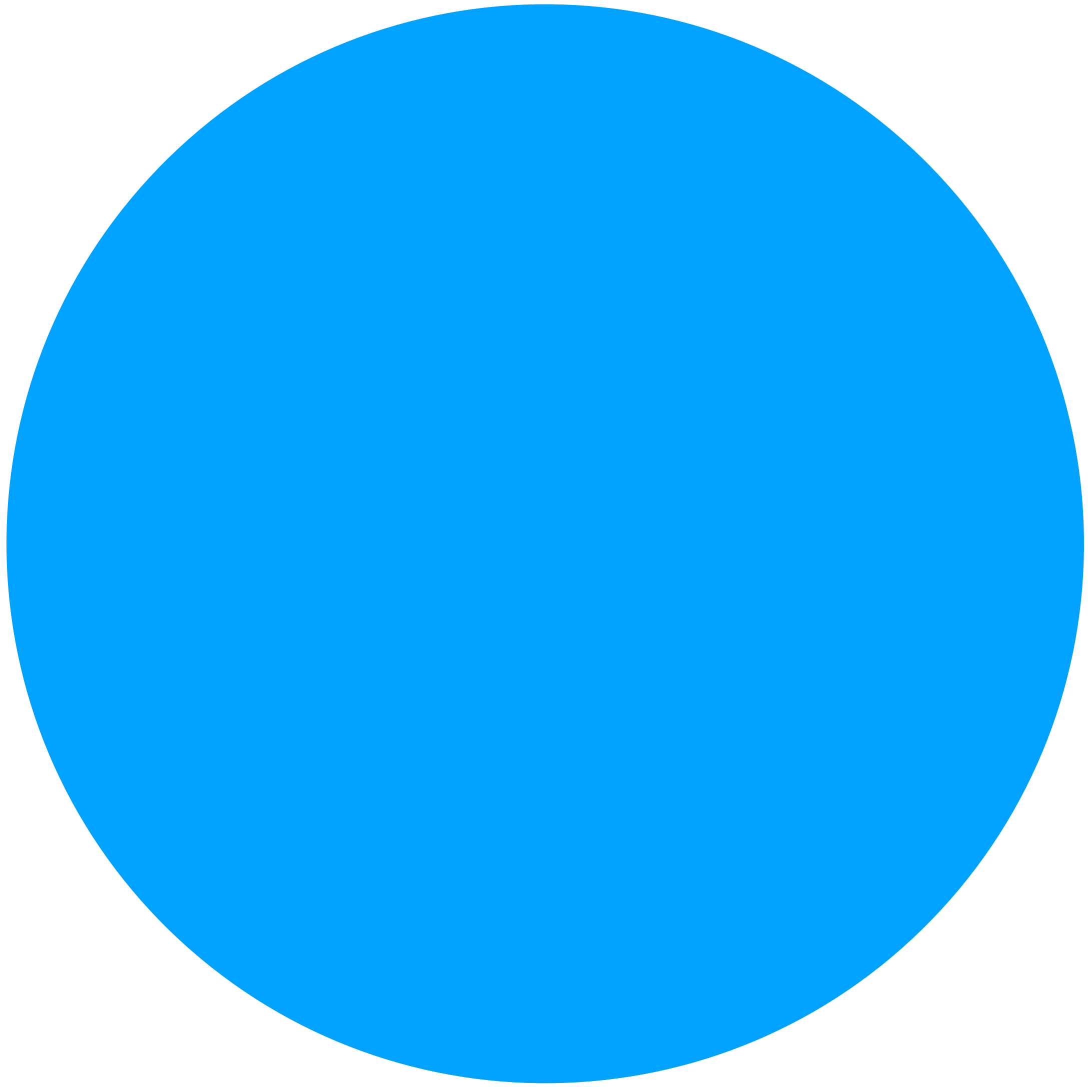
What does it mean to recover the topology ?

(of subsets of euclidean space)



For that we start by a **variational formulations of Delaunay triangulations** (or more generally regular triangulations) that **allows to generalizes it to non-Euclidean context.**

Simplest case: a disk in $\mathbb{R}^d$



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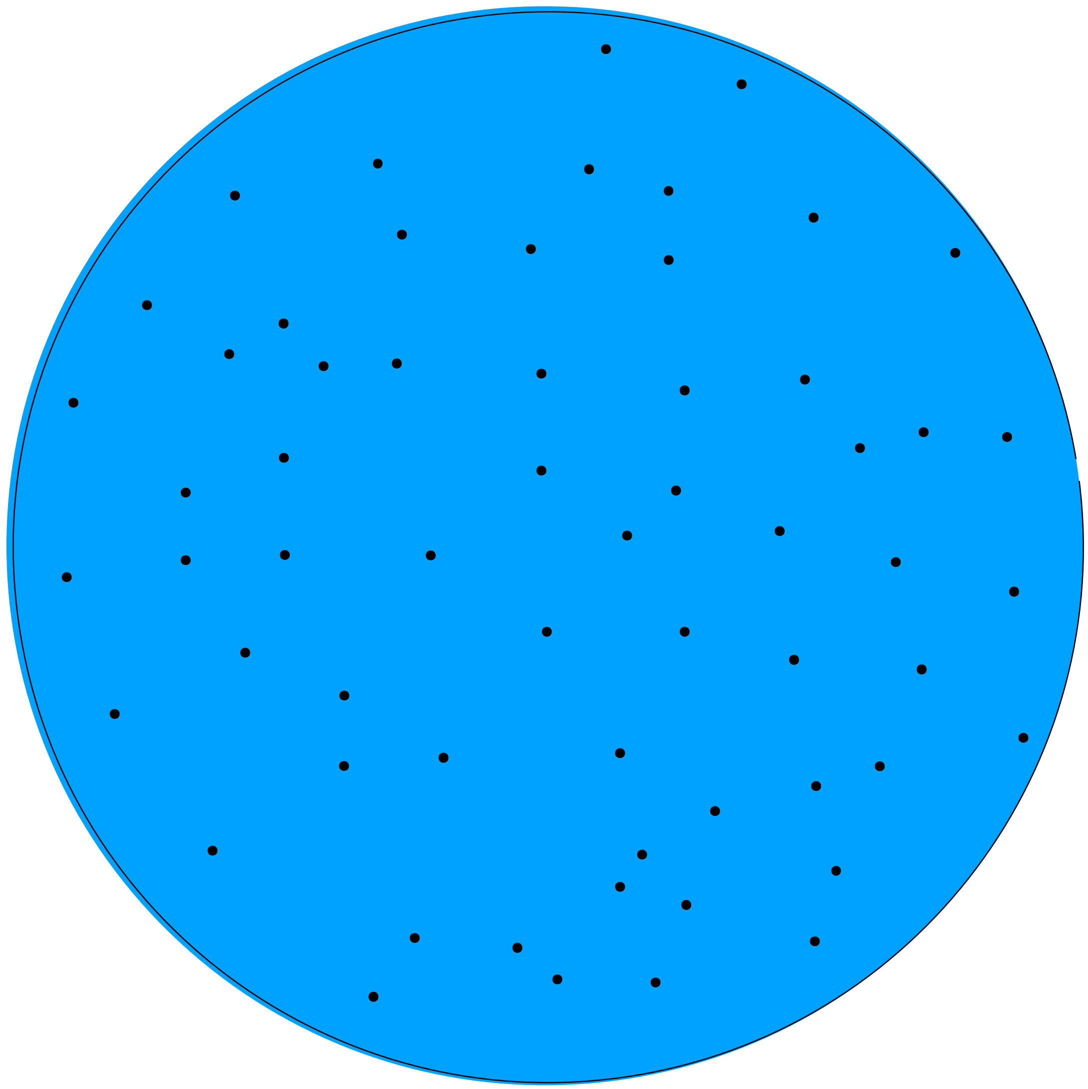
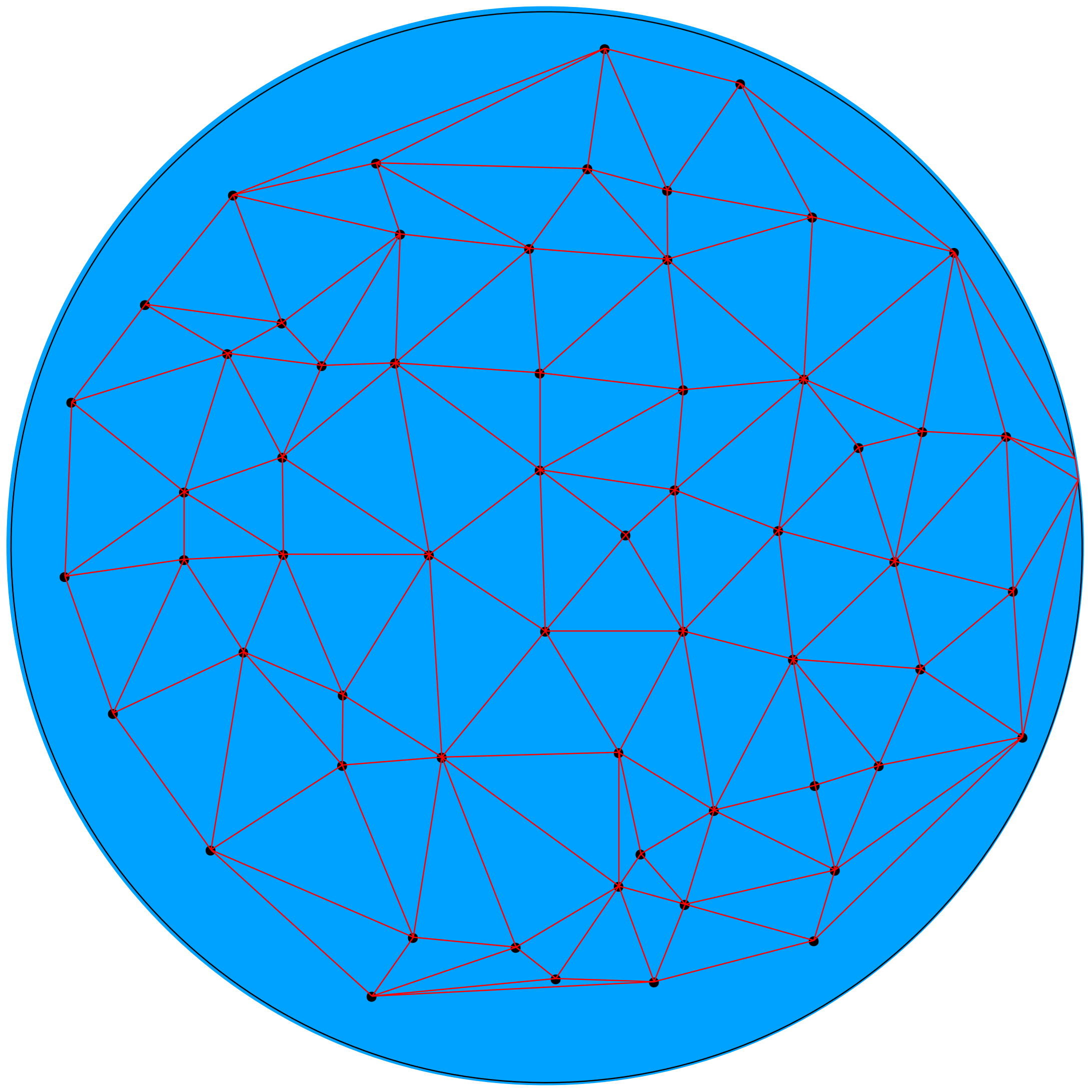


(thanks to D. Attali)

Simplest case: a disk in $\mathbb{R}^d$



Delaunay triangulation

(thanks to D. Attali)

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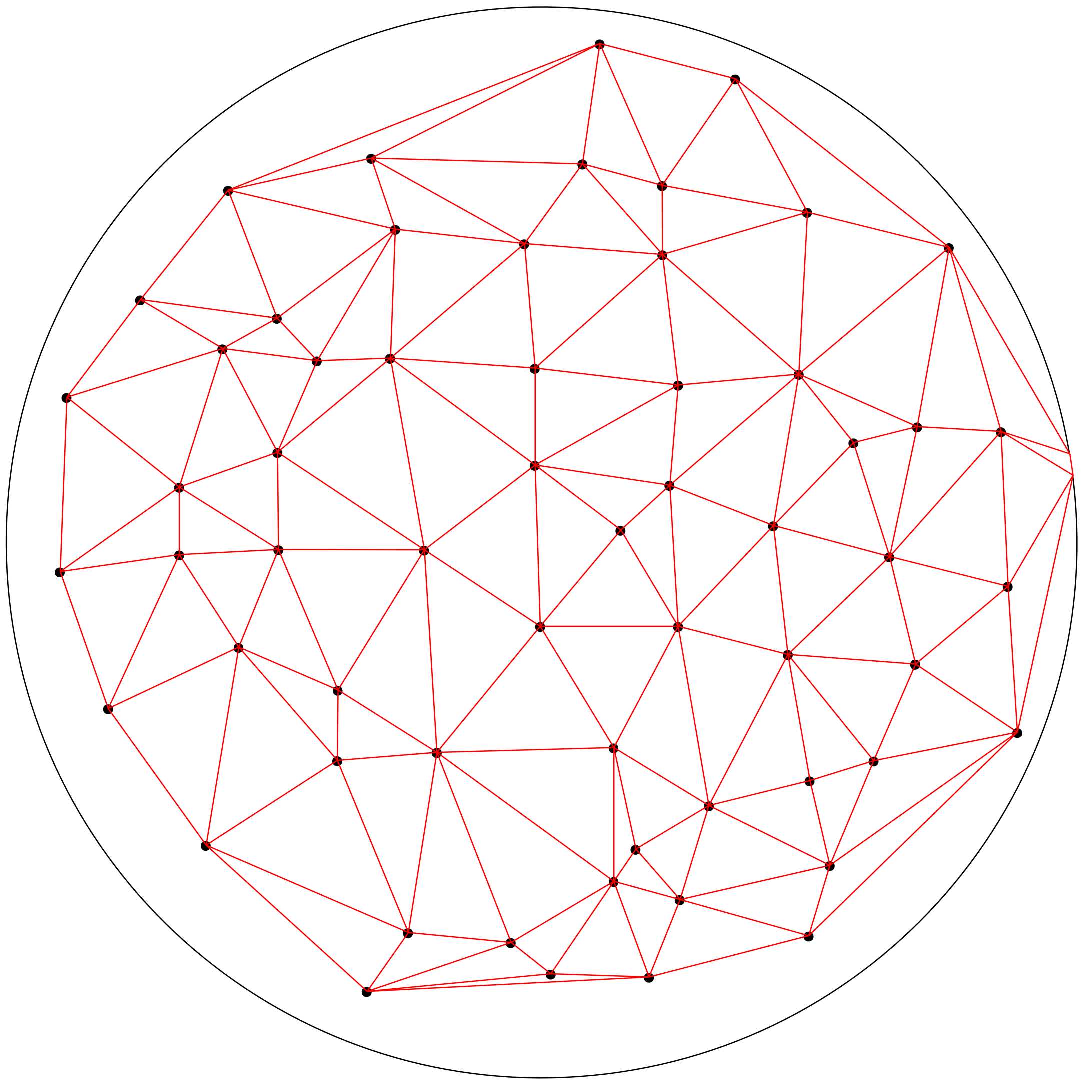

Delaunay triangulation

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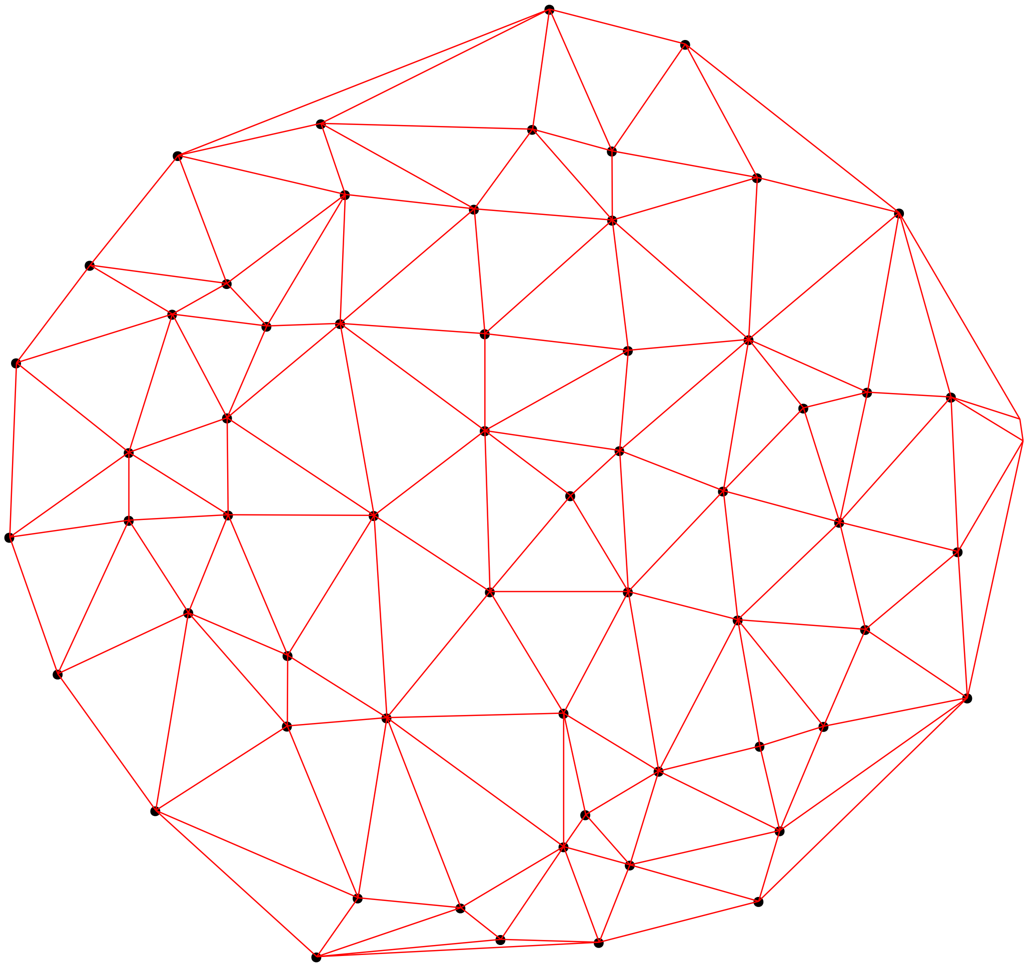
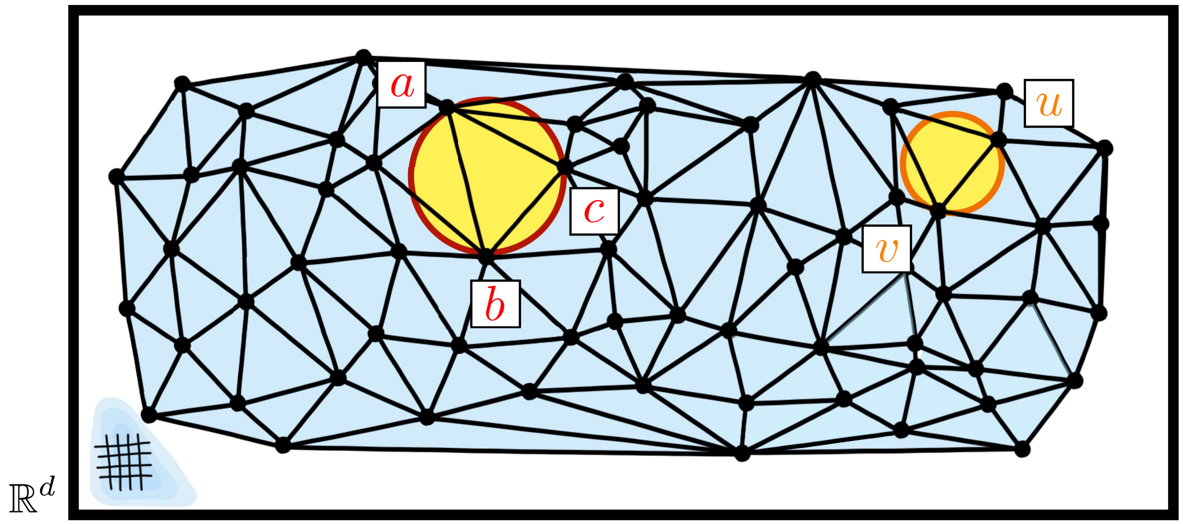


Delaunay triangulation

Delaunay complex

$\sigma \in \mathrm{Del}(P) \iff \exists$ a sphere that circumscribes σ and does not enclose any point of P



P generic

No $(d+2)$ points of P lie on a common $(d-1)$-sphere

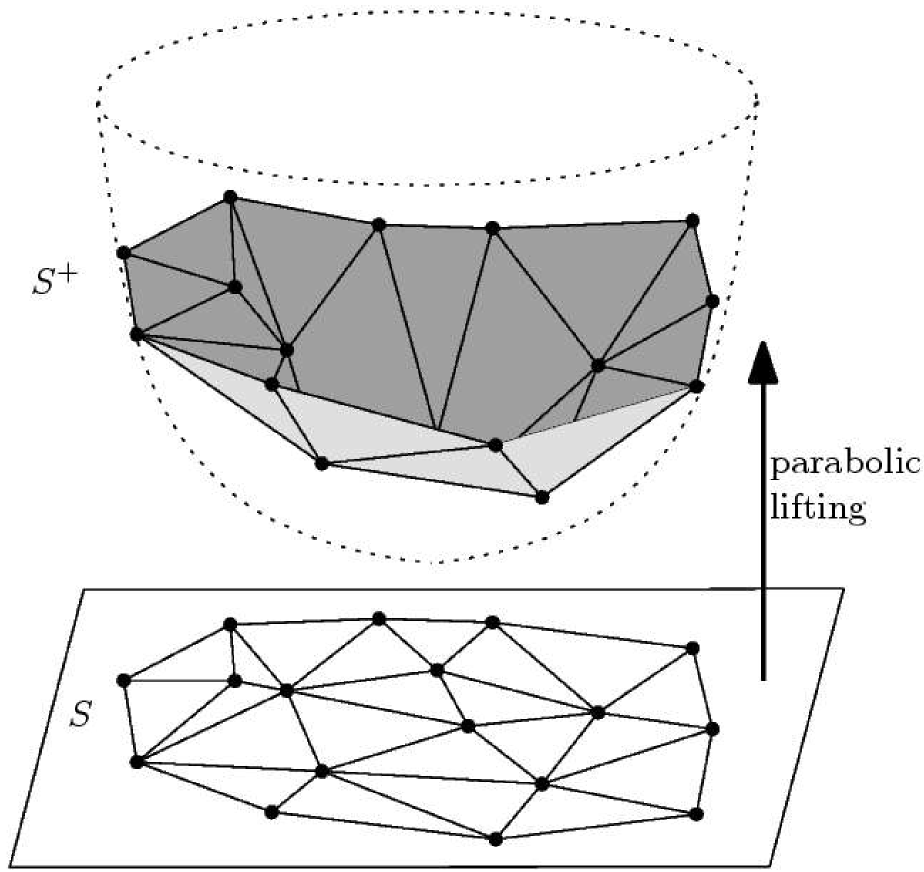
where $d = \dim(\mathrm{aff}\, P)$

$\Downarrow$

Triangulation , i.e. homeomorphic simplicial complex

Delaunay : variational definitions

For each point $u \in \mathbb{R}^n$, we consider its *lifted image* $\hat{u} = (u, \|u\|^2) \in \mathbb{R}^{n+1}$. A classical result says that σ is a Delaunay n-simplex of P if and only if $\hat{\sigma}$ spans an n-face of the lower convex hull of $\hat{P}$.



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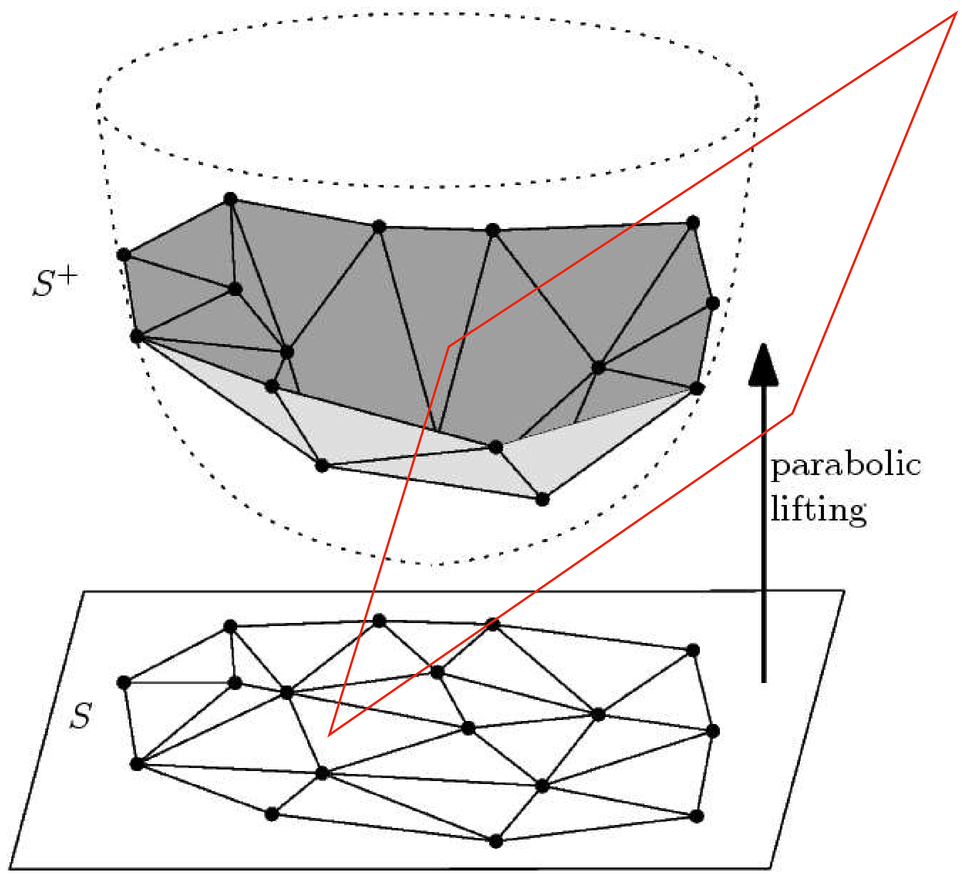

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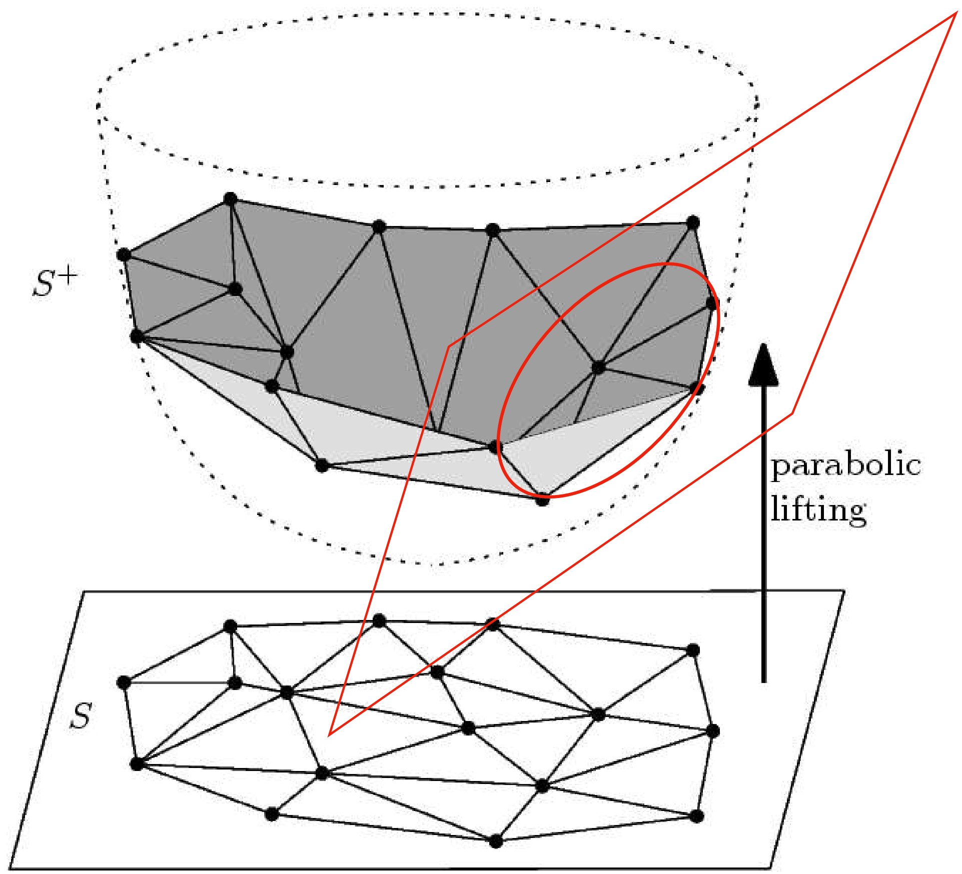

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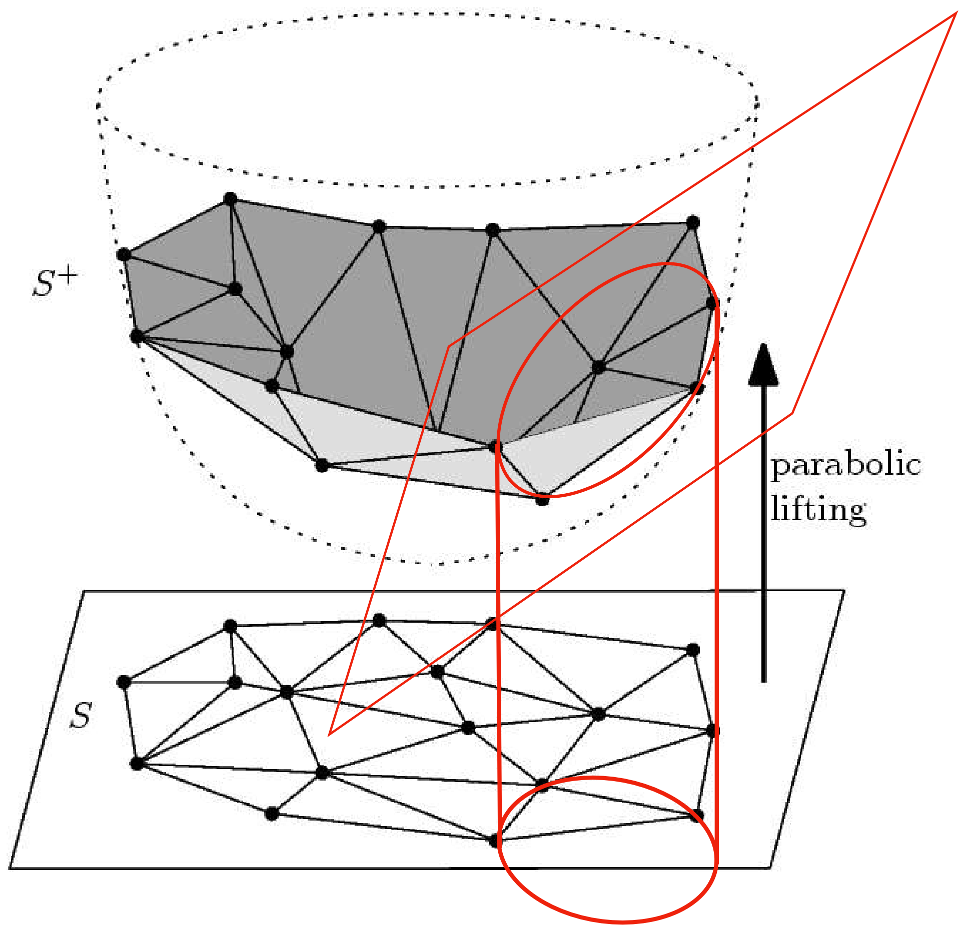
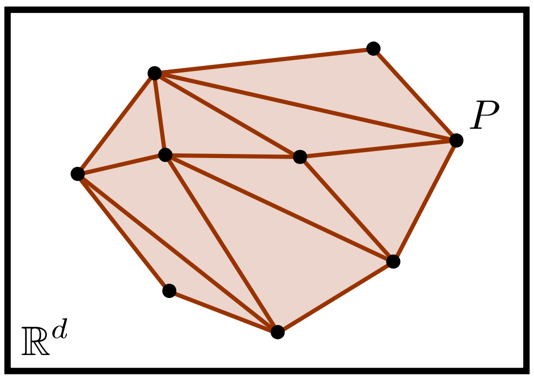
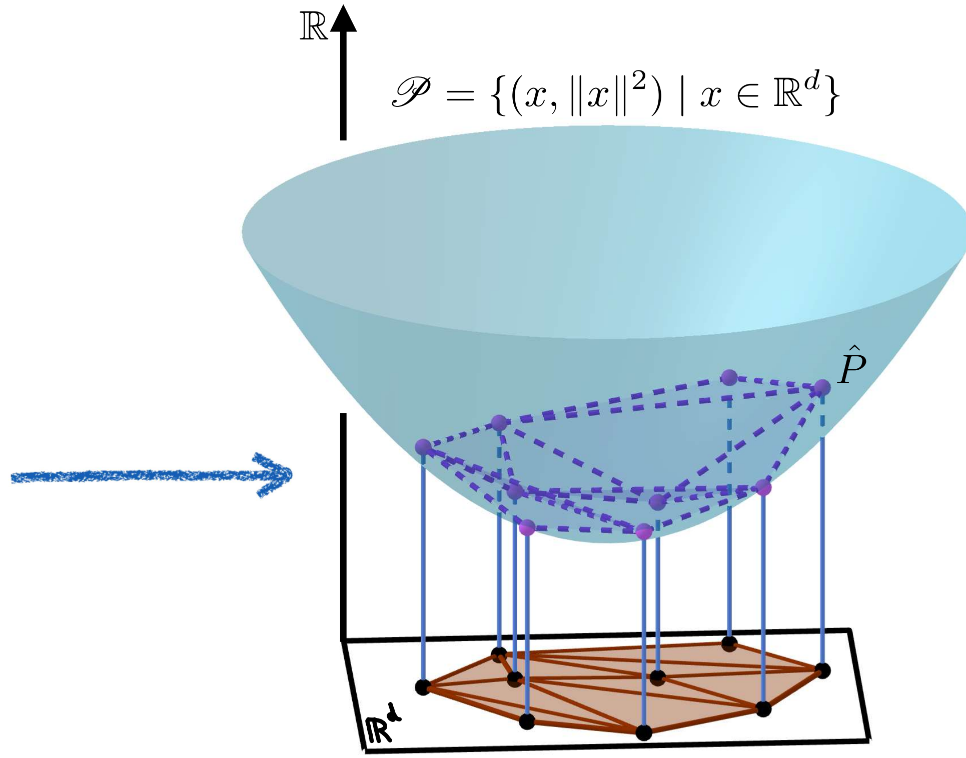


Delaunay : variational definitions



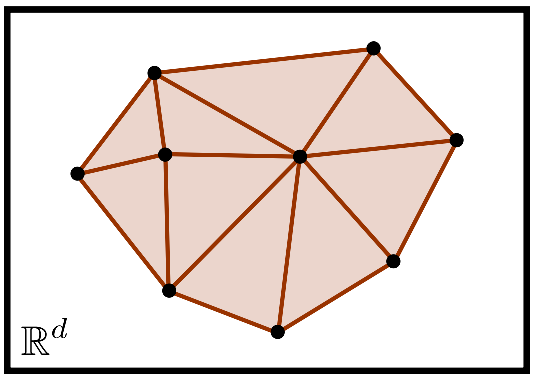
T: triangulation of $\mathrm{Conv}(P)$



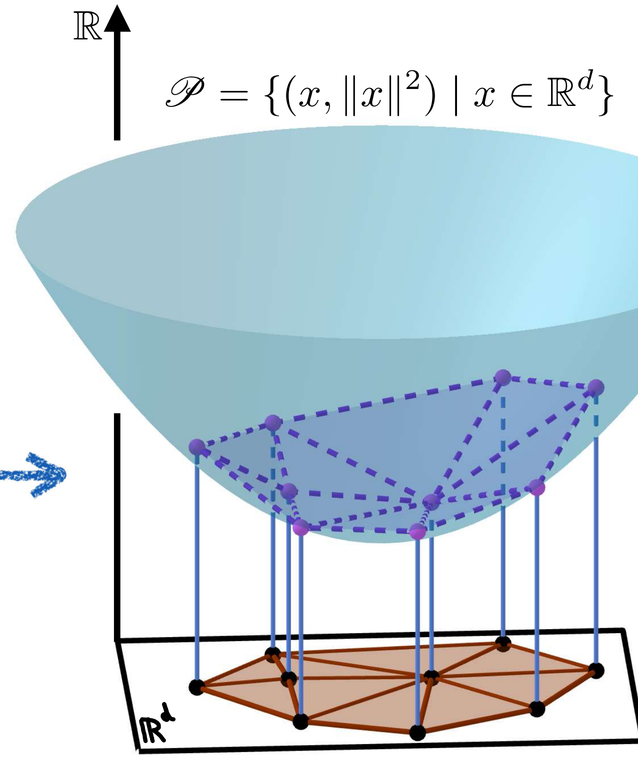
$\hat{T}$: lift of T

- $E_{\mathrm{del}}(T) =$ volume between $\hat{T}$ and paraboloid $\mathscr{P}$.

Delaunay : variational definitions



$T = \mathrm{Del}(P)$



$\hat{T} = \widehat{\mathrm{Del}(P)}$ = triangulation of lower convex hull of $\hat{P}$

- $E_{\mathrm{del}}(T)$ = volume between $\hat{T}$ and paraboloid $\mathscr{P}$.

P generic $\Longrightarrow$ $\mathrm{Del}(P)$ = the triangulation of $\mathrm{Conv}(P)$ with smallest Delaunay energy

Delaunay : variational definitions

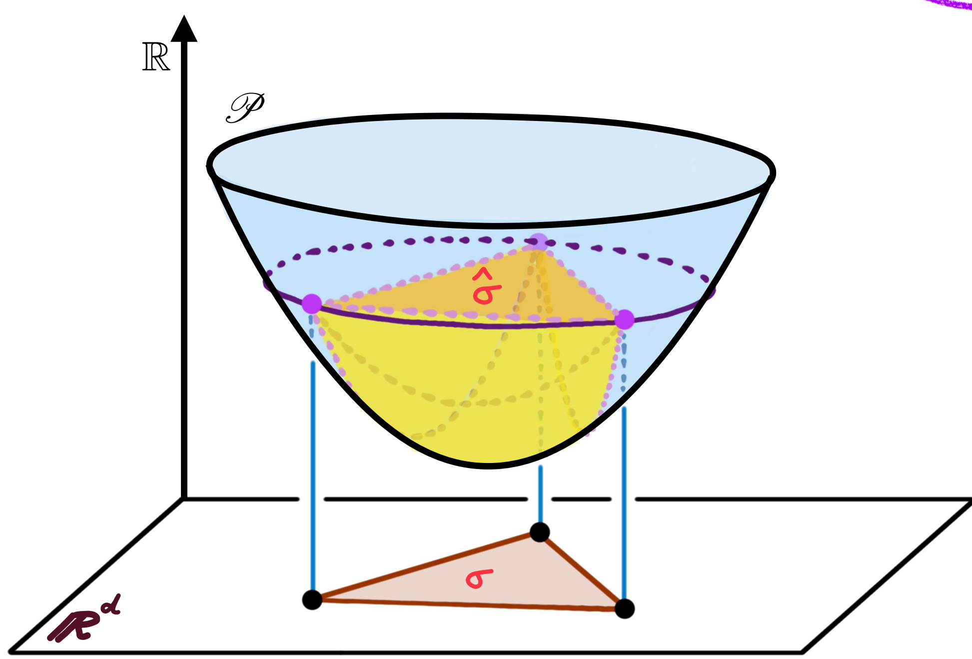
$$E_{\rm del}(T) = \text{volume between } \hat{T} \text{ and paraboloid } \mathscr{P}.$$

$$= \sum_{d\text{-simplex } \sigma \in T} \underbrace{\text{volume between } \hat{\sigma} \text{ and } \mathcal{P}}$$

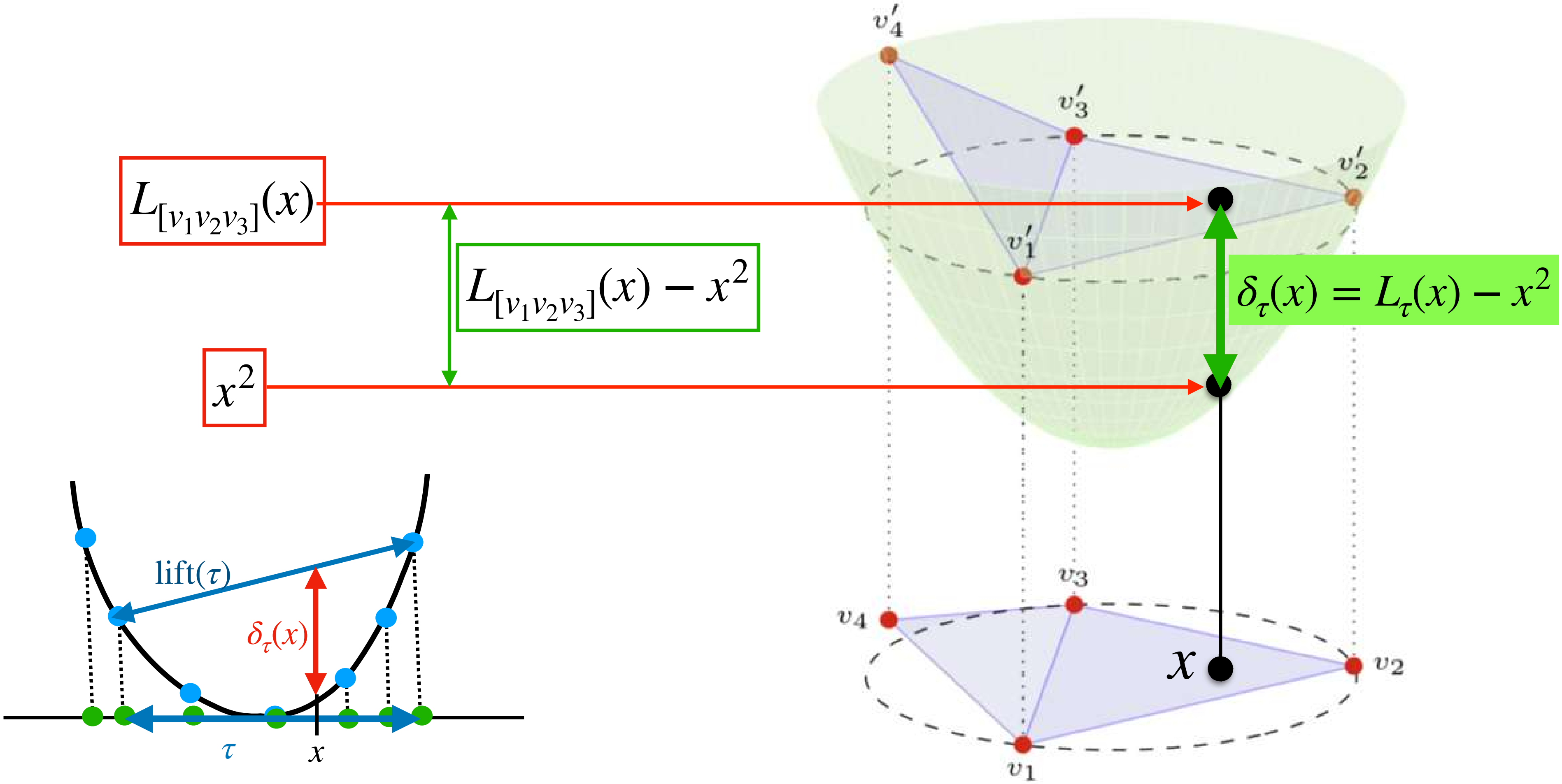
Delaunay weight of σ: $\omega_{\rm del}(\sigma) = \frac{1}{(d+1)(d+2)} \mathrm{vol}(\sigma) \sum_{e \text{ edge of } \sigma} \mathrm{length}(e)^2$

intrinsec expression

[Chen, Holst 2011]



Delaunay : variational definitions

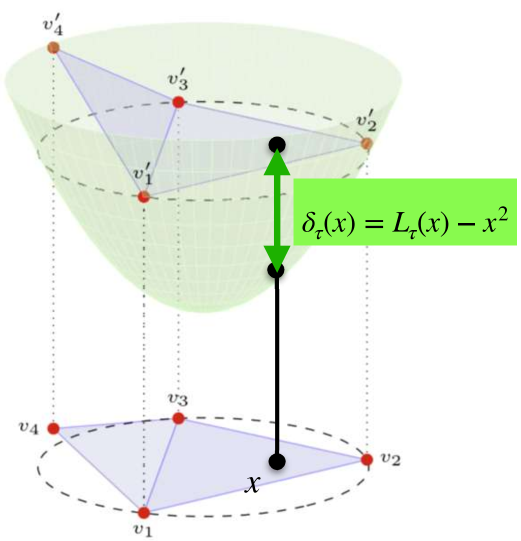


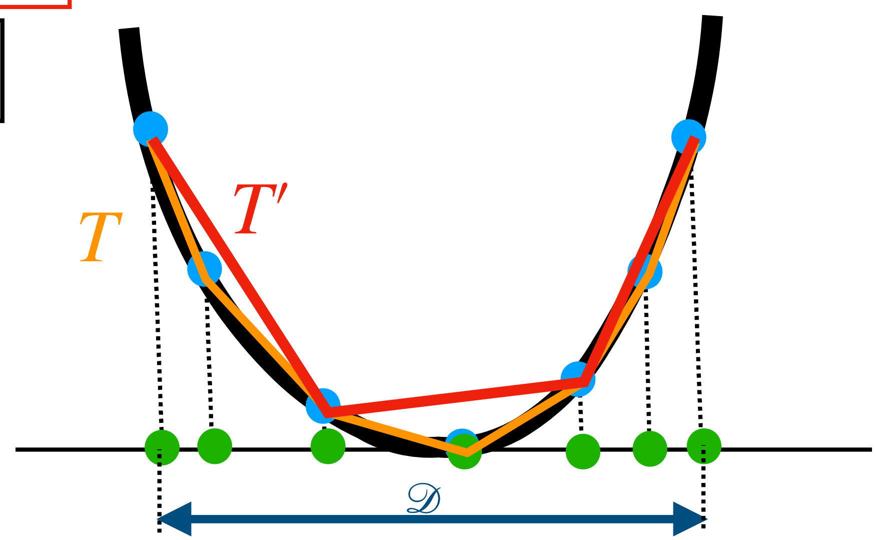
Delaunay : variational definitions

Triangulation T is Delaunay iff.:

$$\forall T', \left(\int_{\mathscr{D}} \delta_T(x)^p dx \right)^{1/p} \leq \left(\int_{\mathscr{D}} \delta_{T'}(x)^p dx \right)^{1/p}$$

Long Chen and Jin-chao Xu. Optimal delaunay triangulations. *Journal of Computational Mathematics*, pages 299–308, 2004.





T minimum along the T' that triangulates $\mathscr{D}$

Delaunay : variational definitions

$$w_p(\tau) = \left(\int_{|\tau|} \delta_\tau(x)^p dx \right)^{\frac{1}{p}}$$

Triangulation T is Delaunay iff.:

$$\forall T', \sum_{\tau \in T} w_p(\tau)^p \leq \sum_{\tau \in T'} w_p(\tau)^p$$

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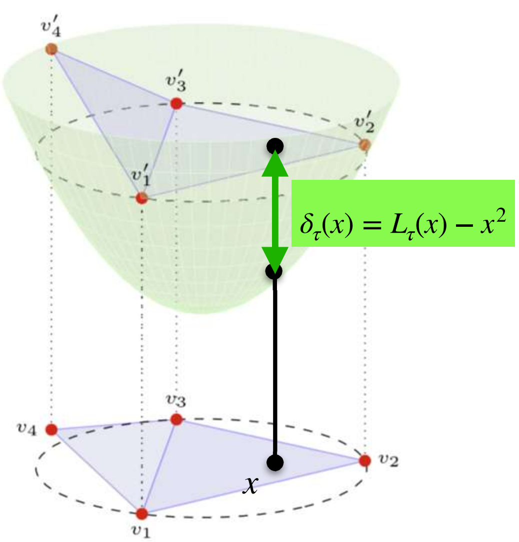

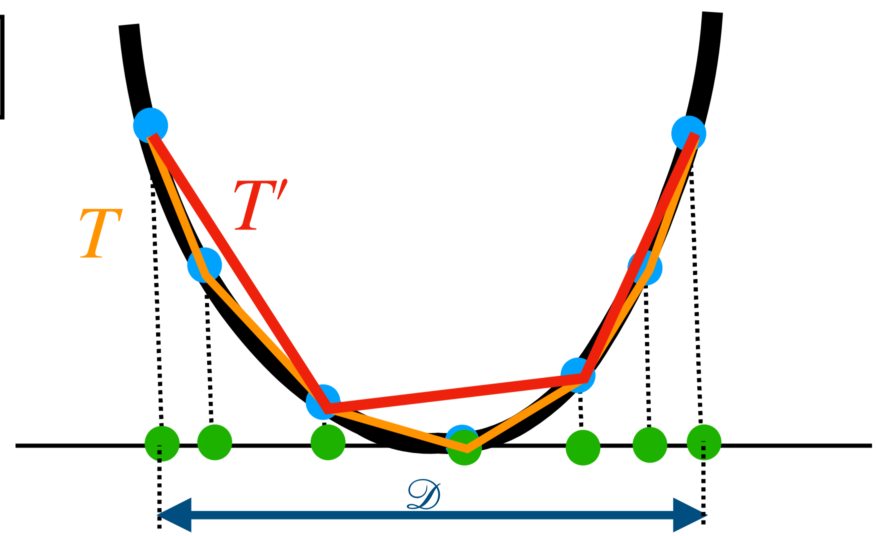

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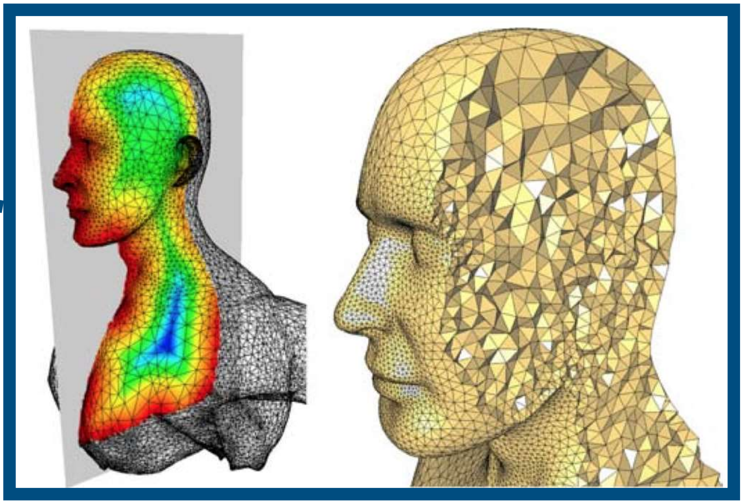
Variational definition of Delaunay
=> triangulation optimization :

Pierre Alliez, David Cohen-Steiner, Mariette Yvinec, and Mathieu Desbrun. Variational tetrahedral meshing. *ACM Transactions on Graphics (TOG)*, 24(3):617–625, 2005.

L. Chen and M. Holst. Efficient mesh optimization schemes based on optimal delaunay triangulations. *Computer Methods in Applied Mechanics and Engineering*, 200(9):967–984, 2011.



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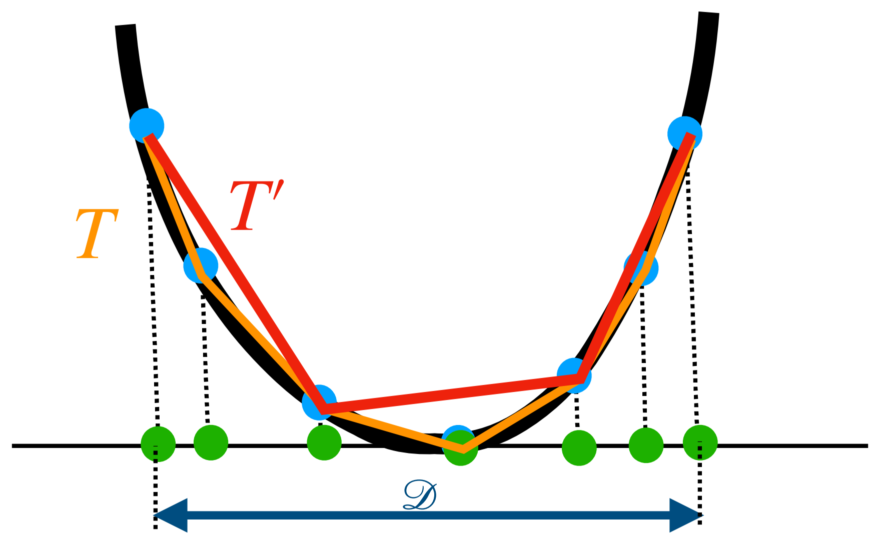

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$$w_p(\tau) = \left(\int_{|\tau|} \delta_\tau(x)^p dx \right)^{\frac{1}{p}}$$

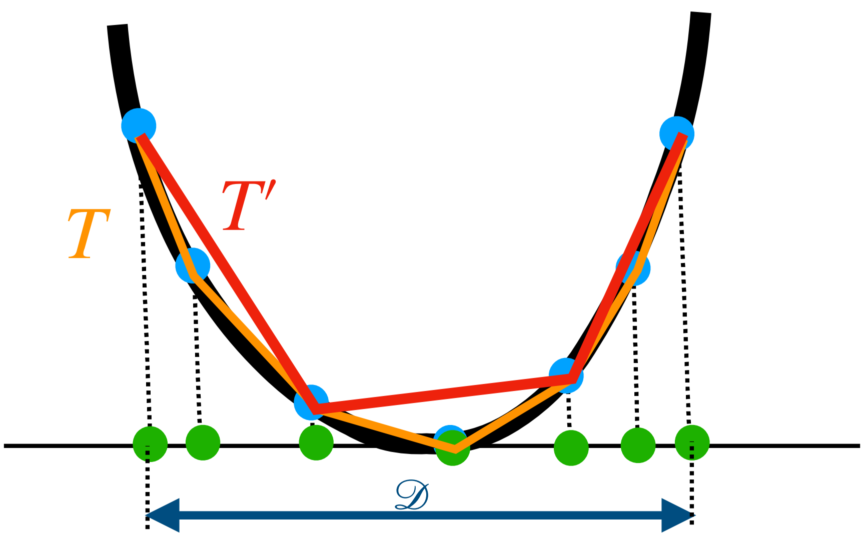
Triangulation T is Delaunay iff.:

$$\forall T', \sum_{\tau \in T} w_p(\tau)^p \leq \sum_{\tau \in T'} w_p(\tau)^p$$

K = $\{$all simplices of dimension ay most $d\}$

The triangulation defines a particular chain in K satisfying the boundary condition $\partial\Gamma = \partial\mathscr{D}$
and:

$$\Gamma(\tau) = \begin{cases} 1 & \text{if } \tau \in T \\ 0 & \text{if } \tau \notin T \end{cases}$$



T minimum along the T' that triangulates $\mathscr{D}$

Delaunay as linear programming

When L[1] minimal chain is Delaunay

$$w_p(\tau) = \left(\int_{|\tau|} \delta_\tau(x)^p dx \right)^{\frac{1}{p}}$$

(Support of) chain Γ is Delaunay iff.:

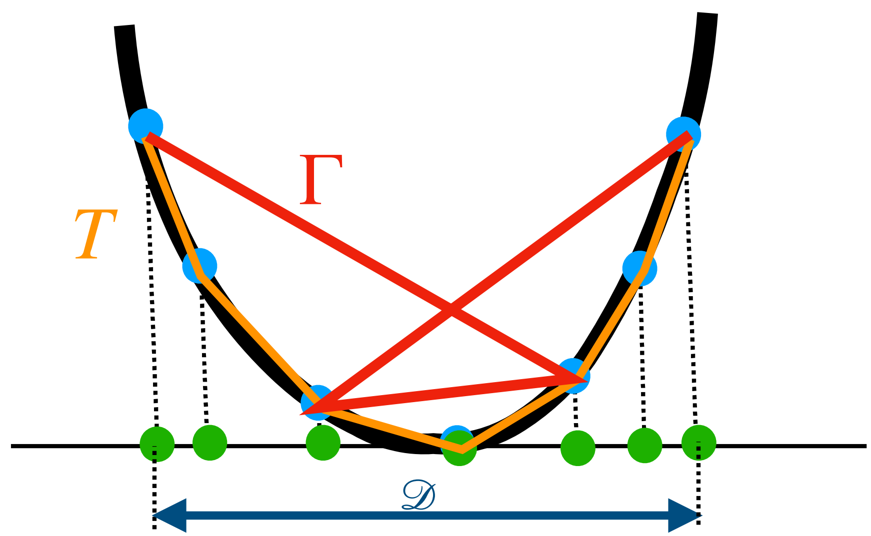
$$\forall \Gamma', \sum_\tau |\Gamma(\tau)| w_p(\tau)^p \leq \sum_\tau |\Gamma'(\tau)| w_p(\tau)^p$$

(among chains Γ' such that $\partial\Gamma' = \partial\mathscr{D}$)

K = $\{$all simplices of dimension ay most $d\}$

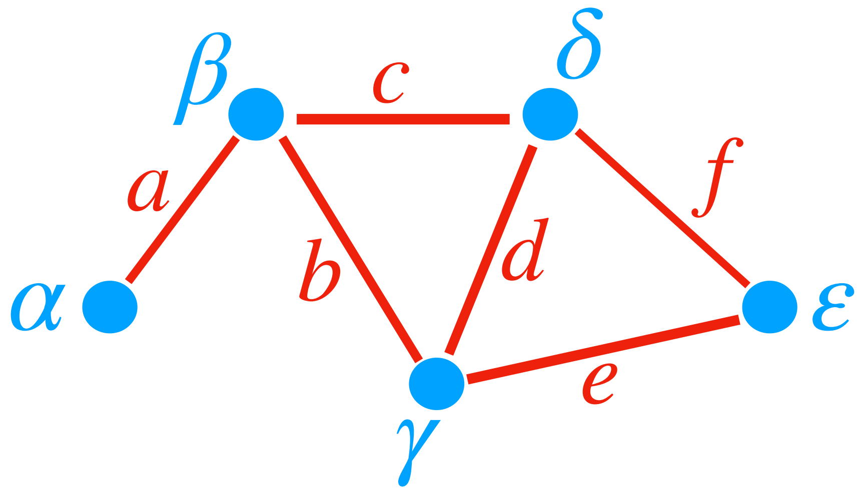
The triangulation defines a particular chain in K satisfying the boundary condition $\partial\Gamma = \partial\mathscr{D}$ and:

$$\Gamma(\tau) = \begin{cases} 1 & \text{if } \tau \in T \\ 0 & \text{if } \tau \notin T \end{cases}$$



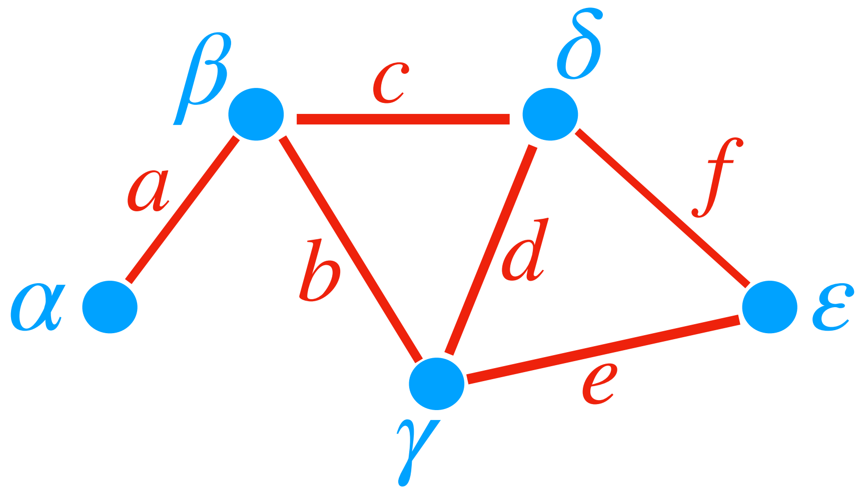
T minimum along the chains Γ such that $\partial\Gamma = \partial\mathscr{D}$

shortest path: linear algebra formulation



Is there a path between α and ε ?

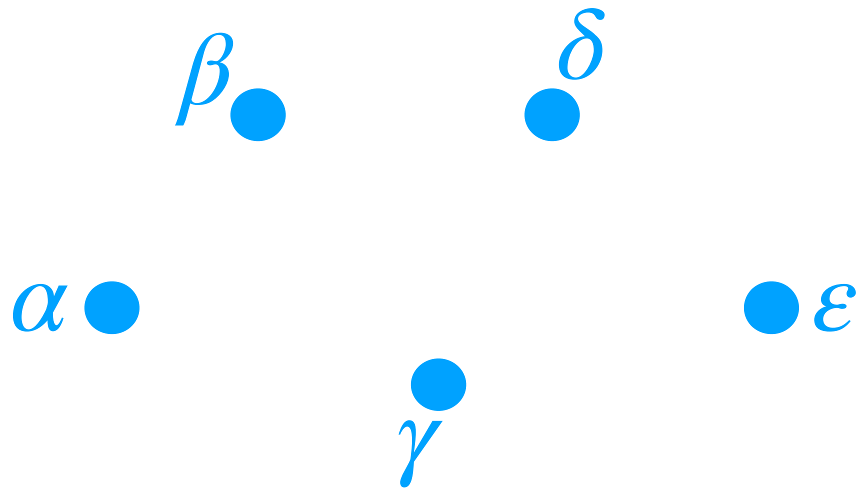
shortest path: linear algebra formulation



Is there a path between α and ε ?

A (linear) **algebra** formulation of this question ?

shortest path: linear algebra formulation



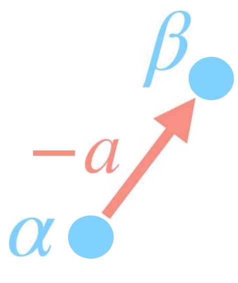
Vector space of 0-chains:

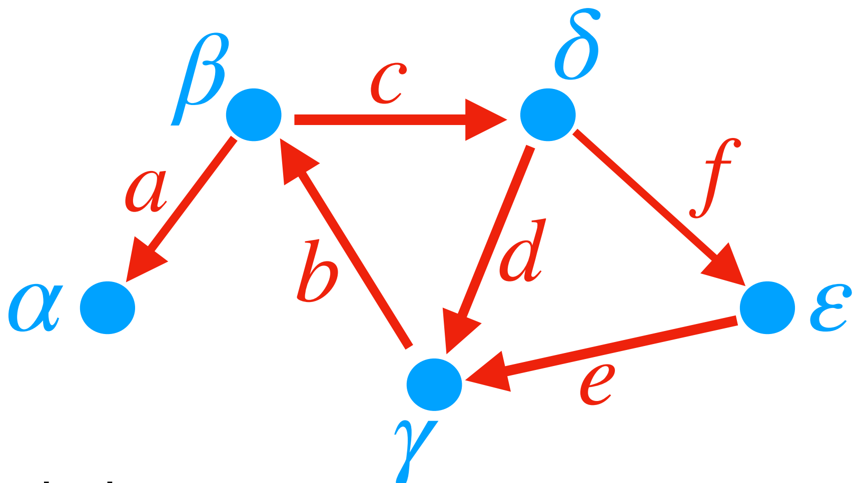
$$C_0 = \left\{ Y_\alpha \alpha + Y_\beta \beta + Y_\gamma \gamma + Y_\delta \delta + Y_\varepsilon \varepsilon \mid Y \in \mathbb{R}^5 \right\}$$

(basis = 0-simplices)

(Think of a sum of weighted Diracs if you prefer)

shortest path: linear algebra formulation





Vector space of 0-chains:

$$C_0 = \left\{ Y_\alpha \alpha + Y_\beta \beta + Y_\gamma \gamma + Y_\delta \delta + Y_\varepsilon \varepsilon \mid Y \in \mathbb{R}^5 \right\}$$

(basis = 0-simplices)

Vector space of 1-chains:

$$C_1 = \left\{ X_a a + X_b b + X_c c + X_d d + X_e e + X_f f \mid X \in \mathbb{R}^6 \right\}$$

(basis = ``oriented" 1-simplices)

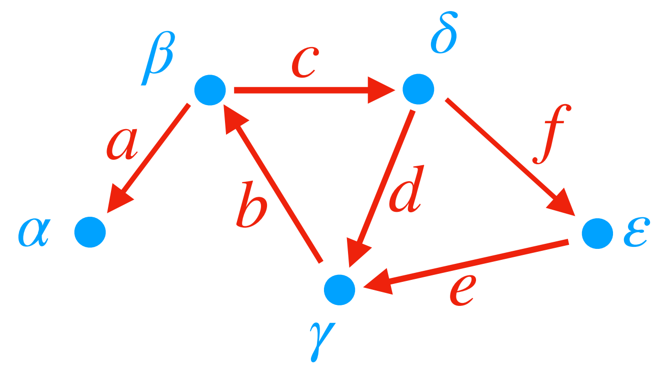
shortest path: linear algebra formulation

$$C_0 = \left\{ Y_\alpha \alpha + Y_\beta \beta + Y_\gamma \gamma + Y_\delta \delta + Y_\varepsilon \varepsilon \mid Y \in \mathbb{R}^5 \right\}$$

(basis = 0-simplices)

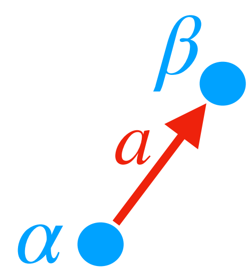
$$C_1 = \left\{ X_a a + X_b b + X_c c + X_d d + X_e e + X_f f \mid X \in \mathbb{R}^6 \right\}$$

(basis = "oriented" 1-simplices)

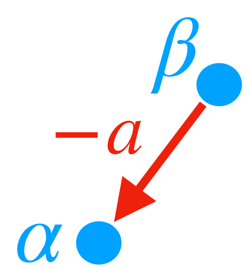


Boundary linear operator:

$$\partial : C_1 \to C_0$$



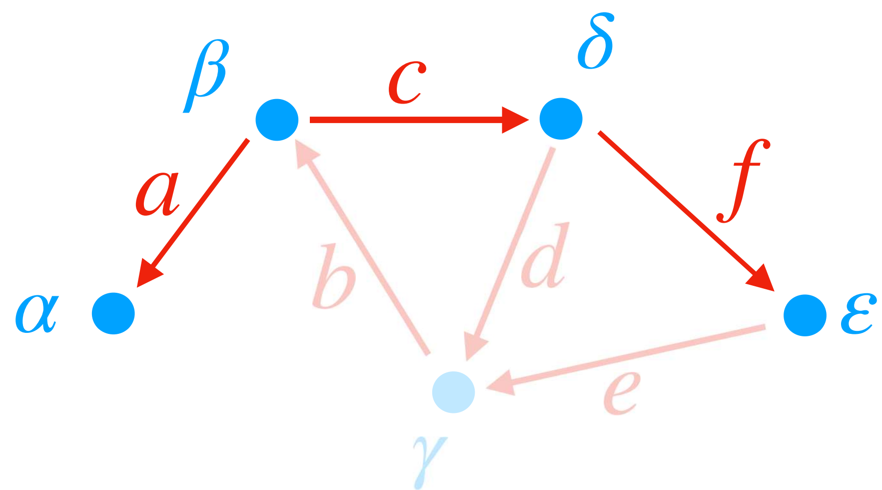
$$\partial a = \beta - \alpha$$



$$\partial(-a) = -\partial a = \alpha - \beta$$

shortest path: linear algebra formulation

$$\partial : C_1 \to C_0$$



$$\partial(-a + c + f) = (\beta - \alpha) + (\delta - \beta) + (\varepsilon - \delta)$$
$$= \varepsilon - \alpha$$

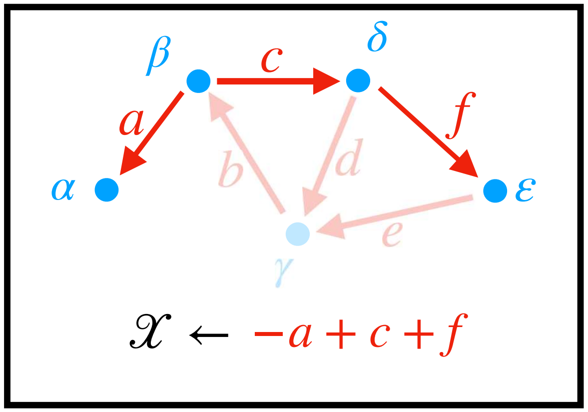
Is there a path between α and ε ?

Yes: $-a + c + f$

shortest path: linear algebra formulation

There a path between α and ε

$$\iff \exists \mathscr{X} \in C_1 \mid \partial \mathscr{X} = \varepsilon - \alpha$$



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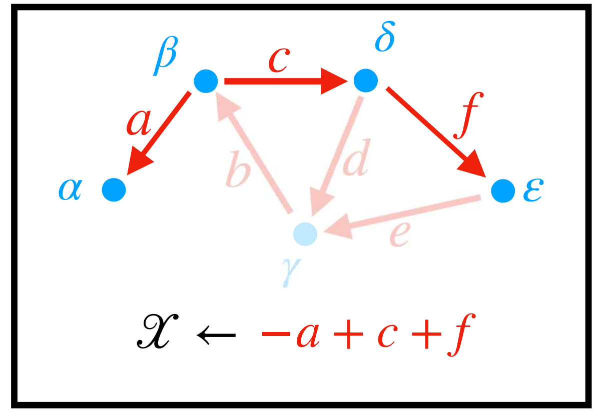
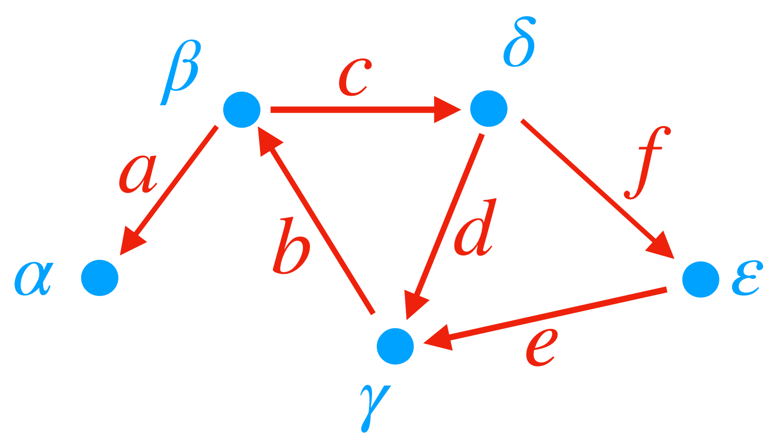
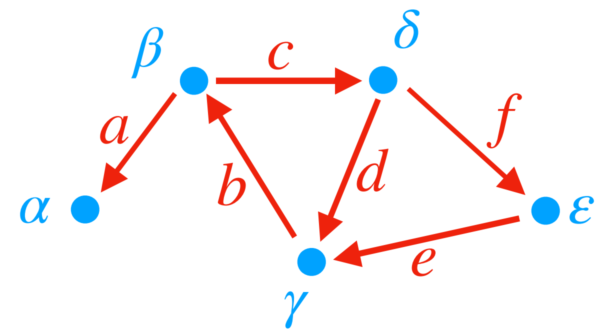


$$\partial = \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \end{array} \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{array}$$



shortest path: linear algebra formulation

There a path between α and ε

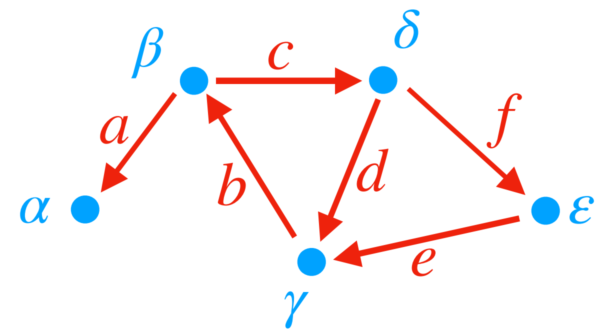


$$\iff \exists \mathscr{X} \in C_1 \mid \partial \mathscr{X} = \varepsilon - \alpha$$

$$\iff \exists \mathscr{X} \in C_1 \mid \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_a \\ X_b \\ X_c \\ X_d \\ X_e \\ X_f \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

shortest path: linear algebra formulation

There a path between α and ε



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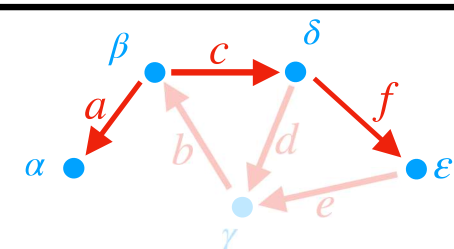
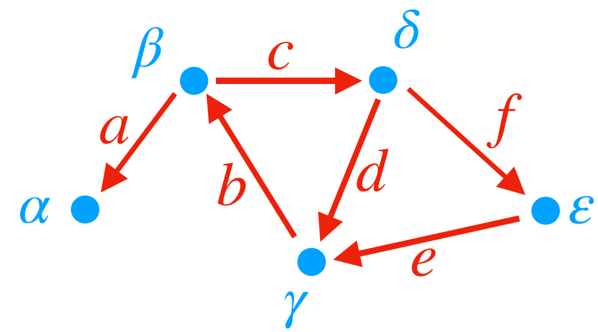


$$\mathscr{X} \leftarrow -a + c + f$$

shortest path: linear algebra formulation



There a path between α and ε

$$\Longleftrightarrow \quad \exists \mathscr{X} \in C_1 \mid \partial \mathscr{X} = \varepsilon - \alpha$$

$$\Longleftrightarrow \quad \exists \mathscr{X} \in C_1 \mid \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_a \\ X_b \\ X_c \\ X_d \\ X_e \\ X_f \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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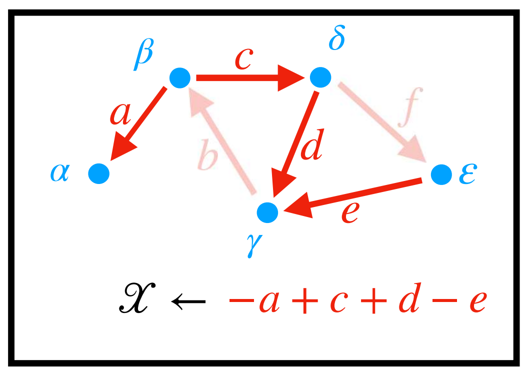


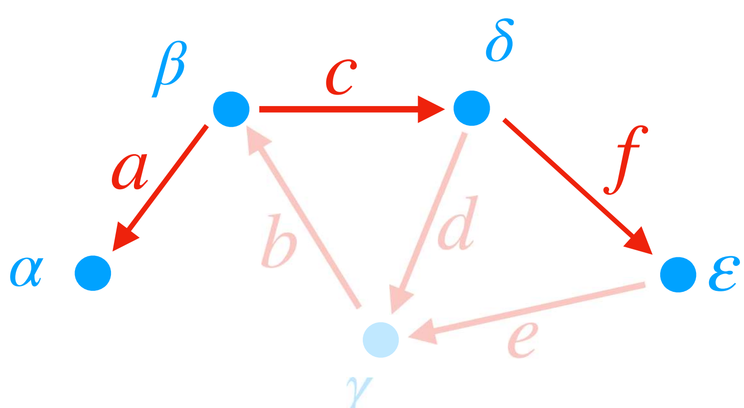
shortest path: linear algebra formulation

Shortest path between α and ε :

Length of a path $\mathscr{X}$ between α and ε :

$$\text{length}(\mathscr{X}) = \sum_{\text{edges}} \left| \mathscr{X}(\text{edge}) \right| \text{ length}(\text{edge})$$

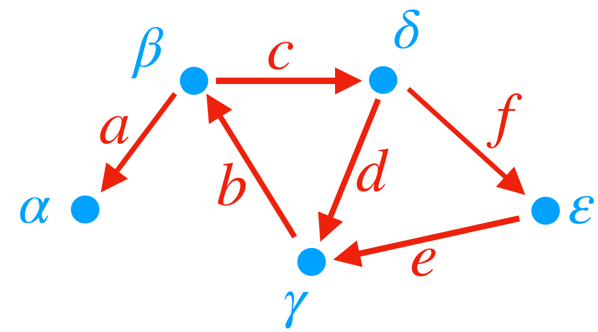
(this is a (weighted) L^1 norm on vector $\mathscr{X}$)



$\mathscr{X} \leftarrow -a + c + f$

shortest path: linear algebra formulation

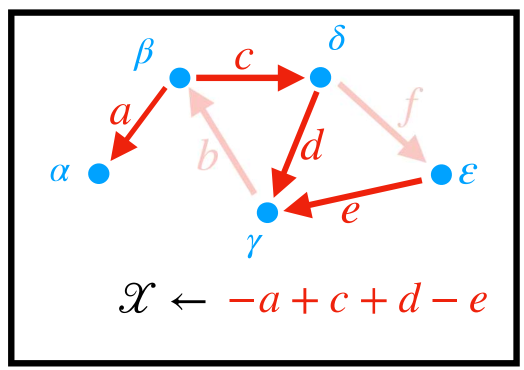
Shortest path between α and ε :



$$\min_{\partial \mathscr{X} = \varepsilon - \alpha} \sum_{\text{edges}} \left| \mathscr{X}(\text{edge}) \right| \text{ weight(edge)}$$

this is a (weighted) L^1 norm

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$



shortest path: linear algebra formulation

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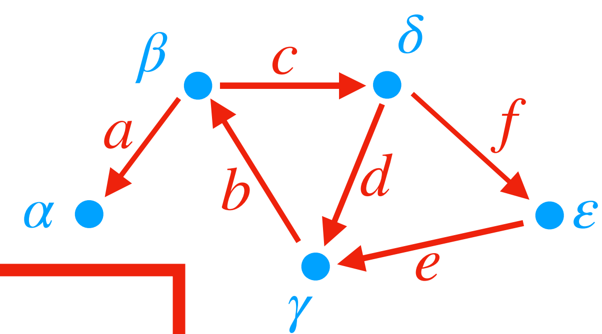
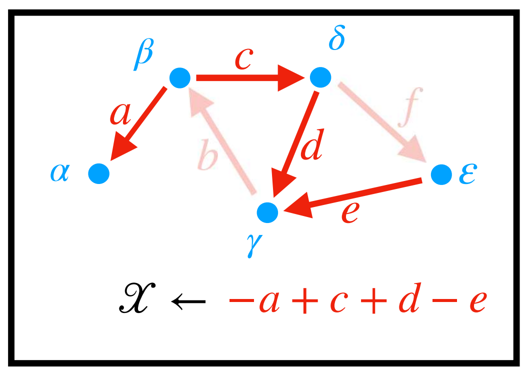


$$\min_{\partial \mathscr{X} = \varepsilon - \alpha} \sum_{\text{edges}} \left| \mathscr{X}(\text{edge}) \right| \, \text{weight}(\text{edge})$$

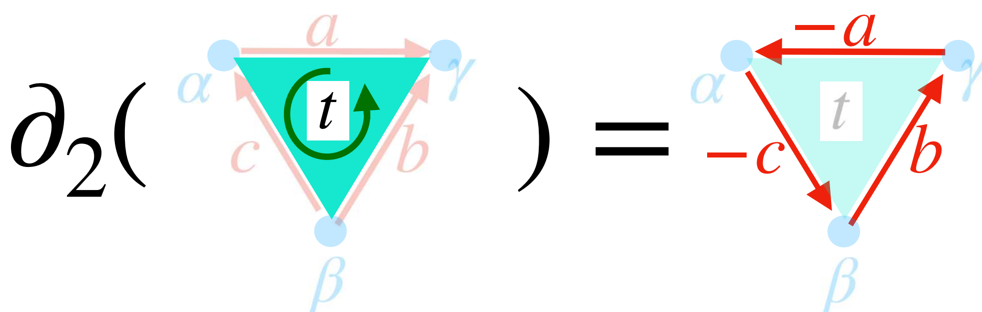
Linear programming (i.e. simplex algorithm when the field is $\mathbb{R}$)

(but Djikstra is much faster !)

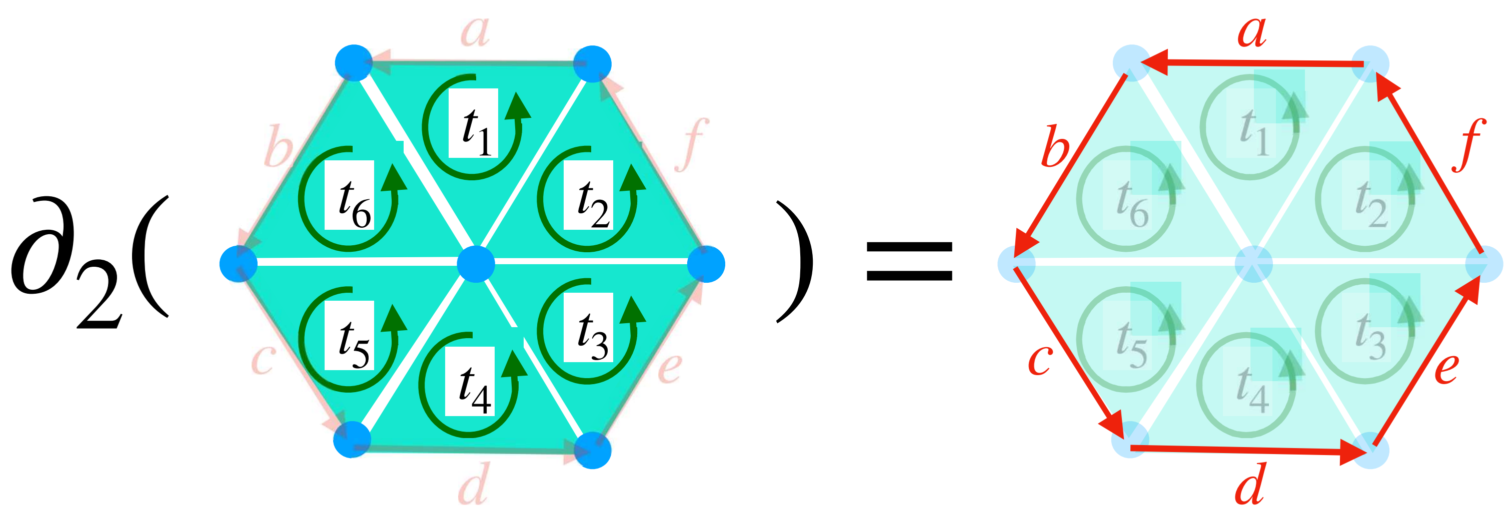
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$



boundary operator ∂_2



$$\partial_2 t = -c + b - a$$



$$\partial_1 \circ \partial_2 = 0$$

$$\partial_2 (t_1 + t_2 + t_3 + t_4 + t_5 + t_6) = a + b + c + d + e + f$$

Delaunay as linear programming

When L¹ minimal chain is Delaunay

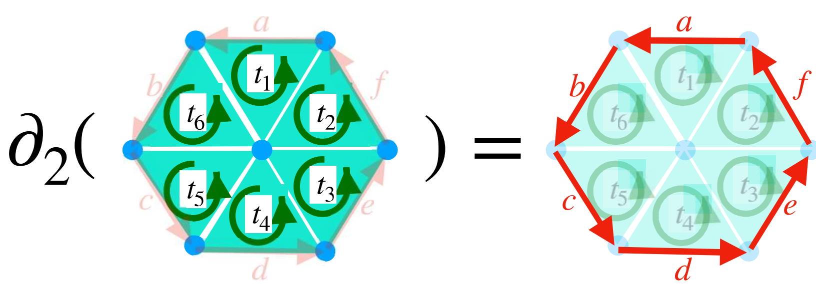
(Support of) chain Γ is Delaunay iff.:

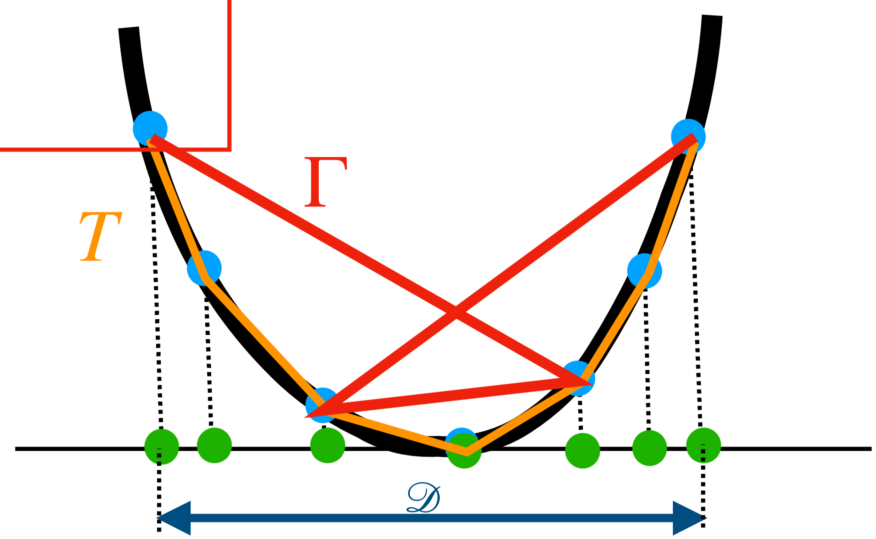
$$\forall \Gamma', \sum_{\tau} |\Gamma(\tau)| w_p(\tau)^p \leq \sum_{\tau} |\Gamma'(\tau)| w_p(\tau)^p$$

(among chains Γ' such that $\partial\Gamma' = \partial\mathscr{D}$)

$$w_p(\tau) = \left(\int_{|\tau|} \delta_\tau(x)^p dx \right)^{\frac{1}{p}}$$

The triangulation defines a particular chain satisfying the boundary condition





T minimum along the chains Γ such that $\partial\Gamma = \partial\mathscr{D}$

Delaunay as linear programming

When L[1] minimal chain is Delaunay

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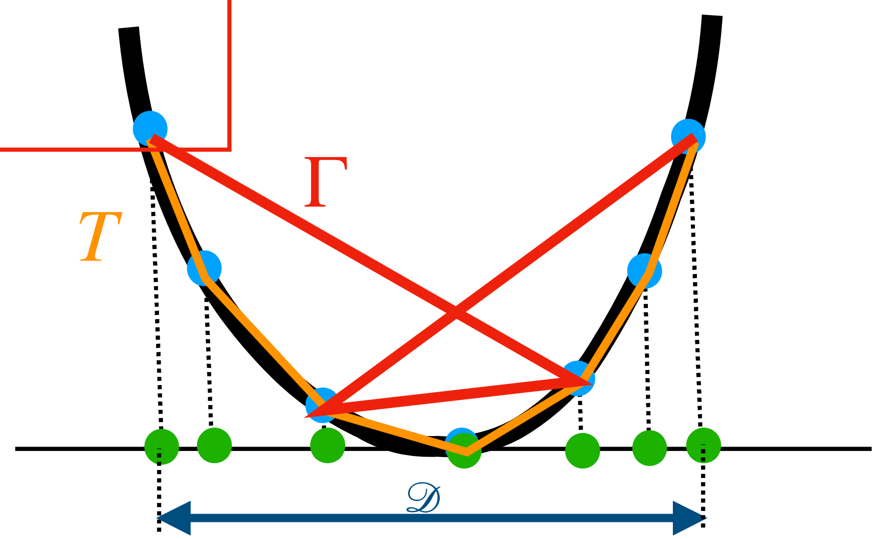
$$w_p(\tau) = \left(\int_{|\tau|} \delta_\tau(x)^p dx \right)^{\frac{1}{p}}$$

Define the following norm on chains:

$$\|\Gamma\|_p = \sum_{\sigma \in K_d} w_p(\tau)^p |\Gamma(\tau)|$$

Still a L^1 norm : exponent p is on the weight, not on the coordinate.



T minimum along the chains Γ such that $\partial \Gamma = \partial \mathscr{D}$

Delaunay as linear programming

Delaunay triangulation

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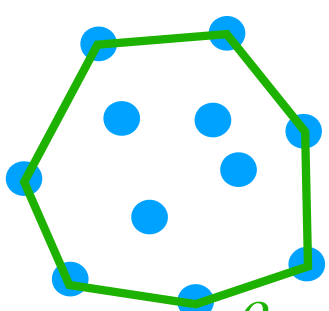
$$w_p(\sigma) = \left(\int_{|\sigma|} \delta_\sigma(x)^p dx \right)^{\frac{1}{p}} = \|\delta_\sigma\|_p$$

(Attali, L., 2016)

Let $\mathbf{P} \subset \mathbb{R}^d$ be a finite set of points.

*Let $\beta_{\mathbf{P}}$ be a (d-1)-**cycle** whose **support** is the boundary of the convex hull of $\mathbf{P}$*

*The support of the chain that minimizes $\Gamma \mapsto \|\Gamma\|_p$ under constraint $\partial\Gamma = \beta_{\mathbf{P}}$ is the **Delaunay triangulation** of $\mathbf{P}$*



Delaunay as linear programming

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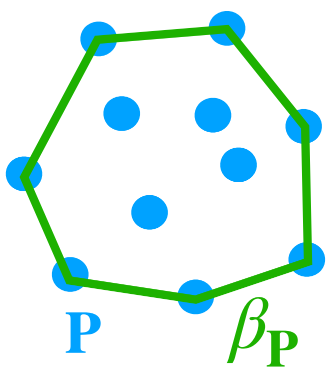
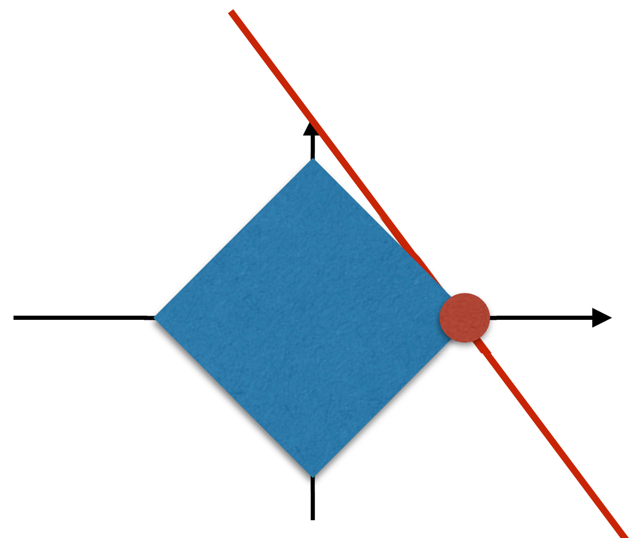
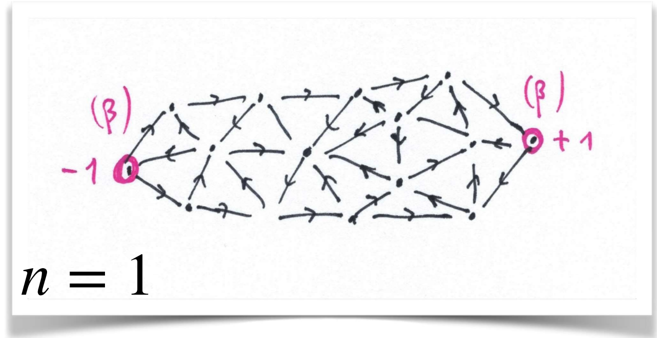




Minimal chain under boundary constraint (real coefficients)

Minimal chain for a given boundary β

$$\underset{x,\ \partial x=\beta}{\arg\min} \|x\|_1$$

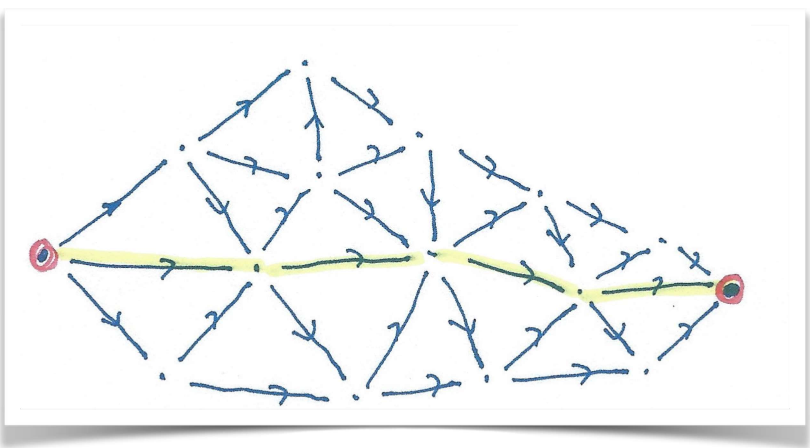


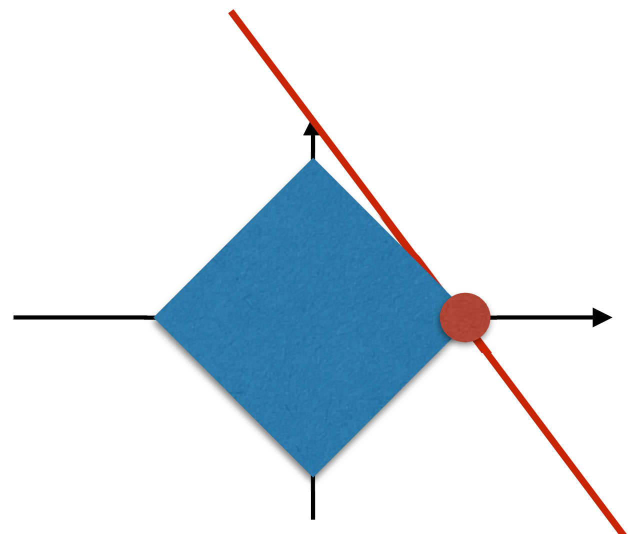
L¹ minima are sparse

Minimizing L¹ norm :
=> **shortest path**

$$\sum_j l_j |I_j|$$

(Path length)





Minimal chain under boundary constraint (real coefficients)

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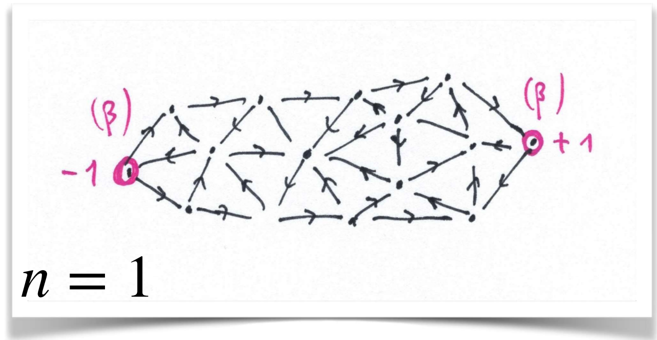
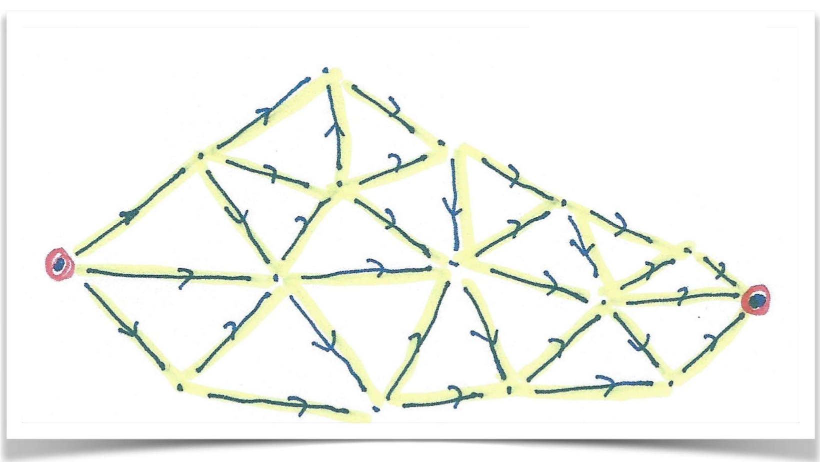
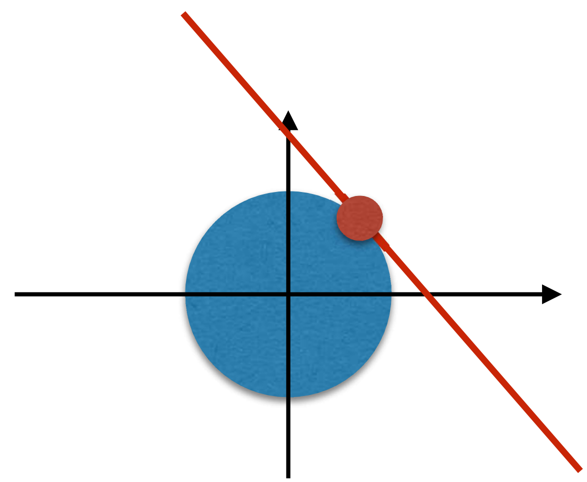
$$\underset{x,\ \partial x = \beta}{\arg\min}\ \|x\|_2$$



L² minima are not sparse

Minimizing L² norm
=> **harmonic form**:





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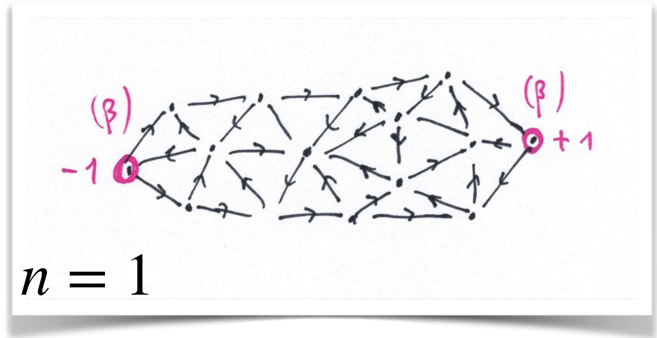
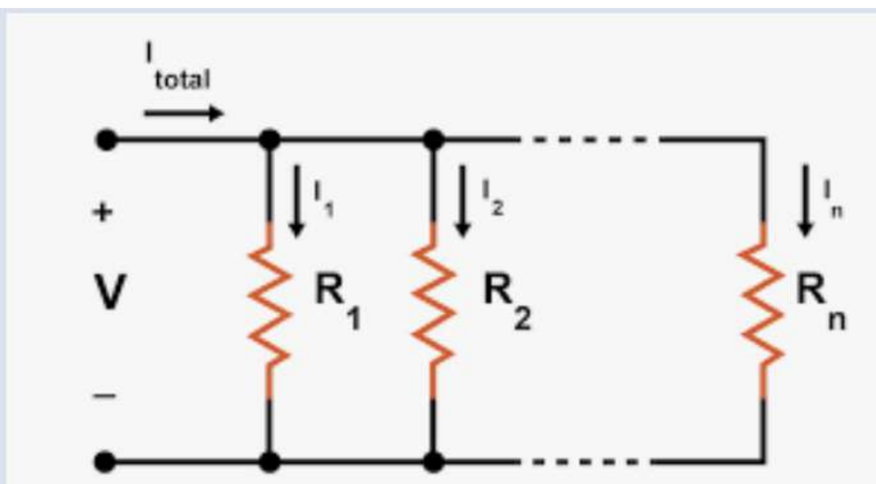


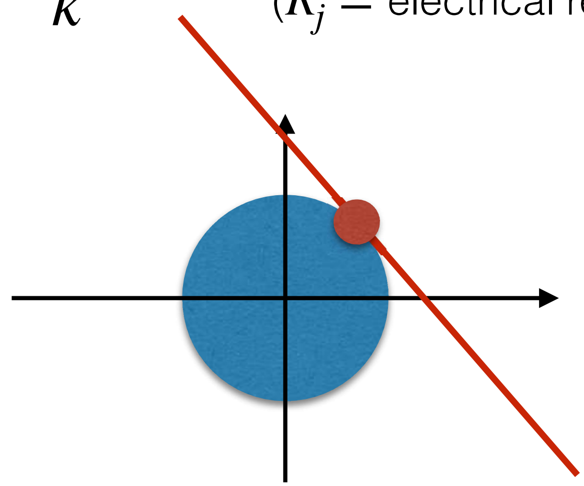
L² minima are not sparse

Minimizing L² norm
=> **harmonic form**:

$$P = \sum_k R_k I_k^2$$

(R_j = electrical resistance)





Delaunay as linear programming

Delaunay triangulation

$$\|\Gamma\|_p = \sum_{\sigma \in K_d} w_p(\tau)^p \, |\Gamma(\tau)|$$

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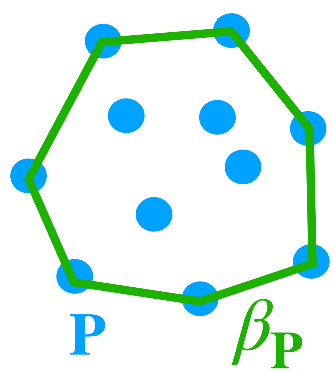

Behavior as $p \to \infty$?

Delaunay order

When lexicographic-minimal chain is Delaunay

Behavior as $p \to \infty$?

$$w_p(\sigma) = \left(\int_{|\sigma|} \delta_\sigma(x)^p dx \right)^{\frac{1}{p}} = \|\delta_\sigma\|_p$$

The weights w_p defines **a preorder $\leq_\infty$ on simplices**:

$$\sigma_1 \leq_\infty \sigma_2 \underset{def.}{\iff} \quad \exists p \in [1, \infty[, \quad \forall p' \in [p, \infty[, \quad w_{p'}(\sigma_1) \leq w_{p'}(\sigma_2)$$

Delaunay order

When lexicographic-minimal chain is Delaunay

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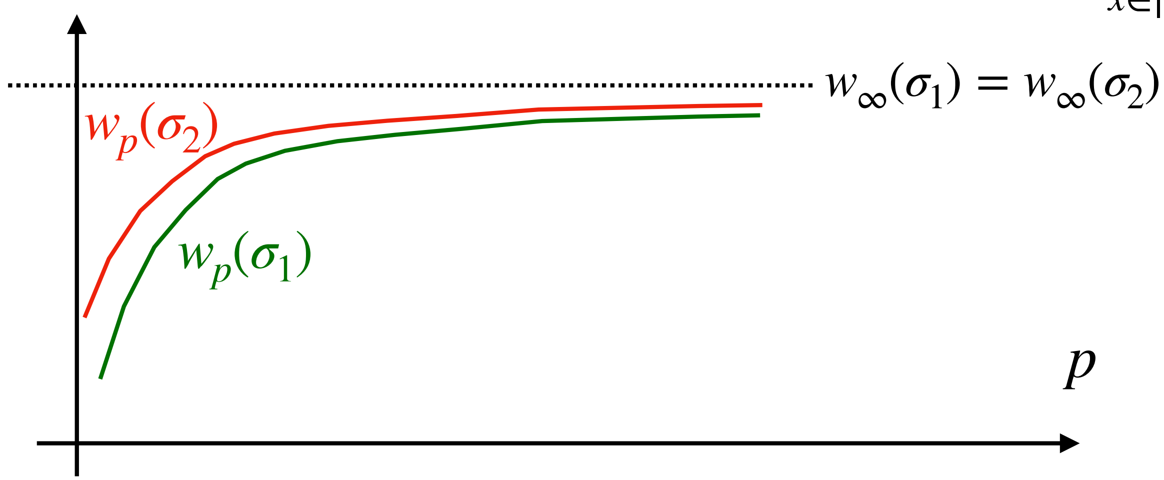
$\leq_\infty$ is a **finer (pre-)order** than comparing $w_\infty = \|\delta_\sigma\|_\infty = \max_{x \in |\sigma|} \delta_\sigma(x) = \lim_{p \to \infty} w_p$



Delaunay order

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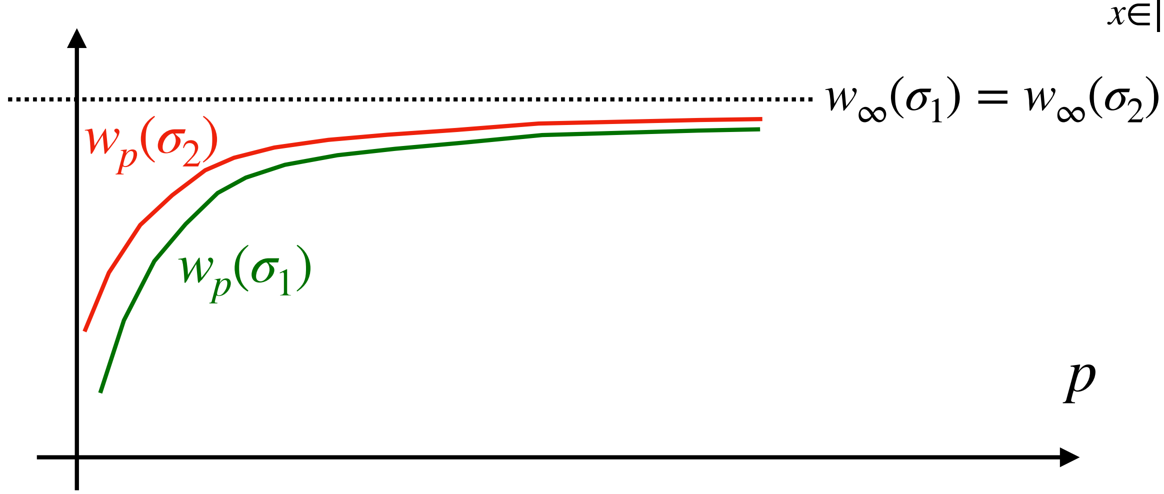

$w_\infty(\sigma_1) = w_\infty(\sigma_2)$
but
$\sigma_1 \leq_\infty \sigma_2$
and
$\sigma_2 \not\leq_\infty \sigma_1$

Delaunay order

When lexicographic-minimal chain is Delaunay

Behavior as $p \to \infty$?

$$w_p(\sigma) = \left(\int_{|\sigma|} \delta_\sigma(x)^p dx \right)^{\frac{1}{p}} = \|\delta_\sigma\|_p$$

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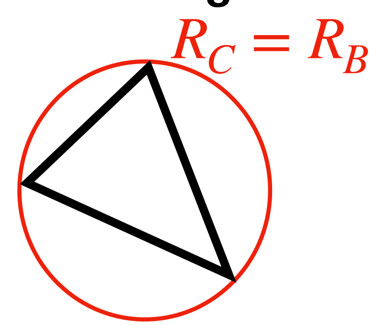
For 2-simplices, under a **generic** condition, one has:

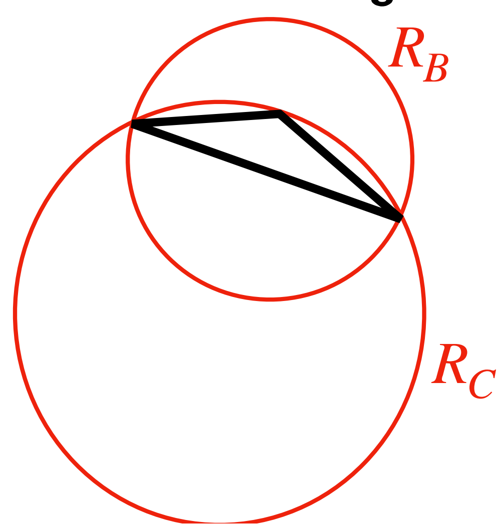
Lemma 7.4. *If Condition 1 holds, $\leq_\infty$ is a total order on the set of 2-simplices of K with:*

$$\sigma_1 \leq_\infty \sigma_2 \iff \begin{cases} \mathrm{R_B}(\sigma_1) < \mathrm{R_B}(\sigma_2) \\ \text{or} \\ \mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2) \quad \text{and} \quad \mathrm{R_C}(\sigma_1) \geq \mathrm{R_C}(\sigma_2) \end{cases}$$

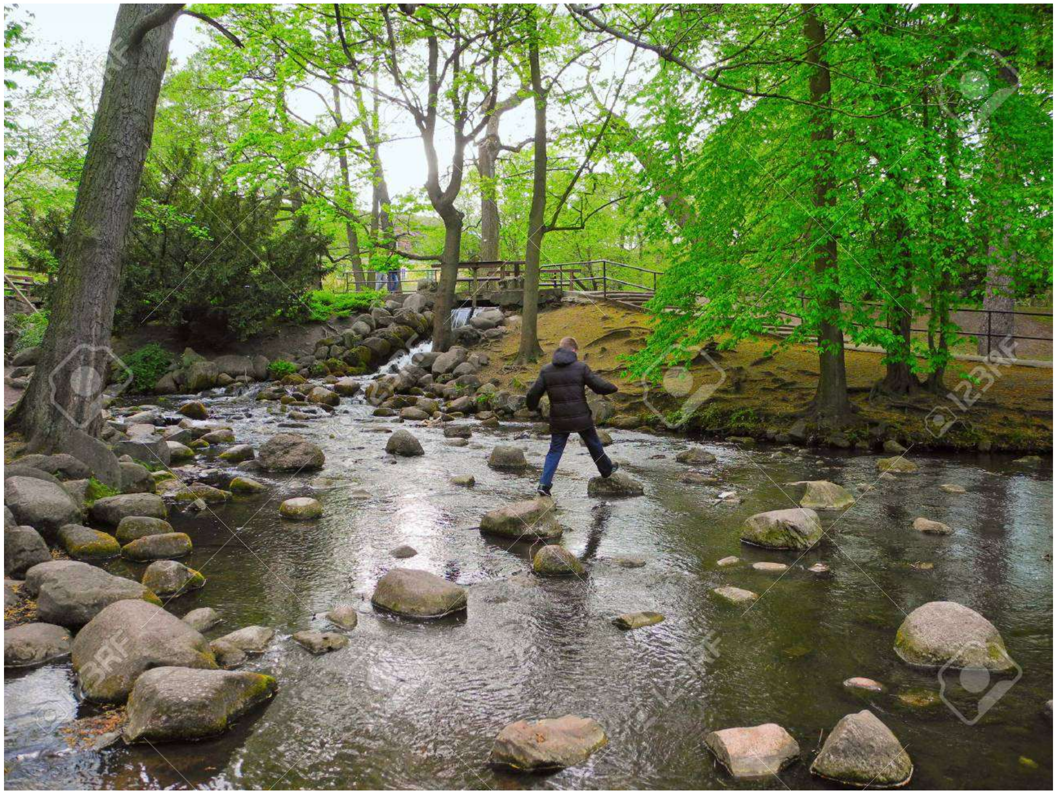
Acute triangle:



Obtuse triangle:

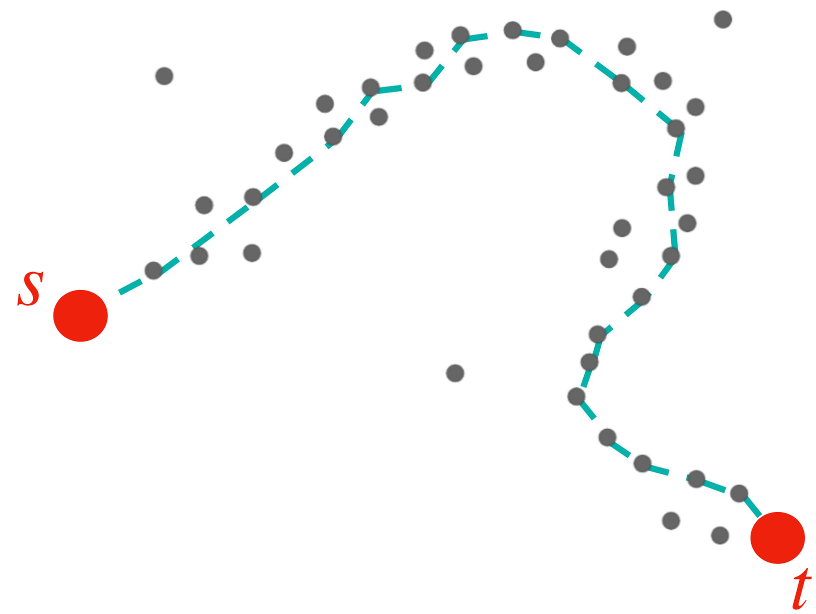


Lexicographic order



Lexicographic order minimal 1-chain

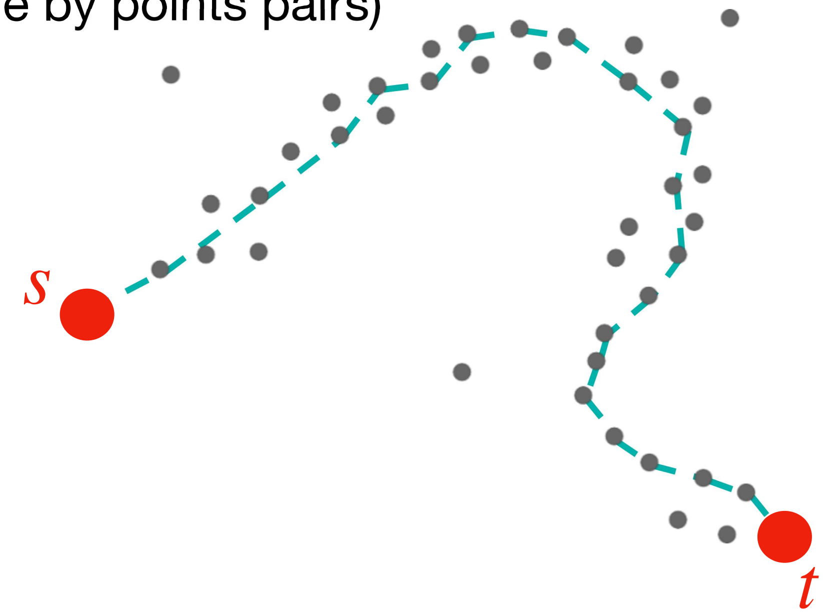
Connect ~~the~~ some dots to form a path between s and t



Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

<u>Objective</u>: find path going through “**densest**” parts of the point cloud.
1D simplicial complex = **Complete graph** (= one edge by points pairs)



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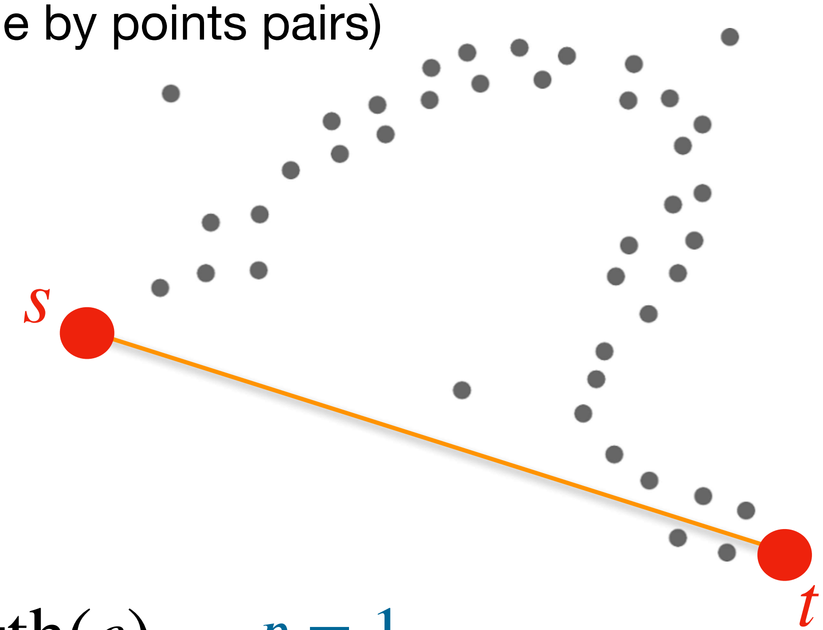

<u>Classic graph problem</u>:
Find minimal path for given edge weights (***Dijkstra****'s algorithm)*

$$\arg\min_{\partial\Gamma = s+t} \sum_{e\in\Gamma} \text{length}(e) \qquad p = 1$$

Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

Objective: find path going through "**densest**" parts of the point cloud.
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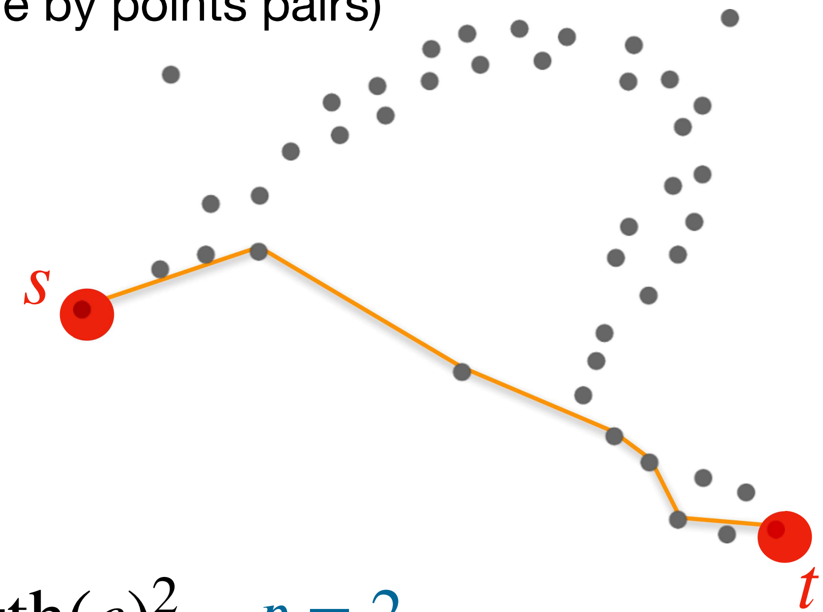

$$\arg \min_{\partial\Gamma = s+t} \sum_{e \in \Gamma} \text{length}(e)^2 \qquad p = 2$$

Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

<u>Objective</u>: find path going through "**densest**" parts of the point cloud.
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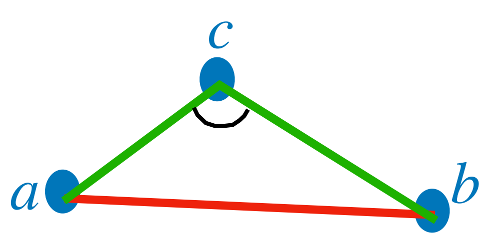
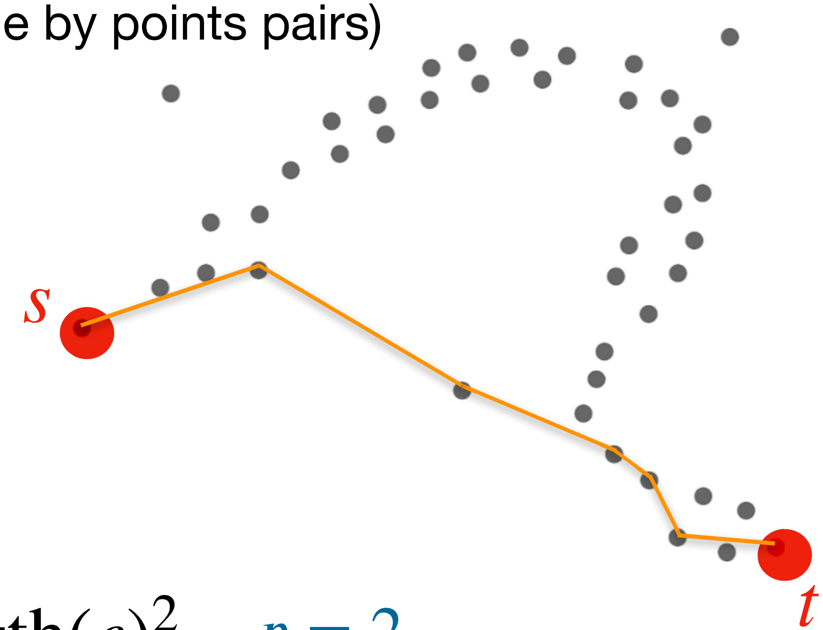
Pythagoras:





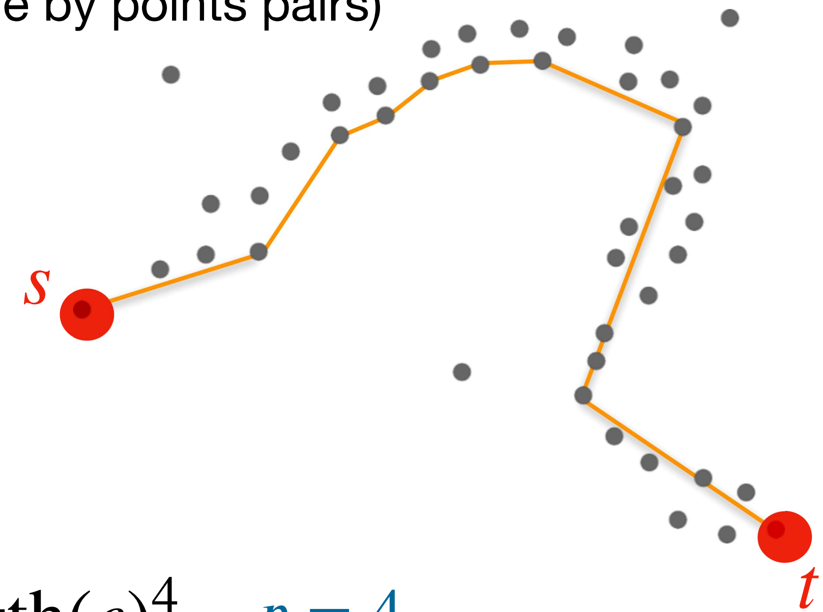
$$\angle acb > \frac{\pi}{2} \Rightarrow ac^2 + cb^2 < ab^2$$

$$\arg \min_{\partial\Gamma = s+t} \sum_{e \in \Gamma} \text{length}(e)^2 \qquad p = 2$$

Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

<u>Objective</u>: find path going through "**densest**" parts of the point cloud.
1D simplicial complex = **Complete graph** (= one edge by points pairs)

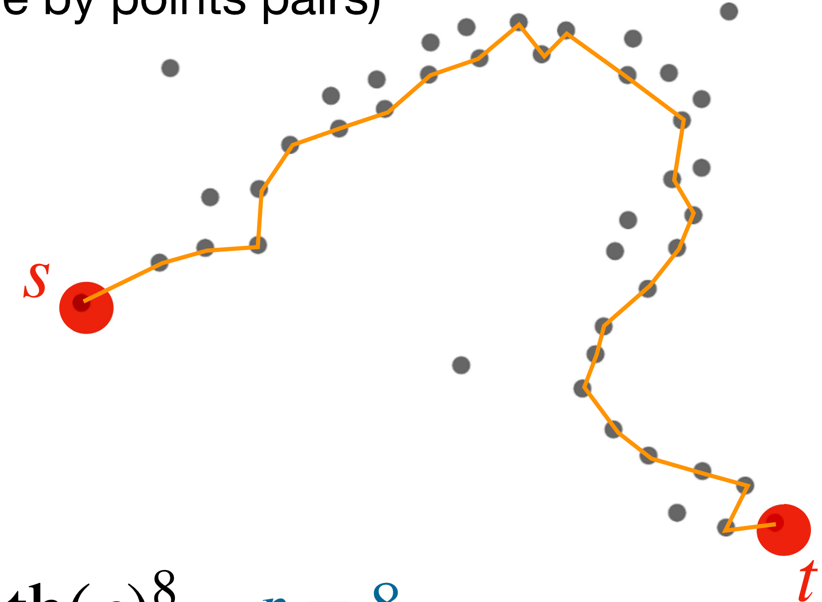


$$\arg\min_{\partial\Gamma = s+t} \sum_{e\in\Gamma} \text{length}(e)^4 \qquad p = 4$$

Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

<u>Objective</u>: find path going through "**densest**" parts of the point cloud.
1D simplicial complex = **Complete graph** (= one edge by points pairs)

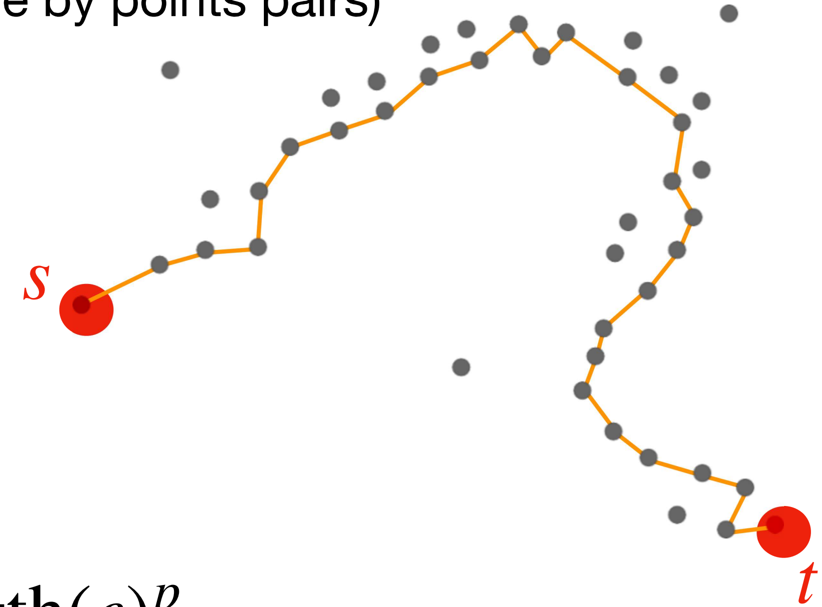


$$\arg \min_{\partial\Gamma = s+t} \sum_{e\in\Gamma} \text{length}(e)^8 \qquad p = 8$$

Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

<u>Objective</u>: find path going through "**densest**" parts of the point cloud.
1D simplicial complex = **Complete graph** (= one edge by points pairs)



$$\arg \min_{\partial\Gamma = s+t} \sum_{e\in\Gamma} \text{length}(e)^p$$

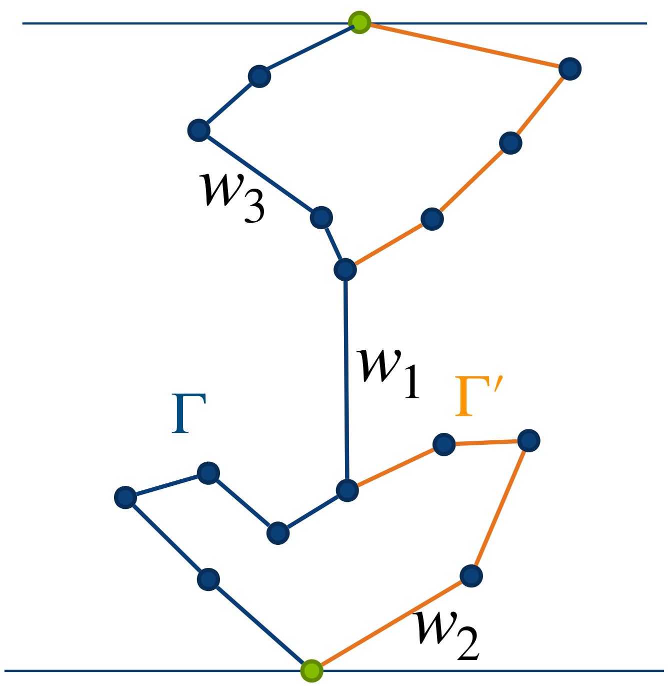
Behavior as $p \to \infty$?

Lexicographic order minimal 1-chain

Limit behavior as $p \to \infty$? : **lexicographic order**

Assume no two edges have same length (generic condition):
Sort edges along decreasing length:

$$w_1 > w_2 > \ldots > w_N \quad \text{, where } w_i = \text{length}(\tau_i)$$



$$\exists p \in \mathbb{N}, \forall i, w_i^p > \sum_{j>i} w_j^p$$

$$\Gamma = \tau_1 + \tau_3 + \ldots$$

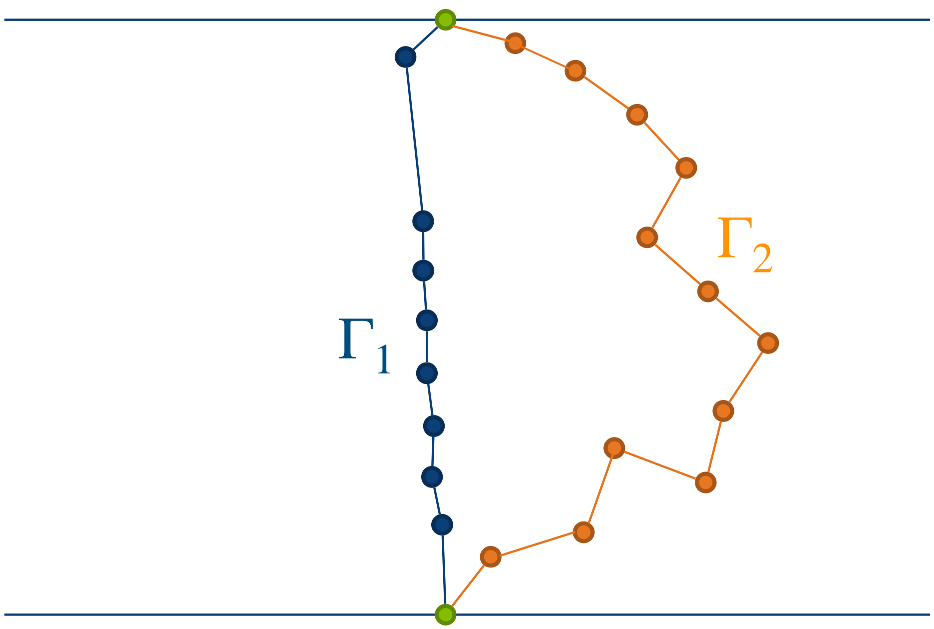
$$\Gamma' = \tau_1 + \tau_2 + \ldots$$

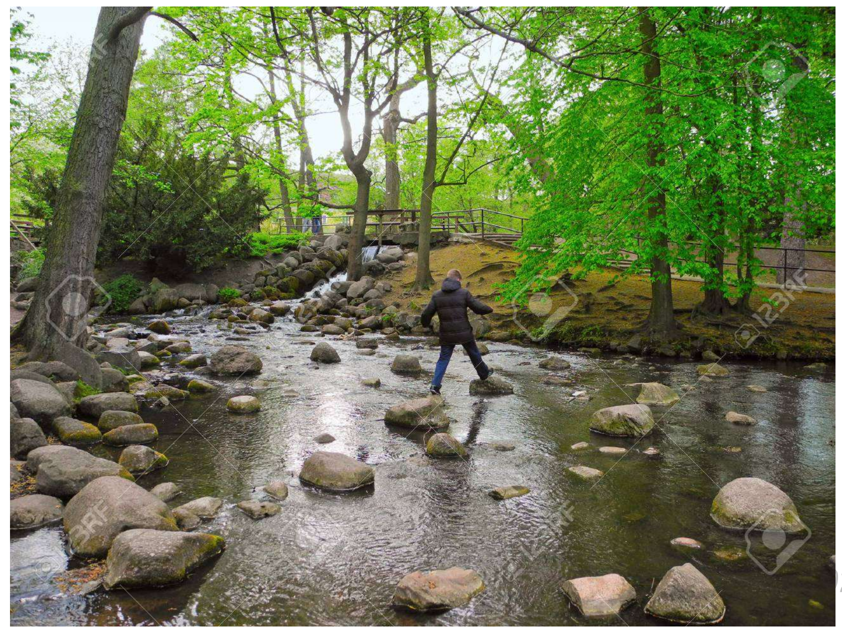
$$\Gamma \sqsubseteq_{lex} \Gamma'$$

Lexicographic order minimal 1-chain

Analogy for lexicographic order: “Rock hopping"

Which path is smaller in the lexicographic order ?





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Lexicographic order minimal 1-chain

Analogy for lexicographic order: “Rock hopping"

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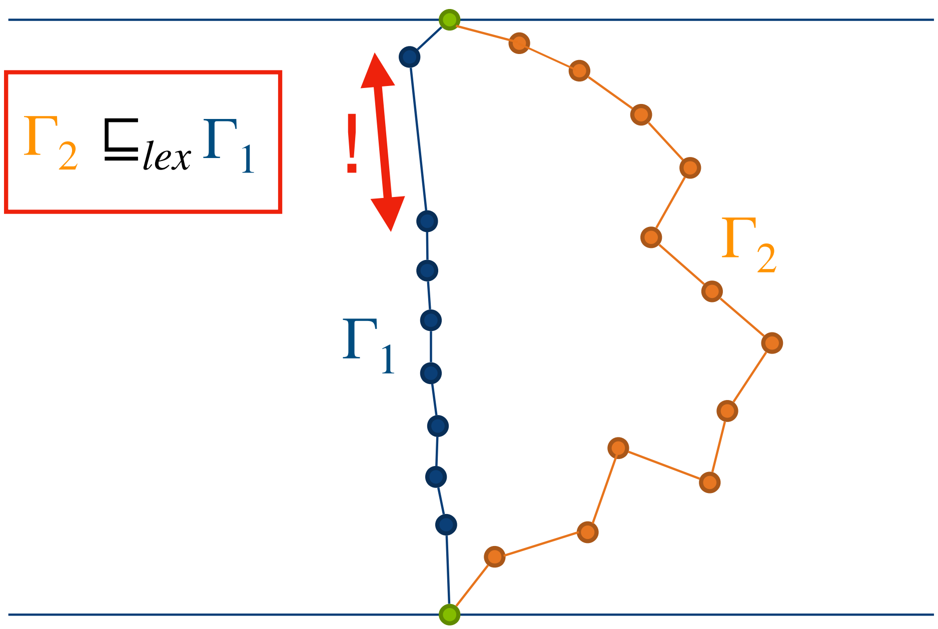

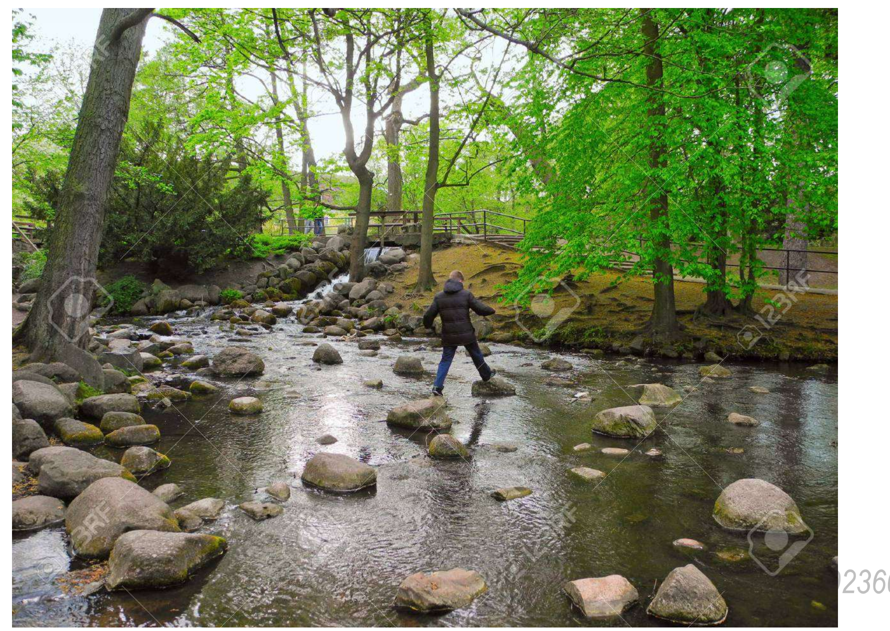

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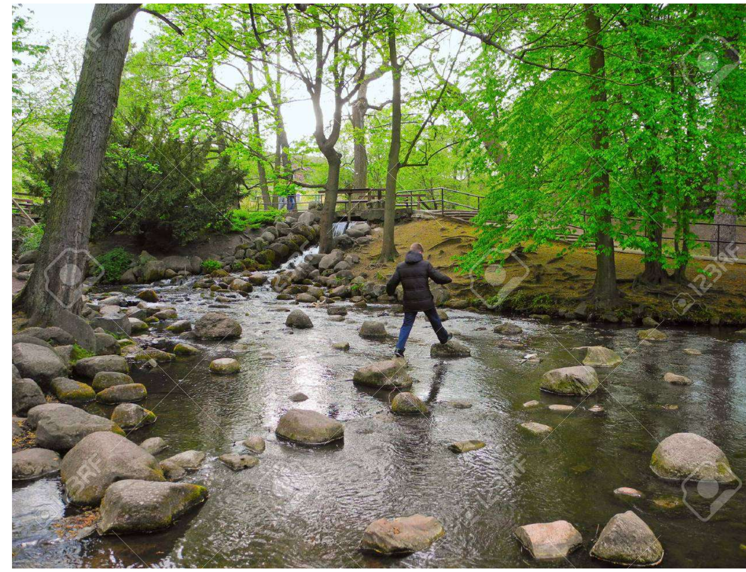
Lexicographic order

When the field is $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, we allow us a small abuse of notation, chains are identified to sets of simplices and :

$$a + b = a - b = (a \cup b) \backslash (a \cap b)$$

vector **sum** (or **difference**) is seen as
set theoretic **symmetric difference**

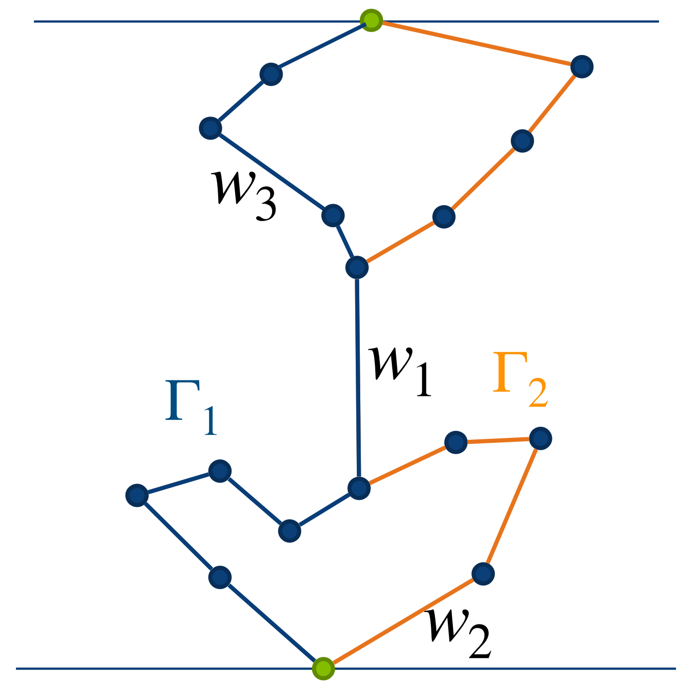
Lexicographic order



$\leq$ defines a **lexicographic order** $\sqsubseteq_{lex}$ on chains:

$$\Gamma \sqsubseteq_{lex} \Gamma' \underset{def.}{\Longleftrightarrow} \begin{cases} \Gamma = \Gamma' \\ \text{or} \\ \sigma_{\max} = \max\{\sigma \in \Gamma - \Gamma'\} \in \Gamma' \end{cases}$$

(With coefficients in $\mathbb{Z}_2$, $\Gamma_1 - \Gamma_2$ (or equivalently $\Gamma_1 + \Gamma_2$) is the *symmetric difference* between Γ_1 and Γ_2 seen as sets)



Delaunay order

When lexicographic-minimal chain is Delaunay

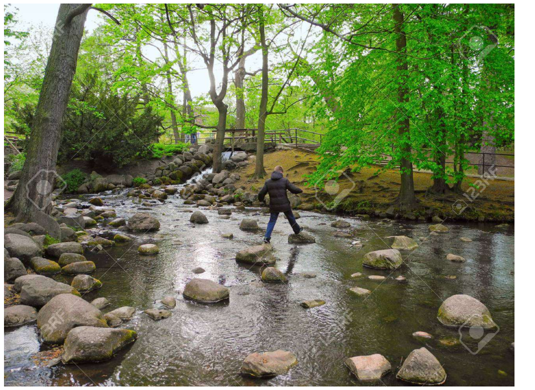
Behavior as $p \to \infty$?

Lemma 7.4. *If Condition 1 holds, $\leq_\infty$ is a total order on the set of 2-simplices of K with:*

$$\sigma_1 \leq_\infty \sigma_2 \iff \begin{cases} \mathrm{R_B}(\sigma_1) < \mathrm{R_B}(\sigma_2) \\ \text{or} \\ \mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2) \quad \text{and} \quad \mathrm{R_C}(\sigma_1) \geq \mathrm{R_C}(\sigma_2) \end{cases}$$

When $\leq_\infty$ is a total order, it defines a **lexicographic order** $\sqsubseteq_{lex}$ on chains:

$$\Gamma_1 \sqsubseteq_{lex} \Gamma_2 \underset{def.}{\iff} \begin{cases} \Gamma_1 = \Gamma_2 \\ \text{or} \\ \sigma_{\max} = \max_{\leq_\infty} \{\sigma \in \Gamma_1 - \Gamma_2\} \in \Gamma_2 \end{cases}$$



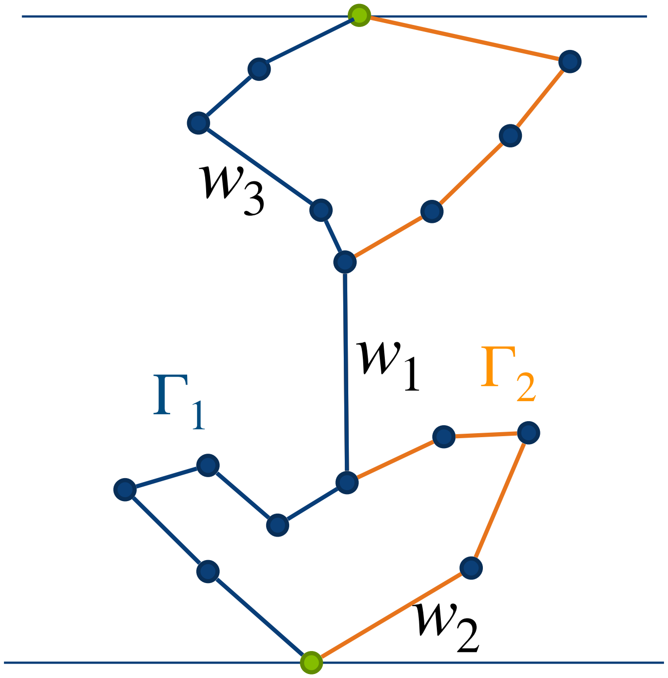
(With coefficients in $\mathbb{Z}_2$, $\Gamma_1 - \Gamma_2$ is the symmetric difference between Γ_1 and Γ_2)

Lexicographic order

$\sigma_0 < \sigma_1 < \ldots < \sigma_{N-1}$

$\leq$ defines a **lexicographic order** $\sqsubseteq_{lex}$ on chains:

$$\Gamma \sqsubseteq_{lex} \Gamma' \underset{def.}{\Longleftrightarrow} \sum_{i=0}^{N-1} \Gamma(\sigma_i)\, 2^i \leq \sum_{i=0}^{N-1} \Gamma'(\sigma_i)\, 2^i$$



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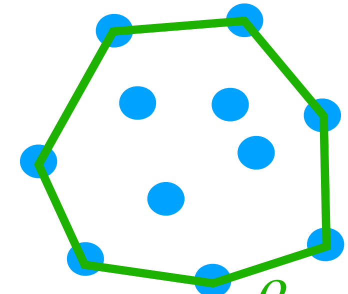
Delaunay triangulation

When lexicographic-minimal chain is Delaunay



Theorem 1 *Let* $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$, *with* $N \geq n+1$, *be weighted points in general position and* $K_{\mathbf{P}}$ *the* n*-dimensional full simplicial complex over* $\mathbf{P}$. *Denote by* $\beta_{\mathbf{P}} \in \boldsymbol{C_{n-1}}(K_{\mathbf{P}})$ *the* $(n-1)$*-chain, set of simplices belonging to the boundary of the convex hull* $\mathcal{CH}(\mathbf{P})$.

Then the simplicial complex $|\Gamma_{\min}|$ *support of*

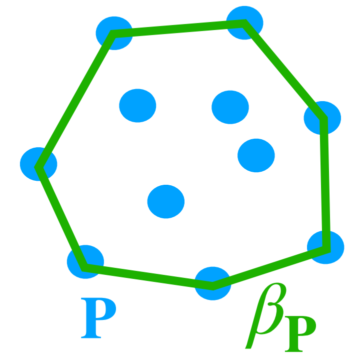
$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \left\{ \Gamma \in \boldsymbol{C_n}(K_{\mathbf{P}}), \partial\Gamma = \beta_{\mathbf{P}} \right\}$$

is the regular triangulation of $\mathbf{P}$.

(Cohen-Steiner, L.,Vuillamy 2020)

Delaunay triangulation

When lexicographic-minimal chain is Delaunay



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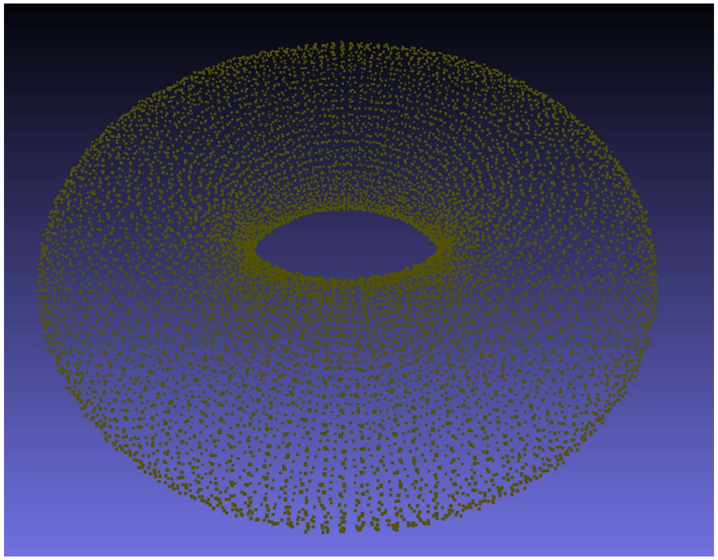
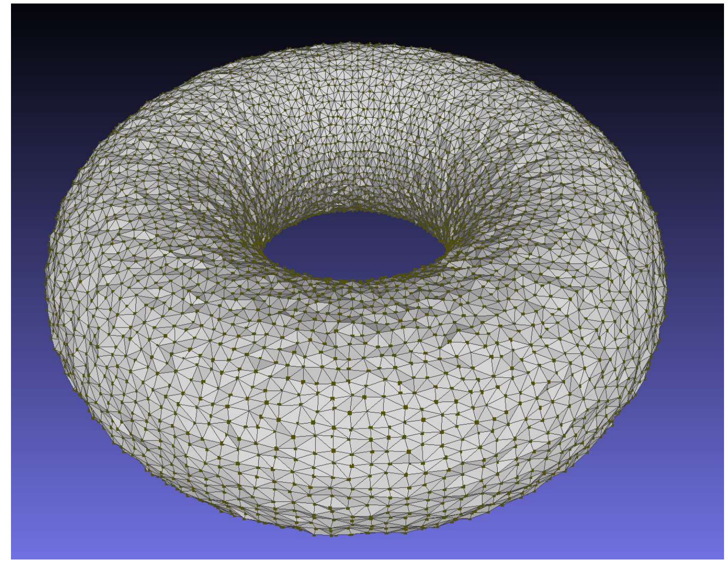
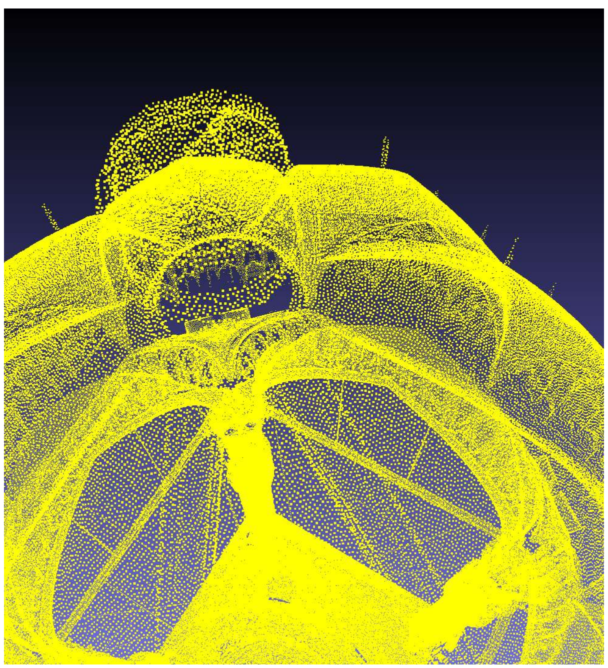
is the regular triangulation of $\mathbf{P}$.

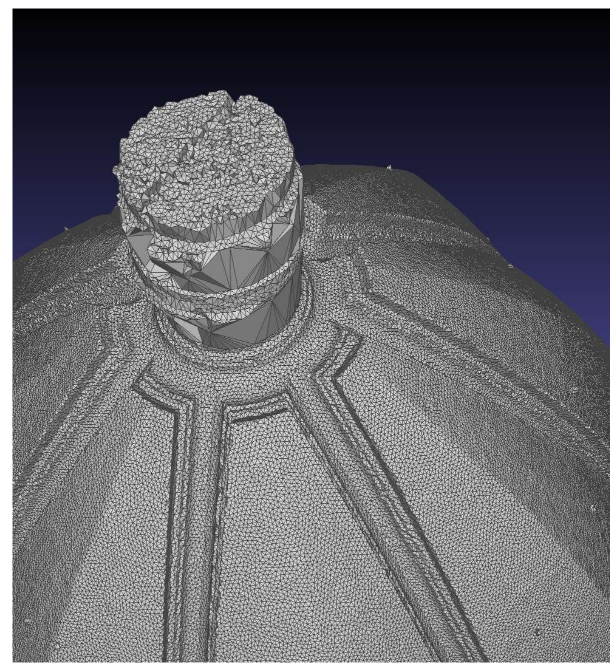
→This extends to smooth (positive reach) 2-manifolds

Topological faithful reconstruction and topological inference









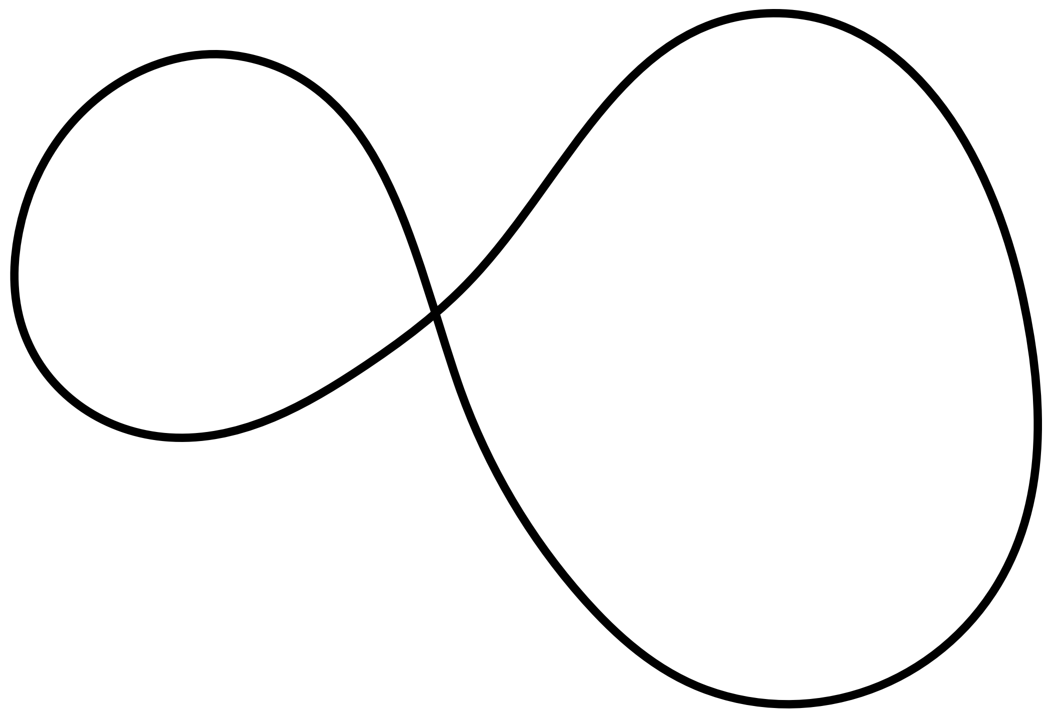
definition : offset of a set

We denote by $S \oplus B(\varepsilon)$ or sometime $S^{\oplus \varepsilon}$
the Minkowski sum of S and a the ball $B(\varepsilon)$ of radius ε

In other words the ε-offset of S
In other words, S « inflated » of ε :

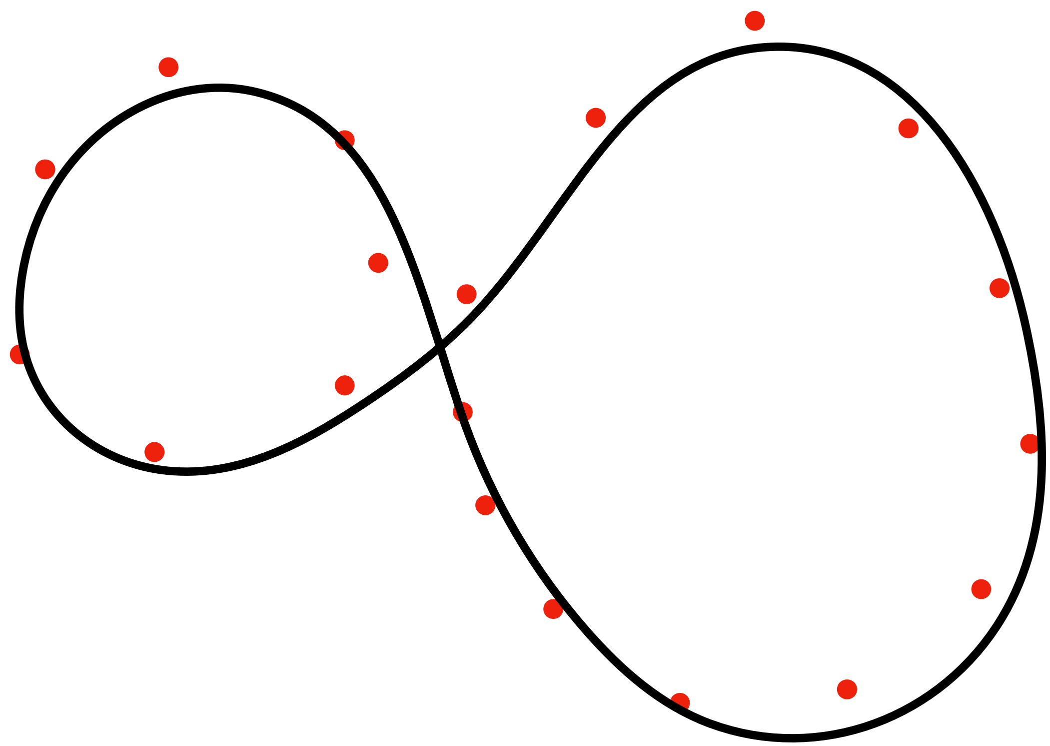
$$S \oplus B(\varepsilon) = S^{\oplus \varepsilon} := \bigcup_{x \in S} B(x, \varepsilon) = \left\{ y \in \mathbb{R}^d \mid d(y, S) \leq \varepsilon \right\}$$

Recovering the homotopy type from a sampling set



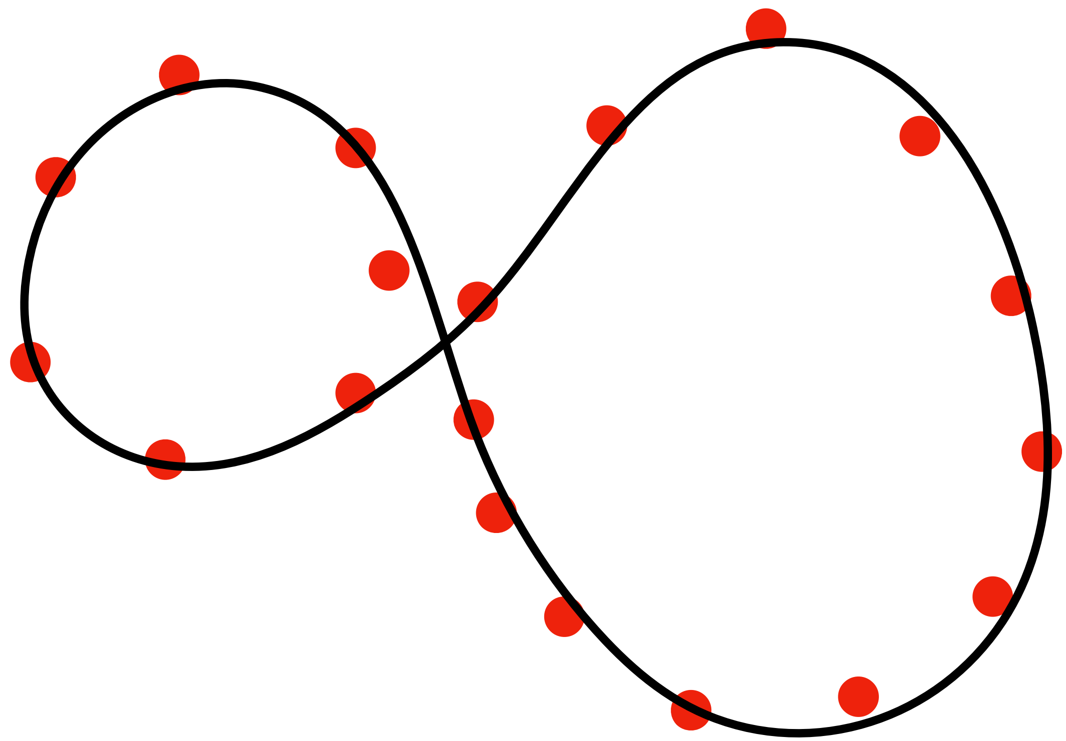
a set

Recovering the homotopy type from a sampling set



a set a sampling S

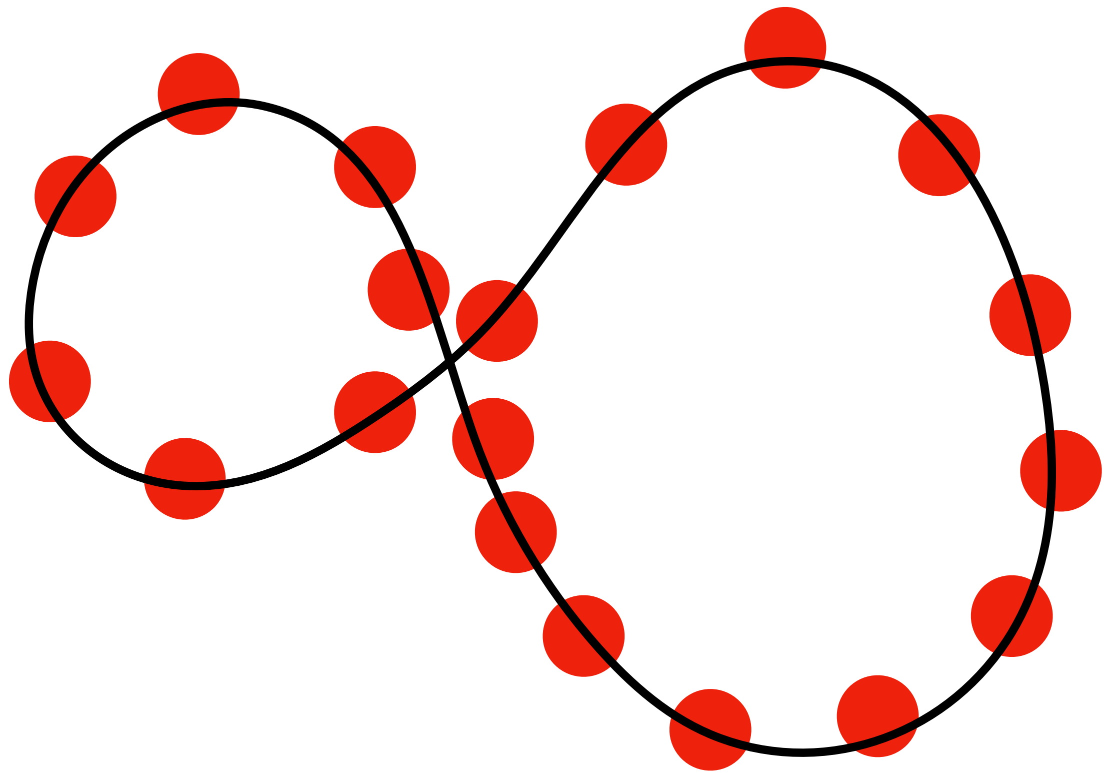
Recovering the homotopy type from a sampling set



$S \oplus B(\varepsilon)$

a set an offset of the sampling

Recovering the homotopy type from a sampling set



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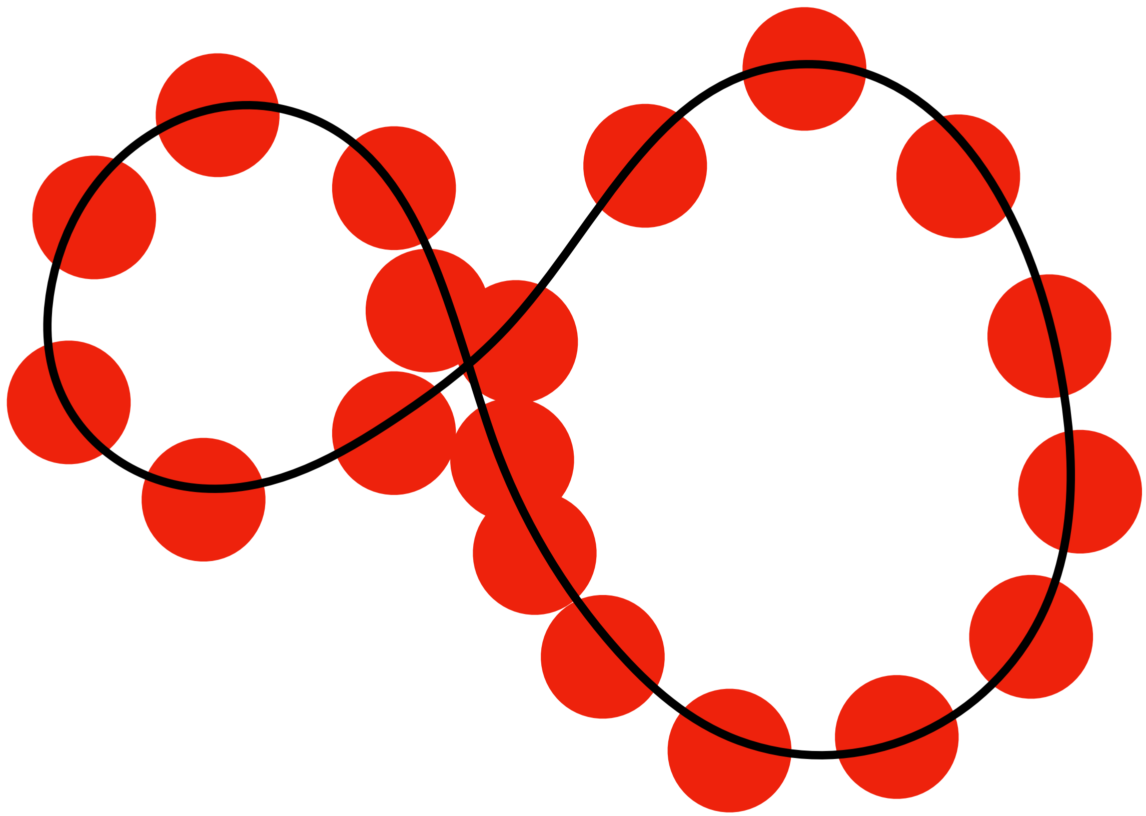

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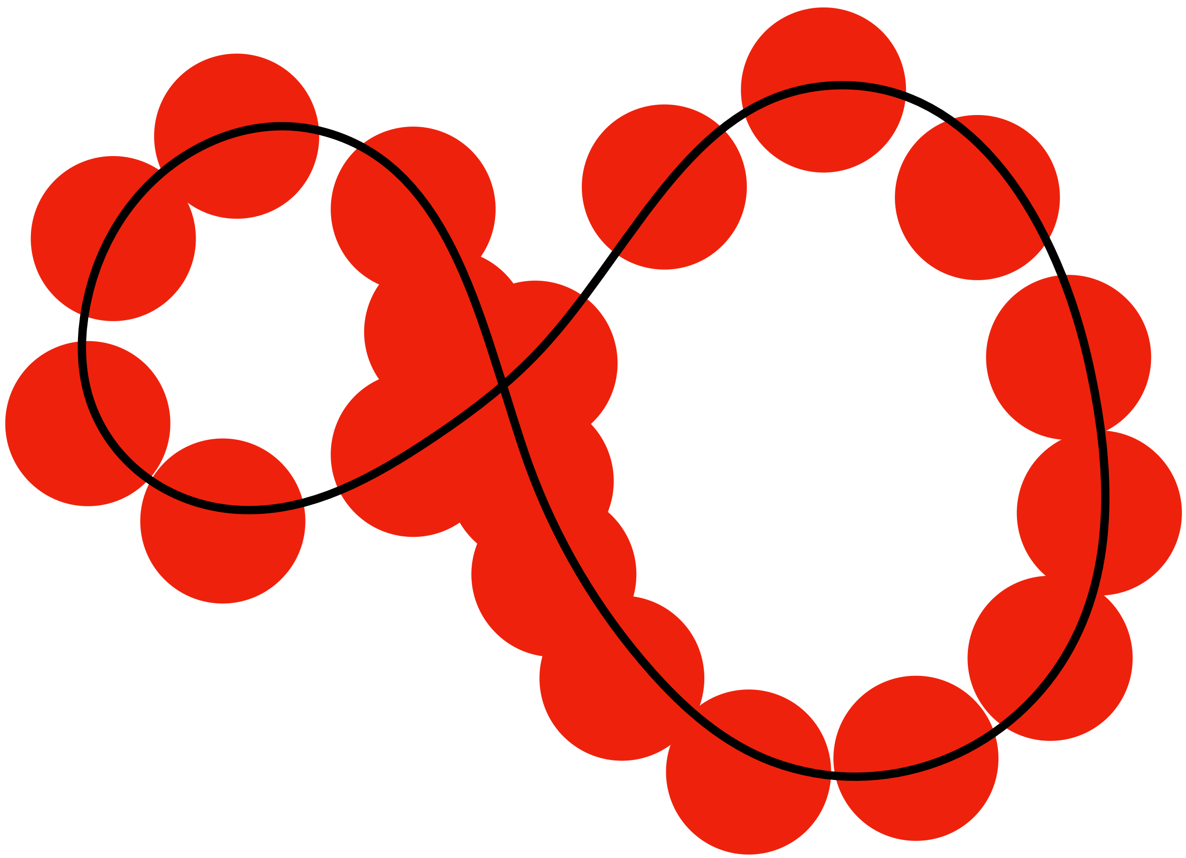

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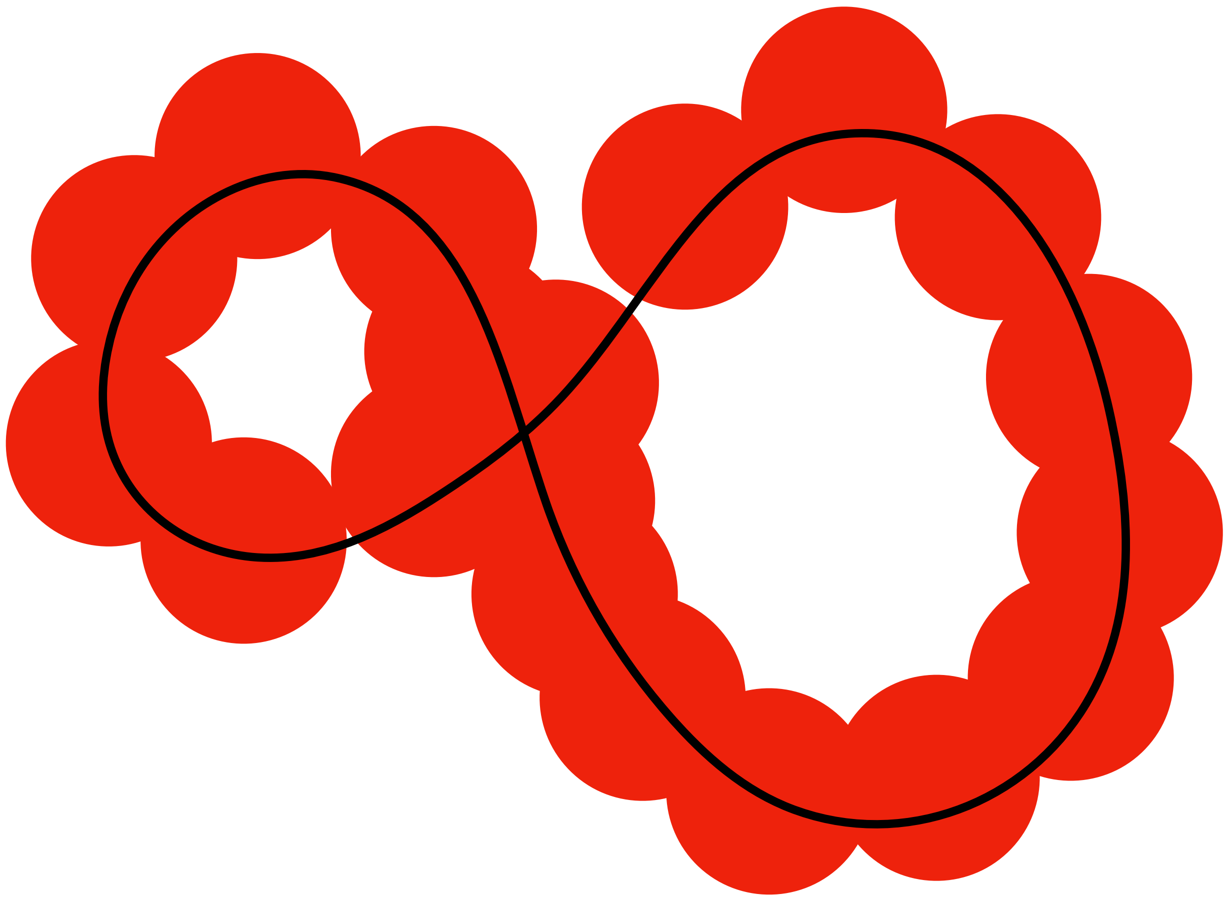

$S \oplus B(\varepsilon)$

a set an offset of the sampling

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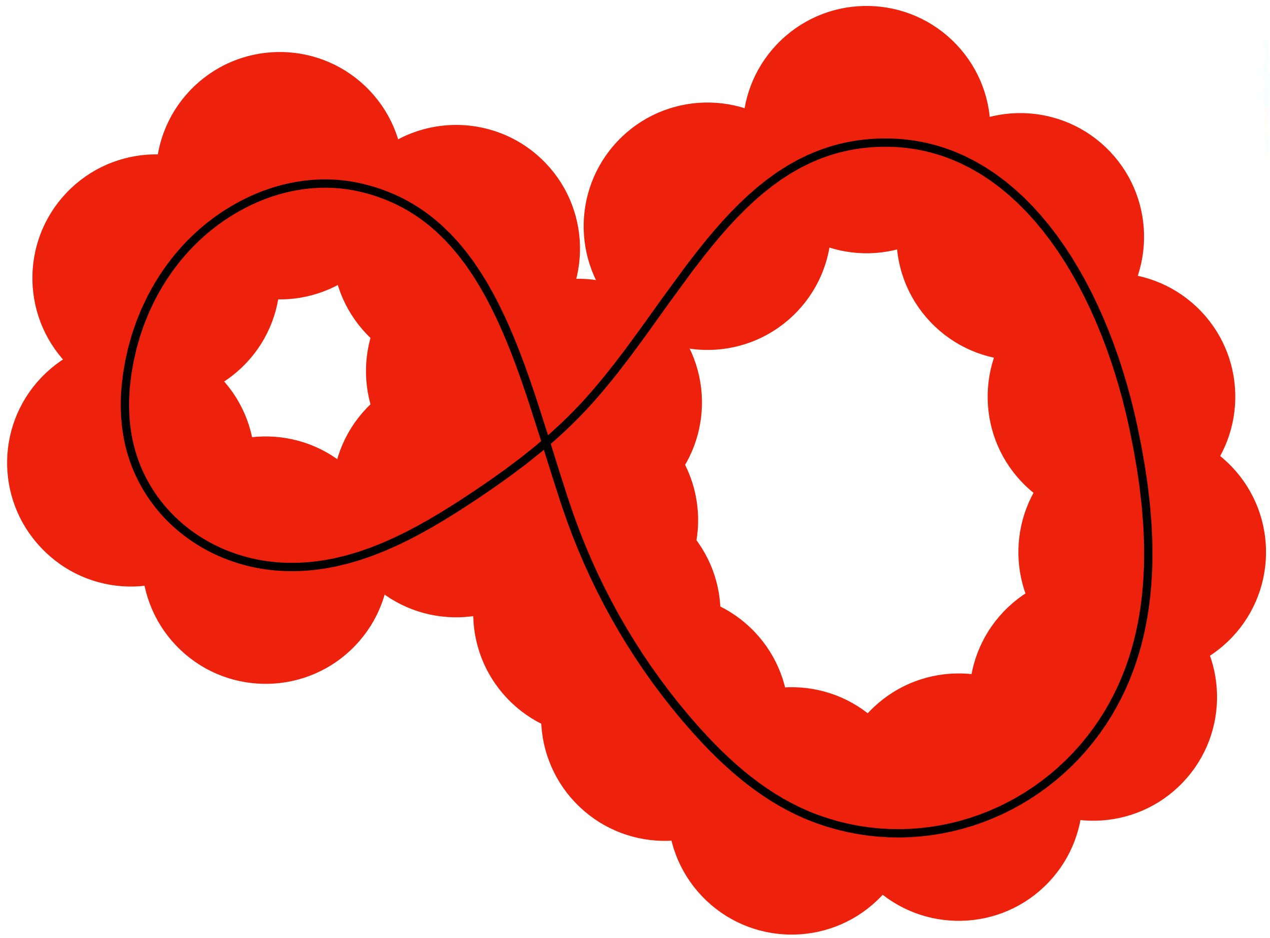
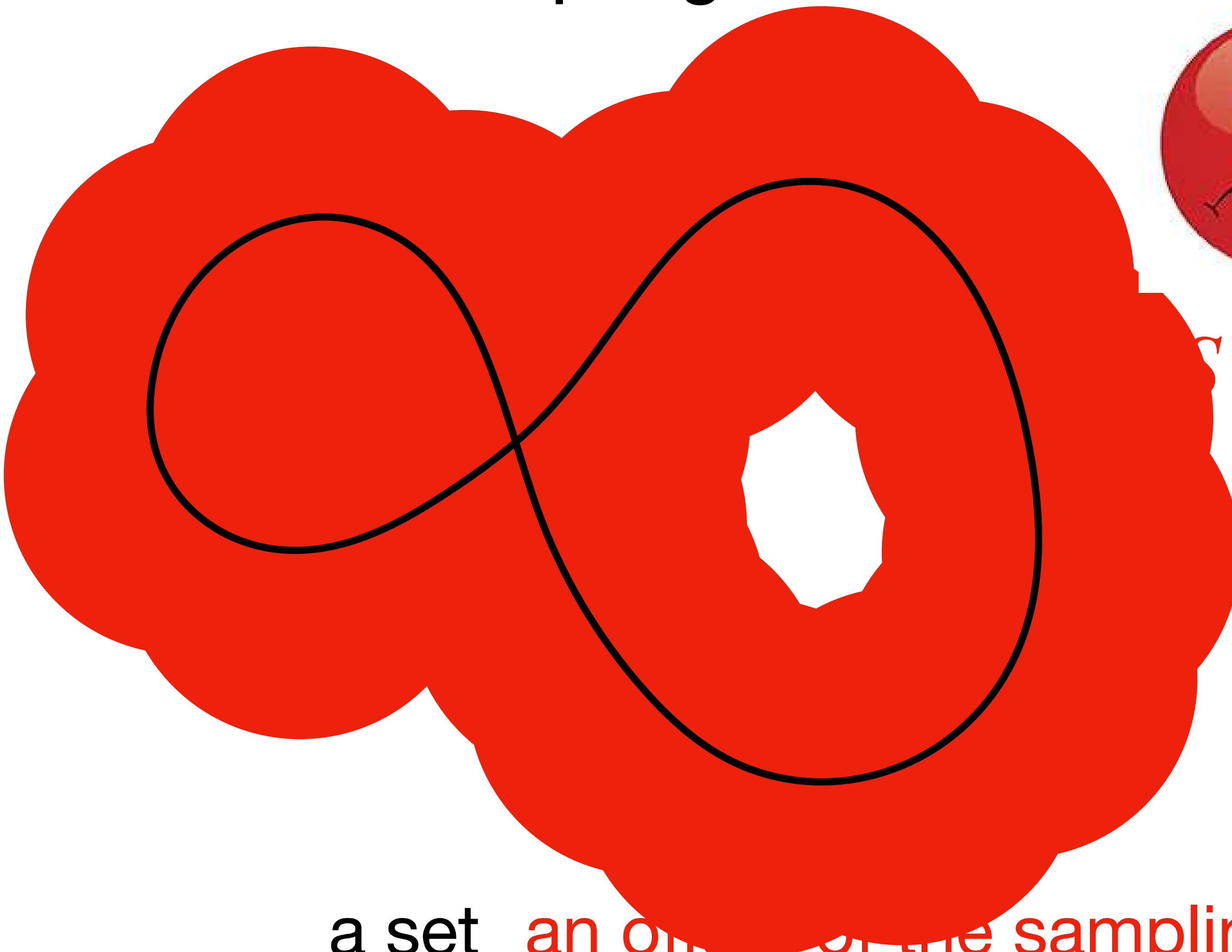


$S \oplus B(\varepsilon)$

a set an offset of the sampling

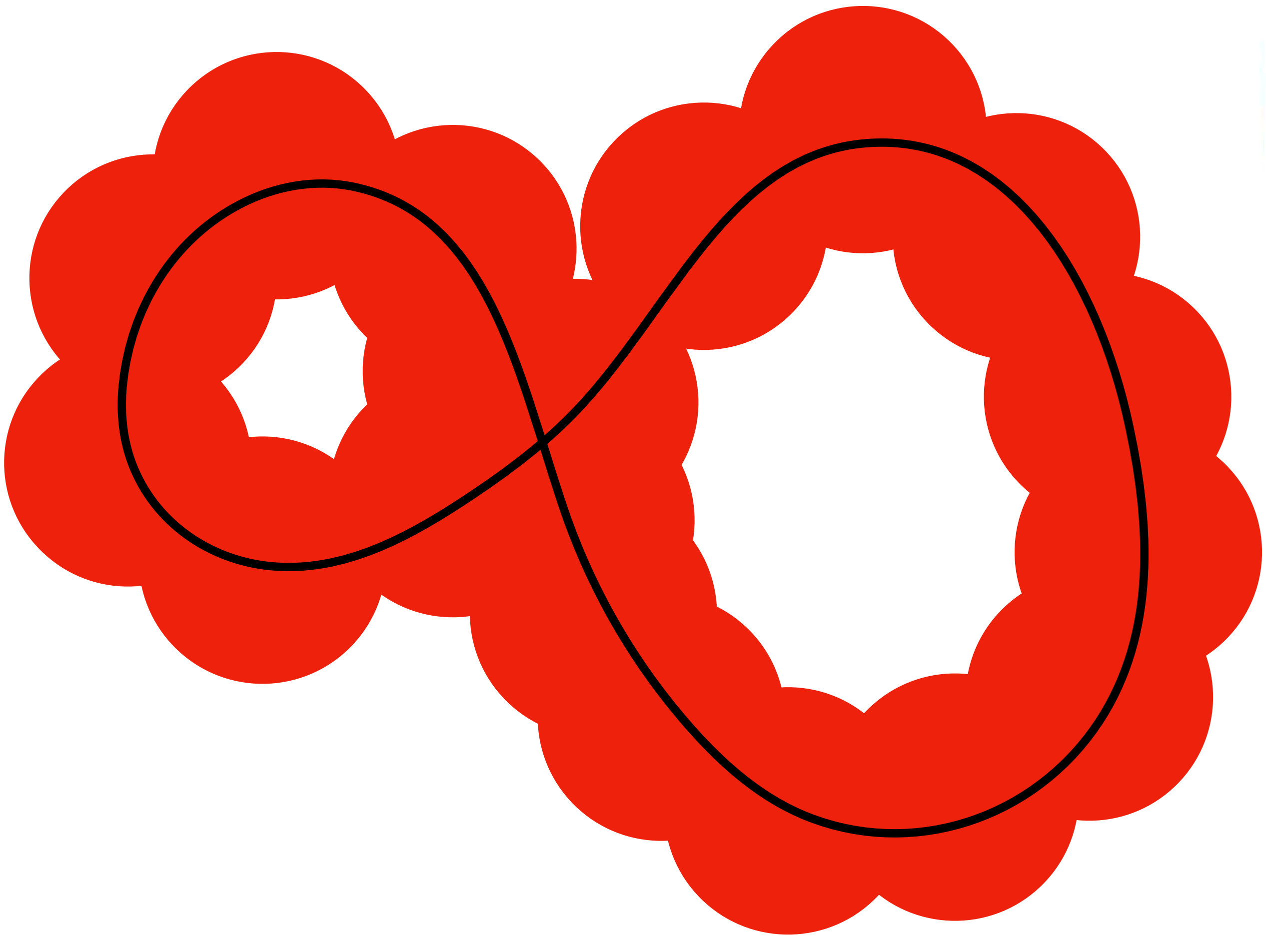
Recovering the homotopy type from a sampling set



$S \oplus B(\varepsilon)$

a set an … sampling

Recovering the homotopy type from a sampling set

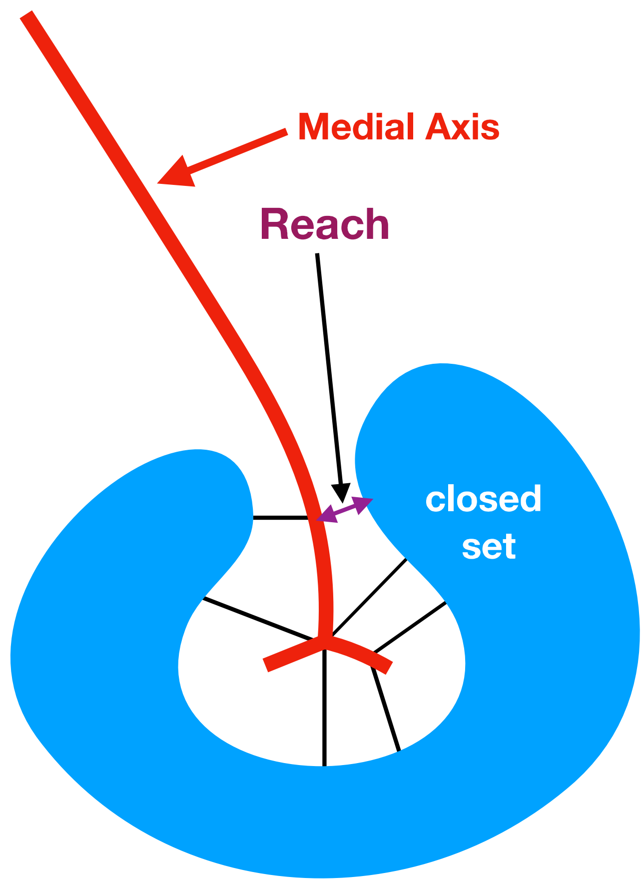


$S \oplus B(\varepsilon)$

a set an offset of the sampling

Medial axis and ***reach***

The ***reach*** of a closed set is the infimum of distances between points in the set and points in its medial axis



Reconstruction Theorem for set with positive reach

$R \leq \text{reach}(S)$

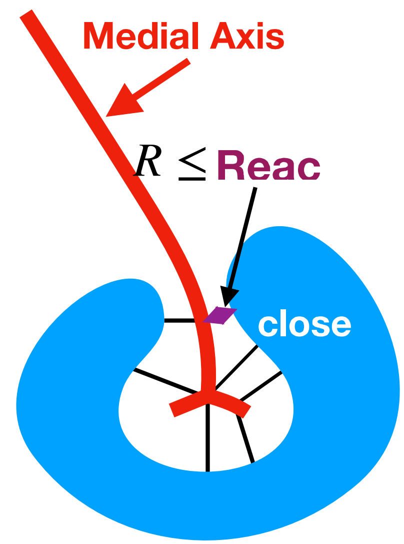
$S \subset P \oplus B(\epsilon)$ **and** $P \subset S \oplus B(\delta)$

General set of positive reach:

If ε *and* δ *satisfy*

$$\varepsilon + \sqrt{2}\,\delta \leq (\sqrt{2} - 1)R,$$

there exists a radius $r > 0$ *such that the union of balls* $P \oplus B(r)$ *deformation-retracts onto* $\mathcal{S}$ *along the closest point projection. In particular,* r *can be chosen as* $r = (R + \varepsilon)/2$

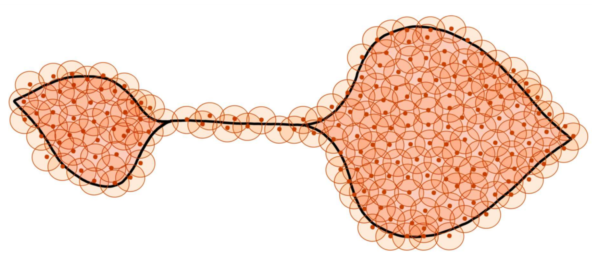


Weaker conditions for manifold of positive reach:

If ε *and* δ *satisfy*

$$(R - \delta)^2 - \varepsilon^2 \geq \left(4\sqrt{2} - 5\right) R$$

These conditions are ***tight*** *for retrieving the homology and homotopy by some offset of the sample*

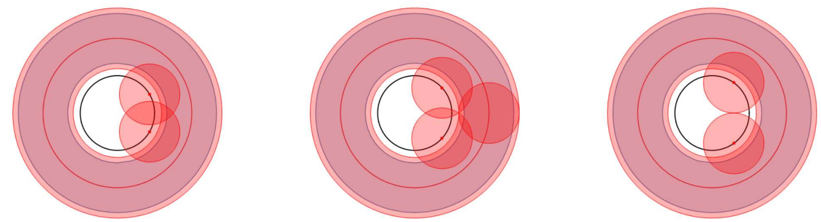


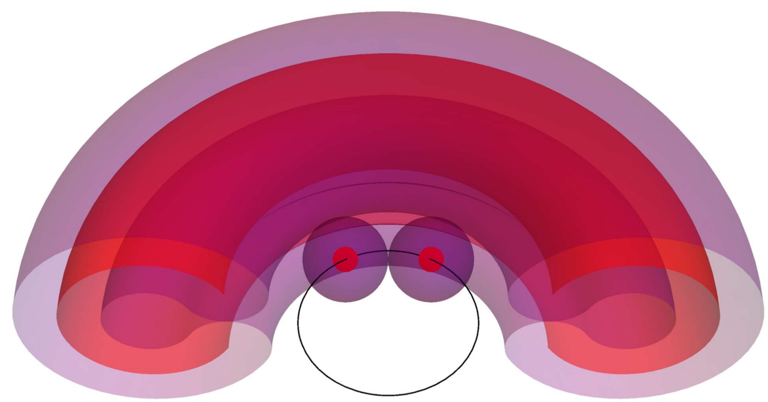
Reconstruction Theorem for set with positive reach

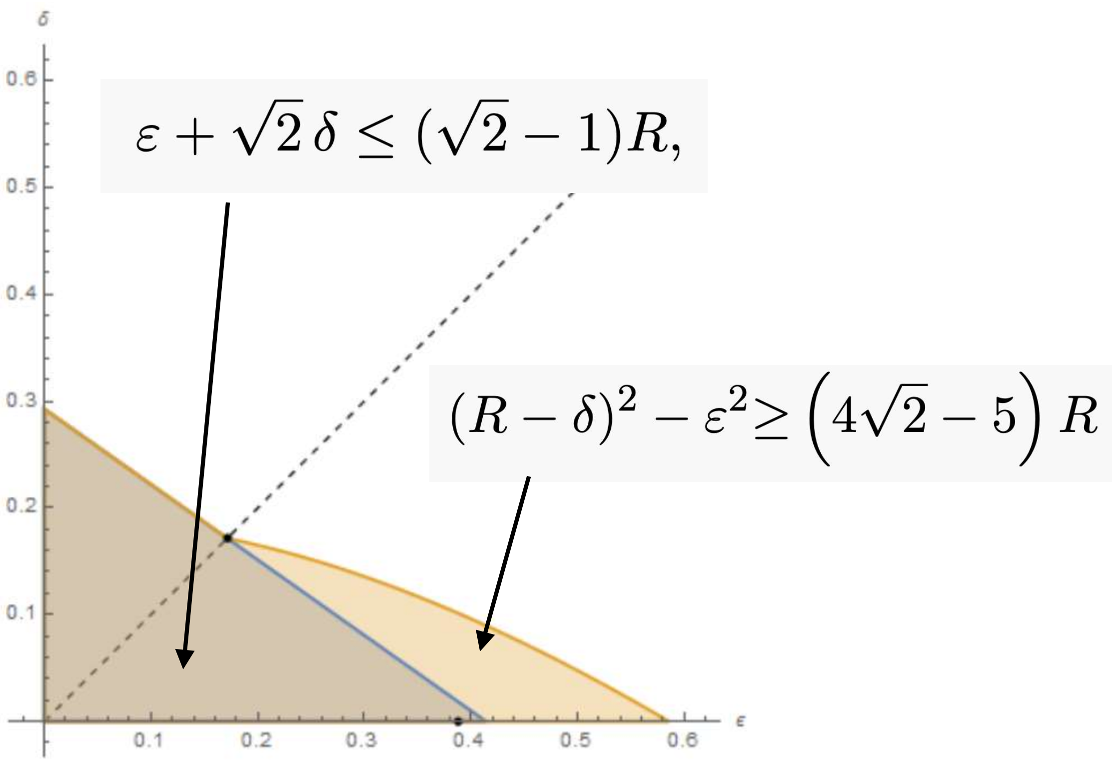
$R \leq \text{reach}(S)$

$S \subset P \oplus B(\epsilon)$ **and** $P \subset S \oplus B(\delta)$

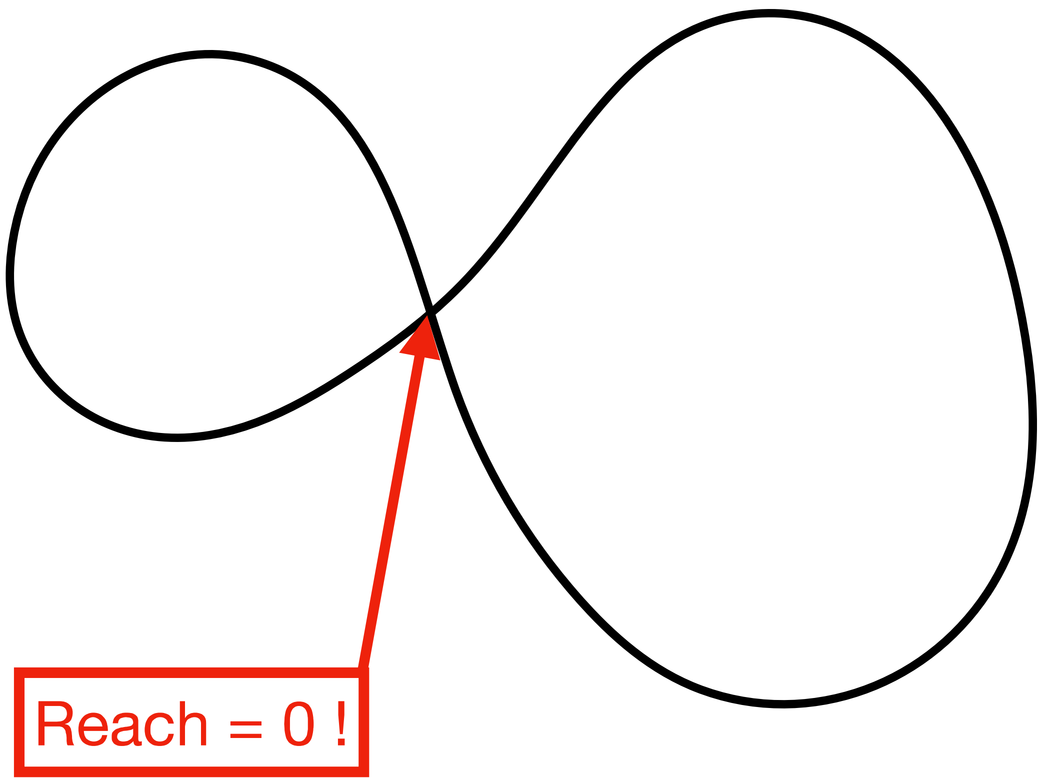
*These conditions are **tight** for retrieving the homology and homotopy by some offset of the sample*







Recovering the homotopy type from a sampling set



Regularity measures guaranteed homotopy type recovering from samples

Reach and medial axis

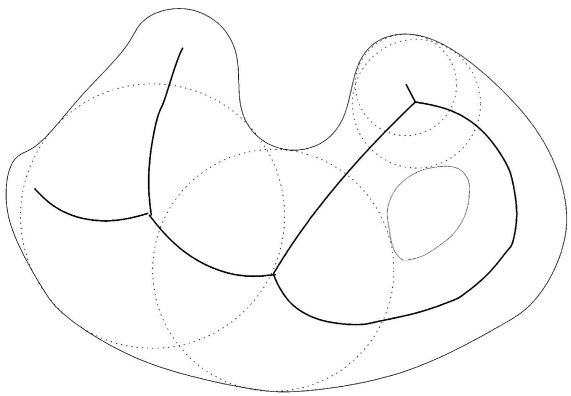
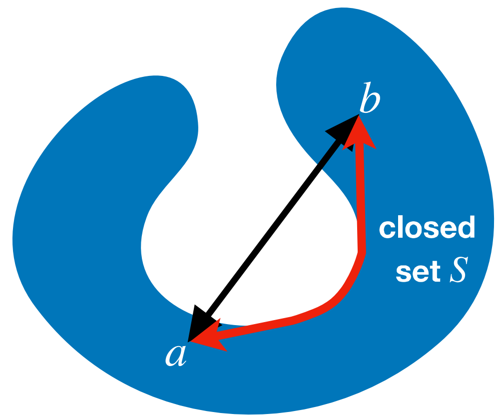


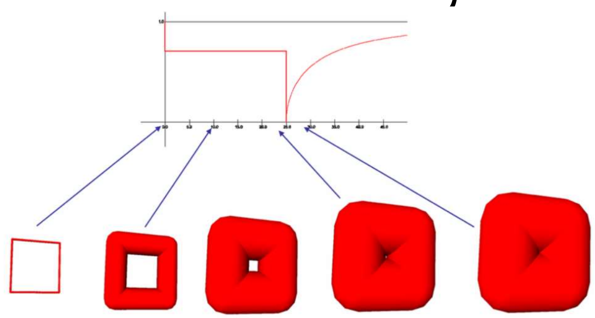
Fig. 1. A set and its medial axis.

Metric distorsion

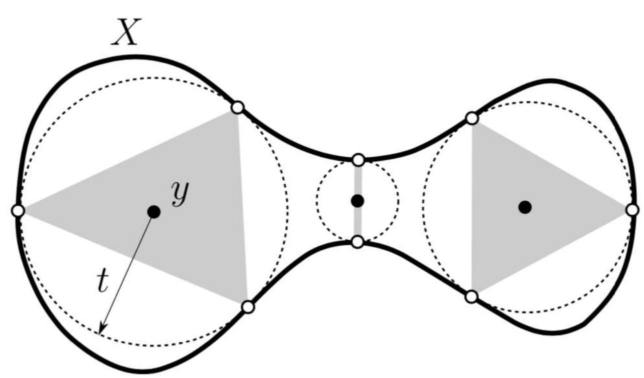


(smooth objects)

Critical function and μ-reach



Convexity defect



(not necessarily smooth)

When a simplicial complex over a point sample **recovers** the **homotopy type**

P. Niyogi, S. Smale, and S. Weinberger. Finding the homology of submanifolds with high confidence from random samples. *Discrete Comput. Geom.*, 39:419–441, 2008.

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(optimal)

reach
(smooth concavities)

Critical function &
μ-reach
(non-smooth)

F. Chazal, D. Cohen-Steiner, and A. Lieutier. A sampling theory for compact sets in Euclidean space. *Discete Comput. Geom.*, 41:461–479, 2009.

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(best known constants for Cech complexes)

Convexity defects

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thanks to
Antoine Commaret's PhD

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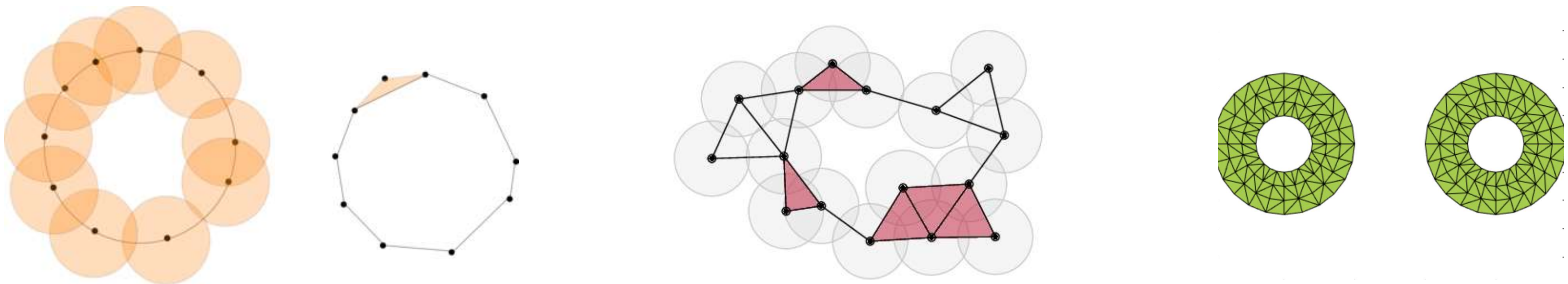
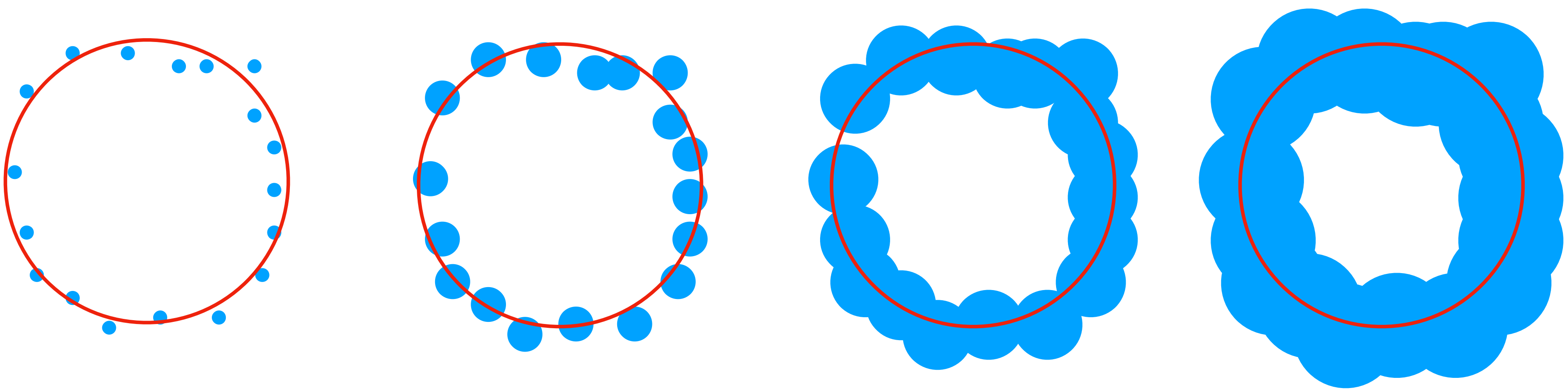
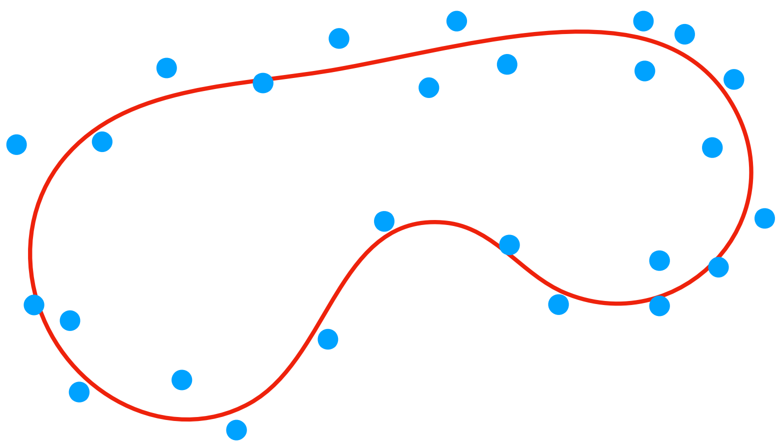
Čech complex and α-complex

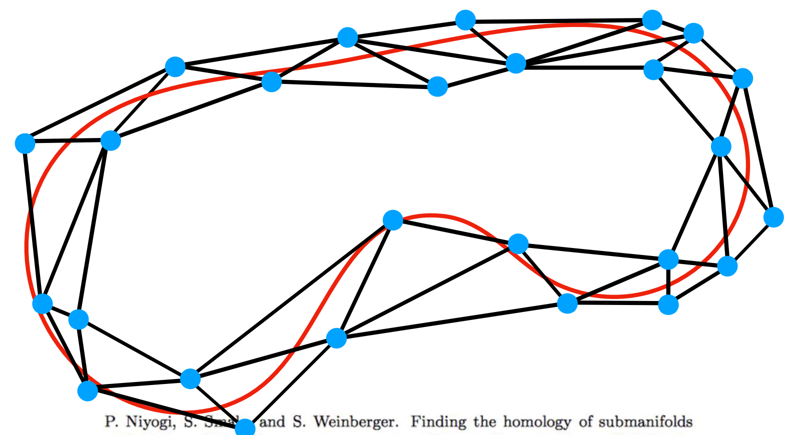


By the nerve Theorem, both Čech complex and α-complex have the homotopy type of the corresponding union of balls



From sample to homotopy type (alpha-complex, Cech-complex,..)





OK

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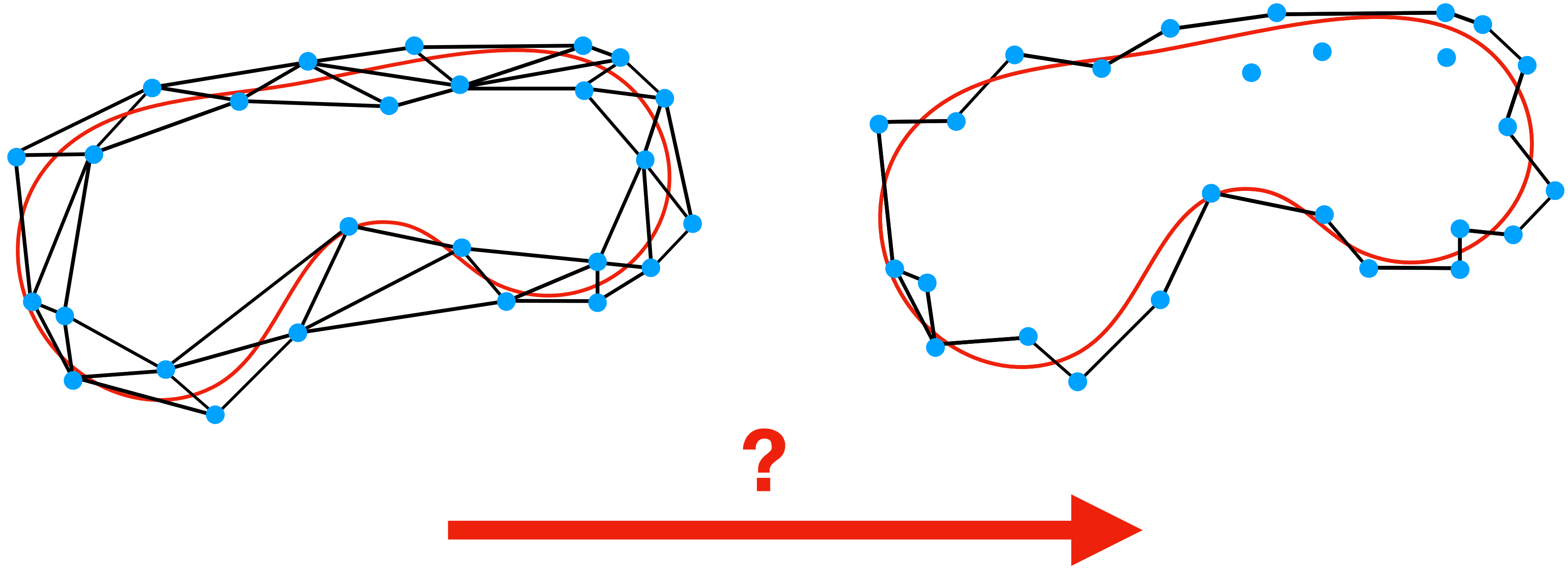
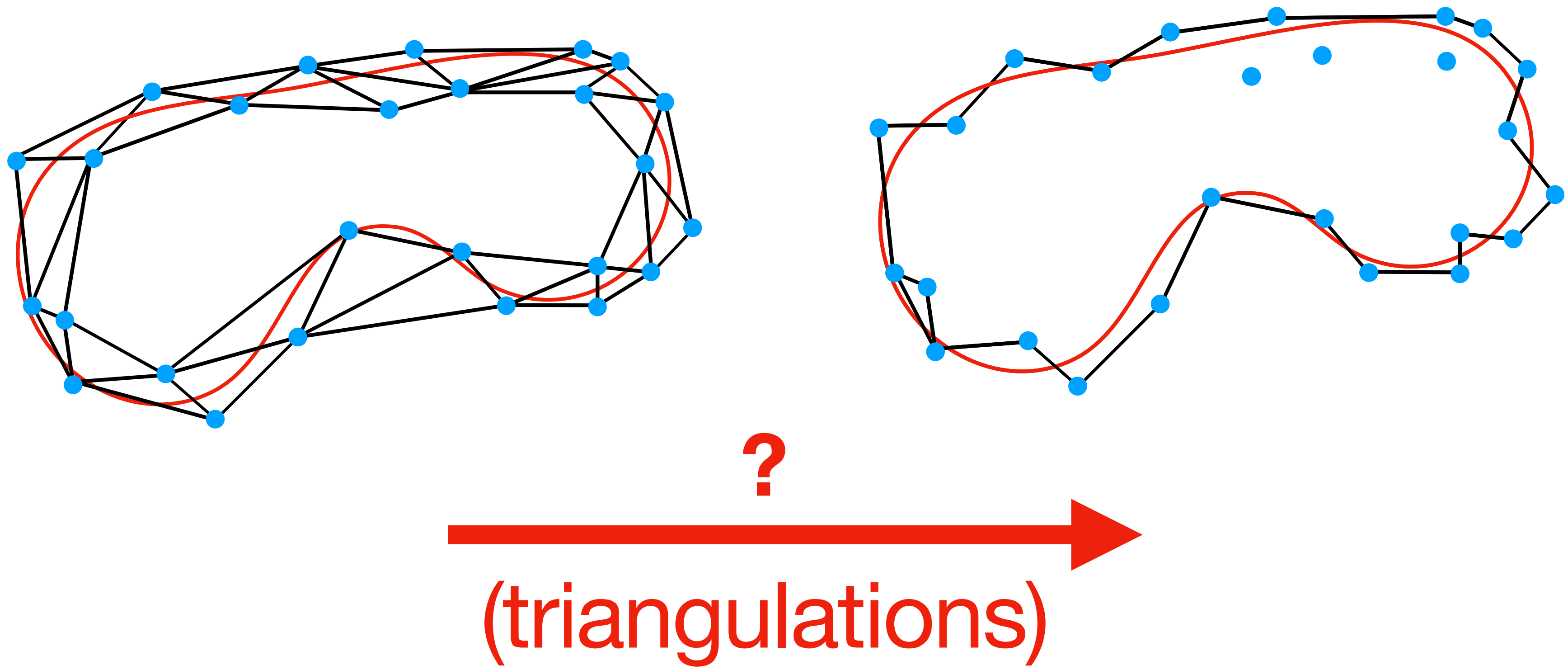
From **homotopy type** to **triangulations** (= homeomorphisms)



From homotopy type to homeomorphisms



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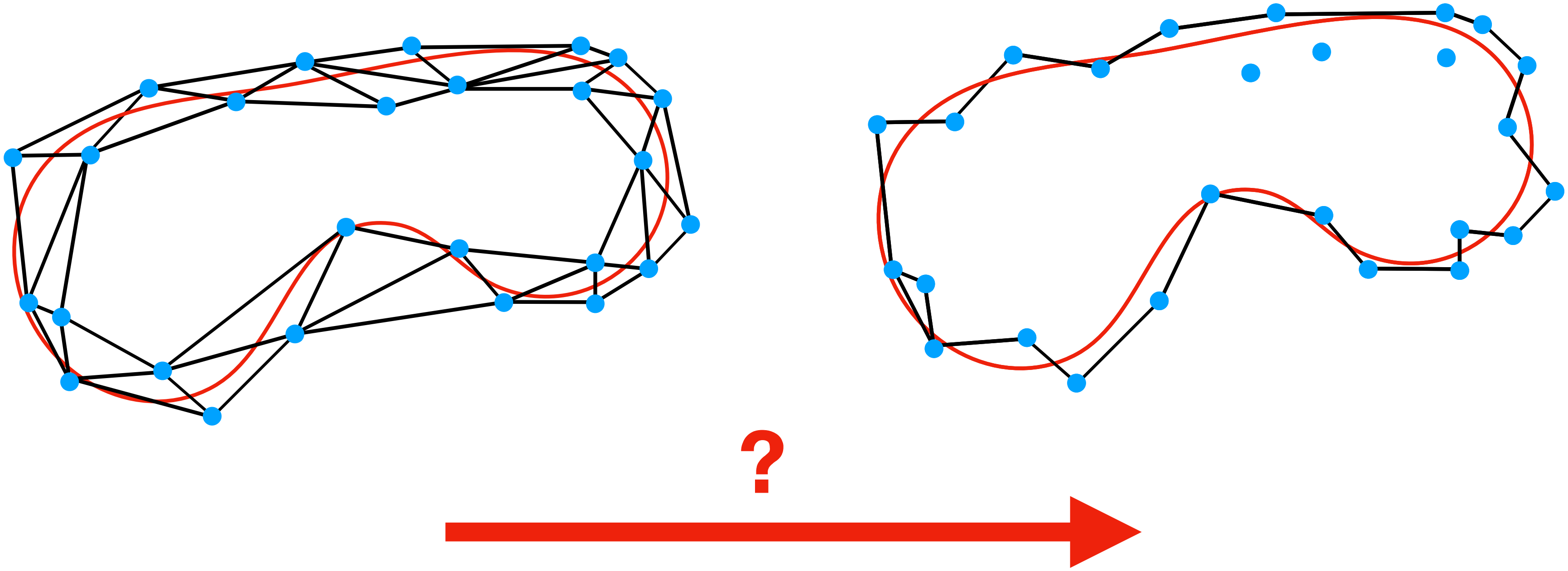
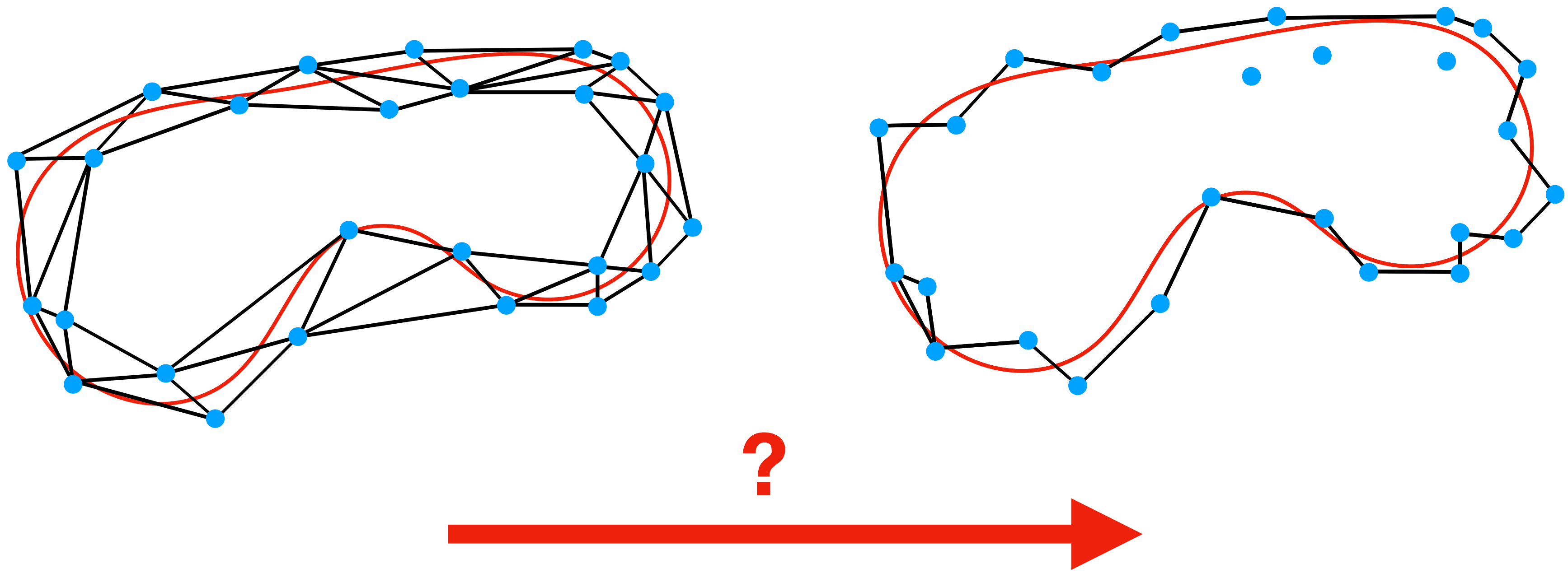


(homological approaches to triangulations)

From homotopy type to homeomorphisms

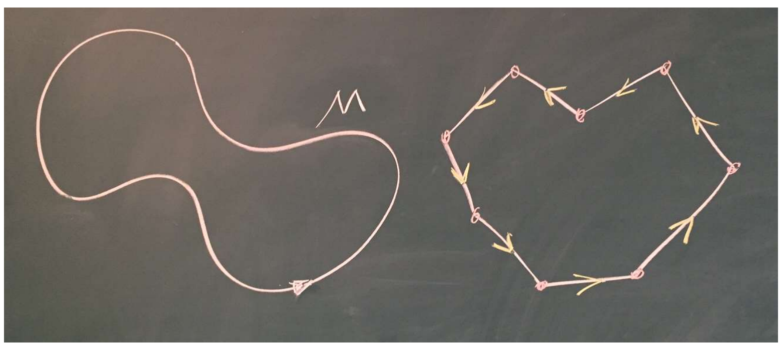


(homological approaches to triangulations)

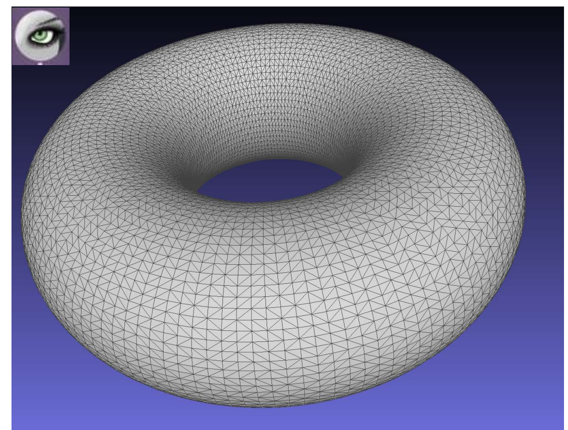
Fundamental class

(orientable and non-orientable, with/without boundary)

If M is a **connected compact orientable** d-manifold, its d-homology group is one dimensional: $\dim \mathbf{H}_d(M^d) = 1$ and a generator of it is called the **Fundamental class:**.



$$\dim \mathbf{H}_1(M^1) = 1$$

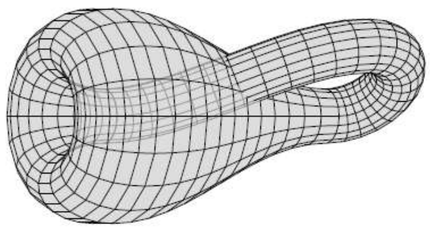


$$\dim \mathbf{H}_2(M^2) = 1$$

Fundamental class

(orientable and non-orientable, with/without boundary)

If M is a **connected compact orientable** d-manifold, its d-homology group is one dimensional and a generator of it is called the **Fundamental class**.



$$\dim \mathbf{H}_d(M^d, \mathbb{Z}_2) = 1$$

If the coefficients field is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, this is also true for non-orientable (compact, connected) manifolds.

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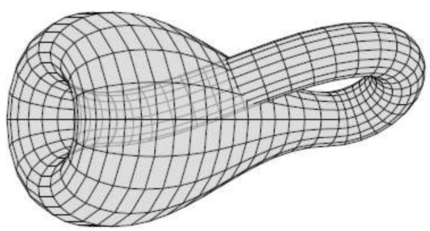

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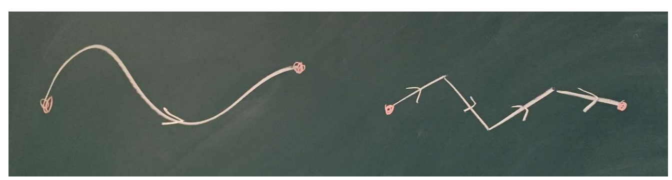
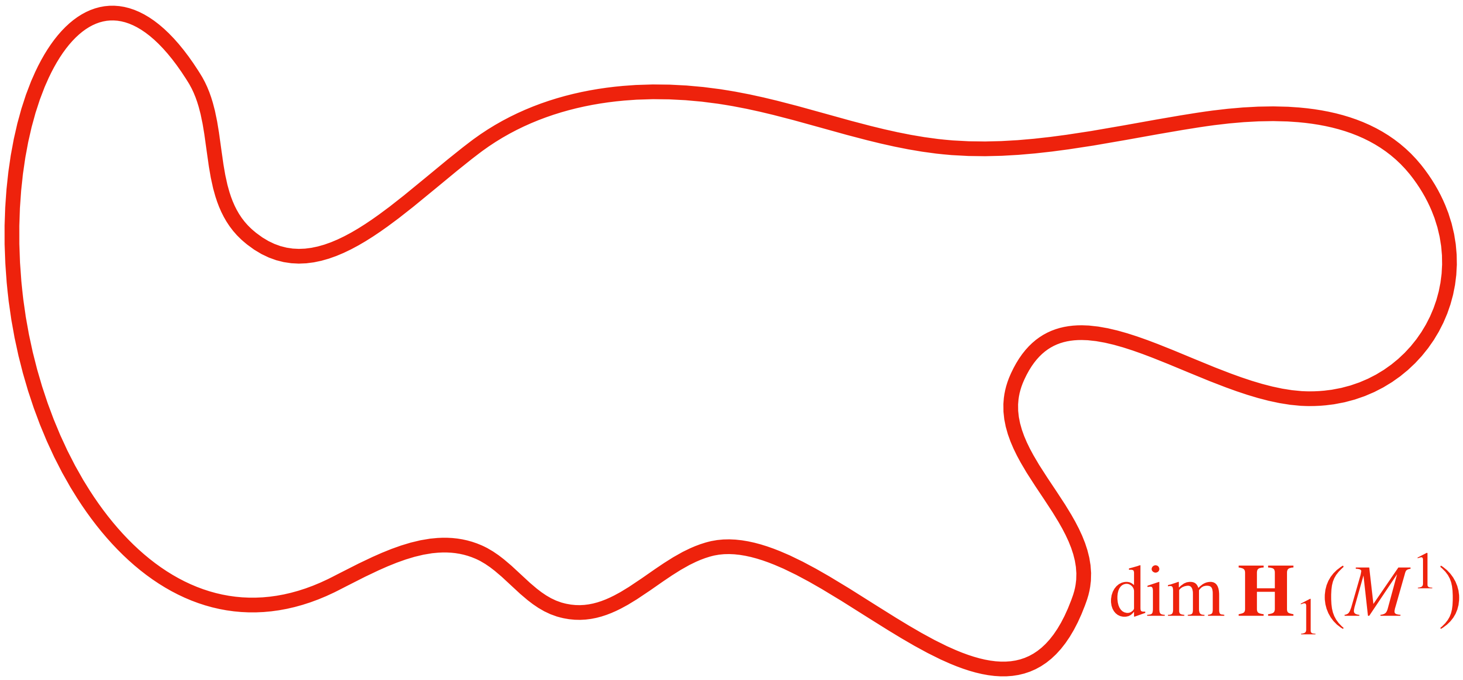
If the coefficients field is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, this is also true for non-orientable (compact, connected) manifolds.

For manifolds with boundaries, this generalizes with relative homology:

$$\dim \mathbf{H}_d(M, \partial M, \mathbb{Z}_2) = 1$$



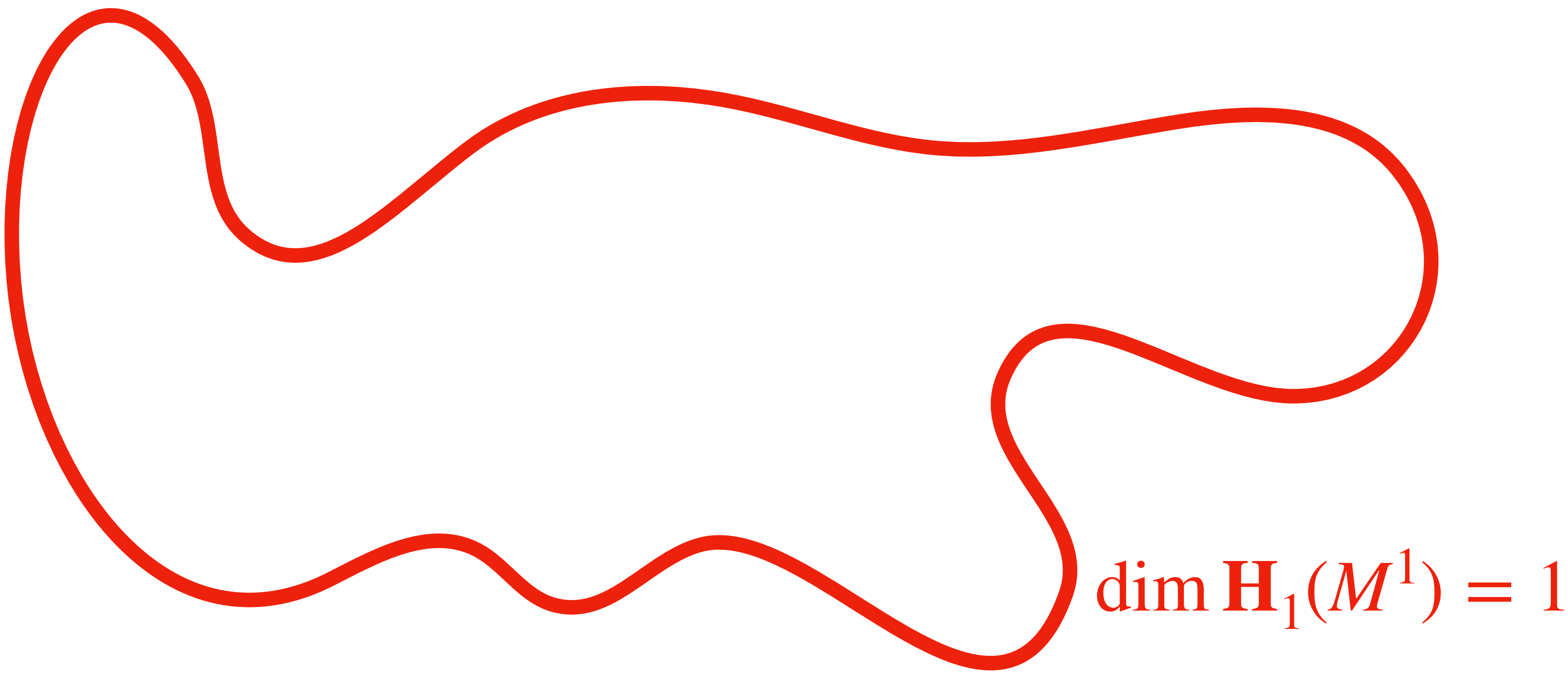
Manifold fundamental homology class



$$\dim \mathbf{H}_1(M^1) = 1$$

Manifold fundamental homology class

$\mathscr{M}$ **connected** $\Rightarrow \dim \mathbf{H}_d(\mathscr{M}^d) = 1$



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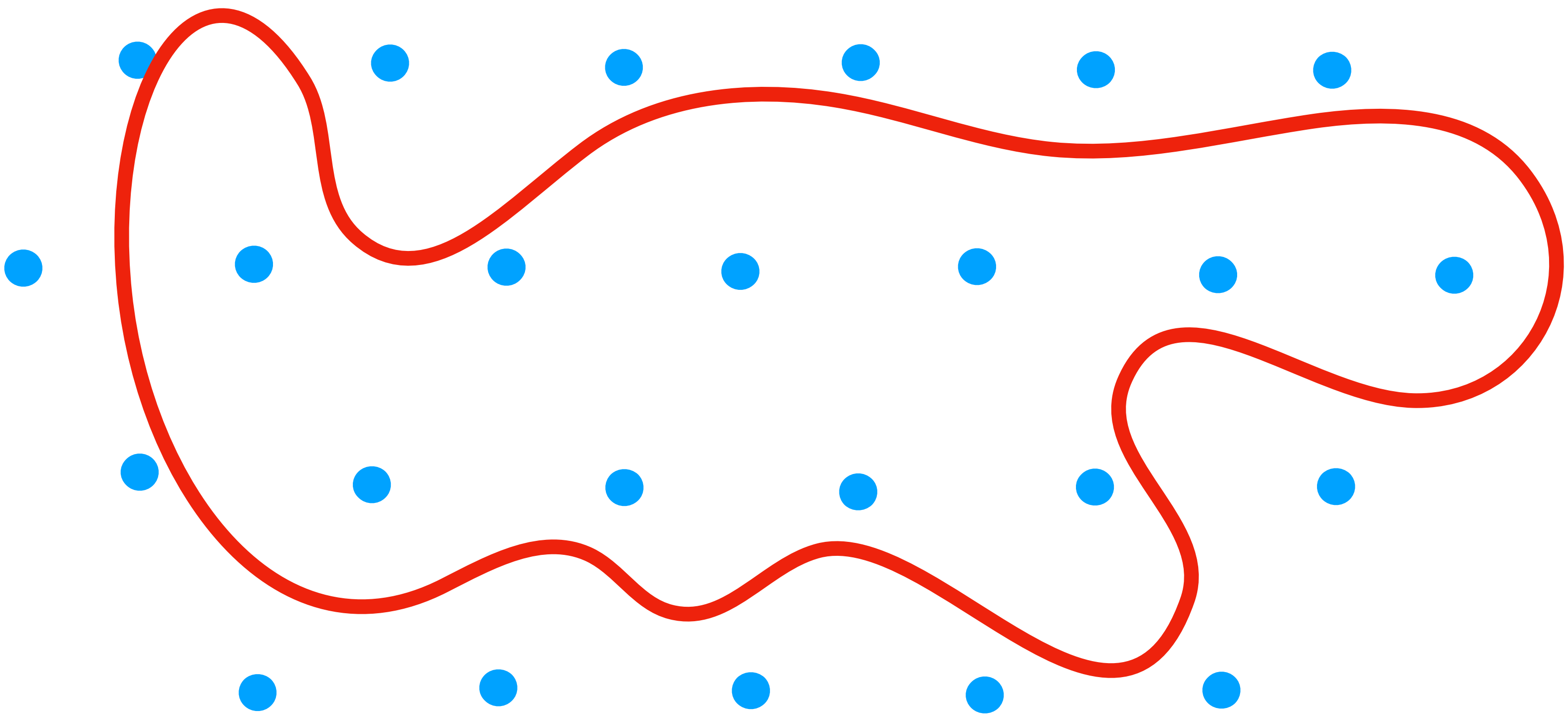

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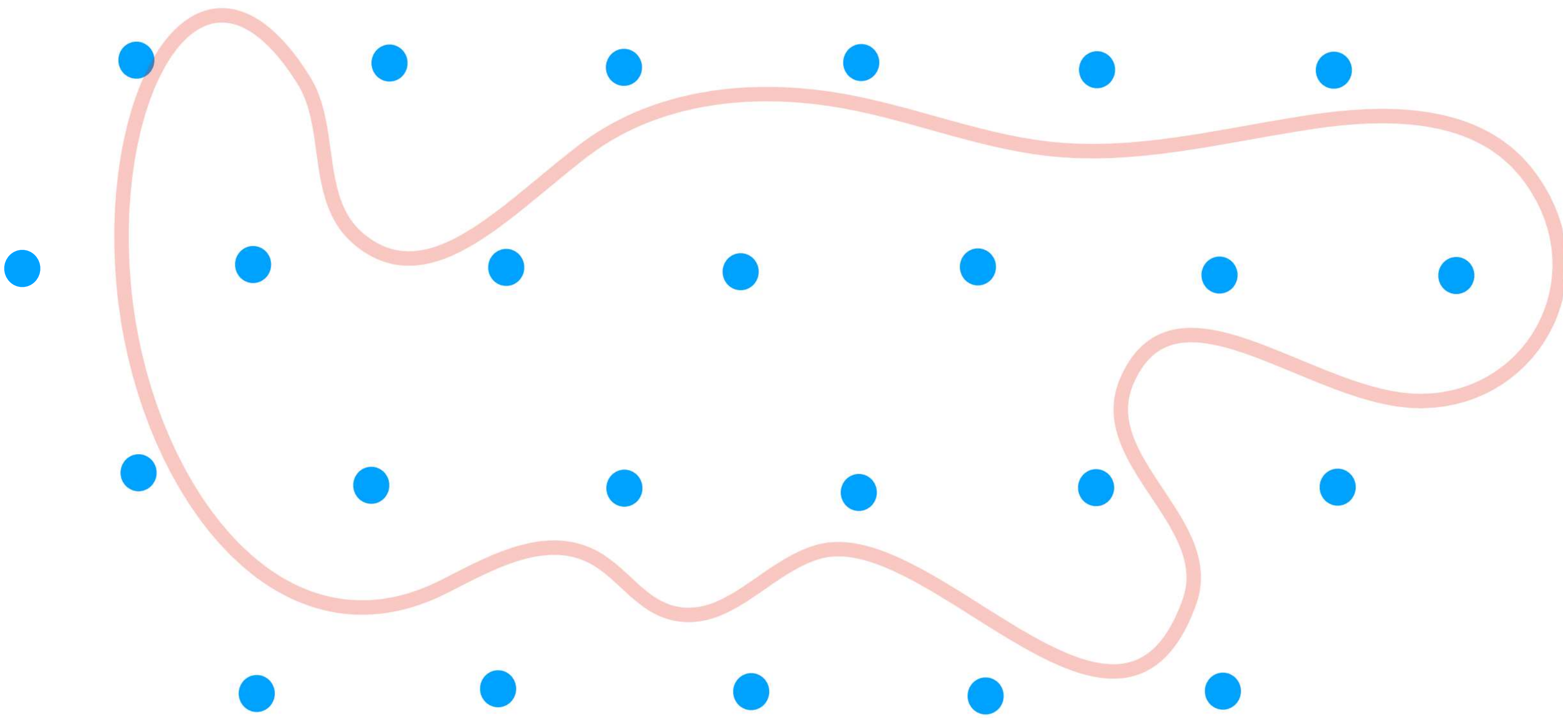

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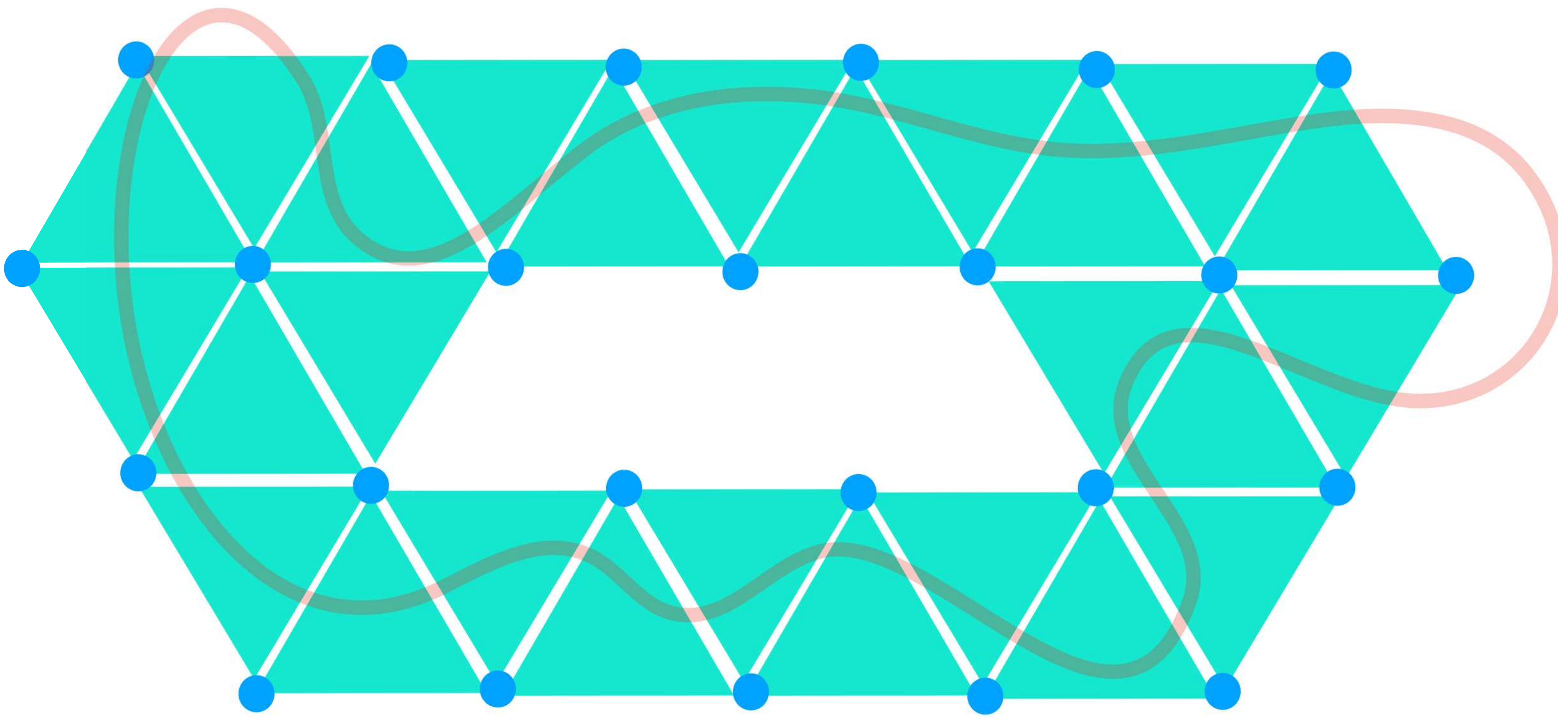

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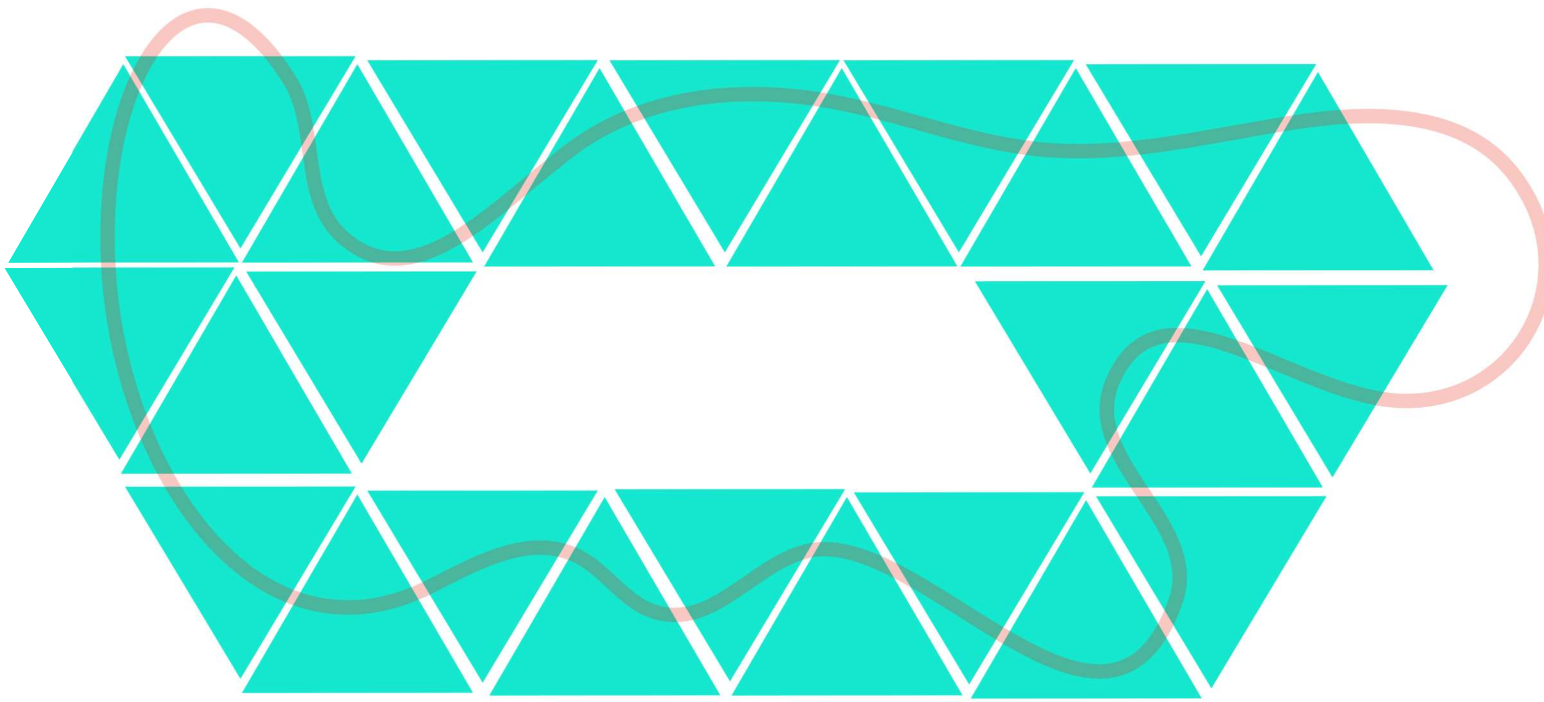

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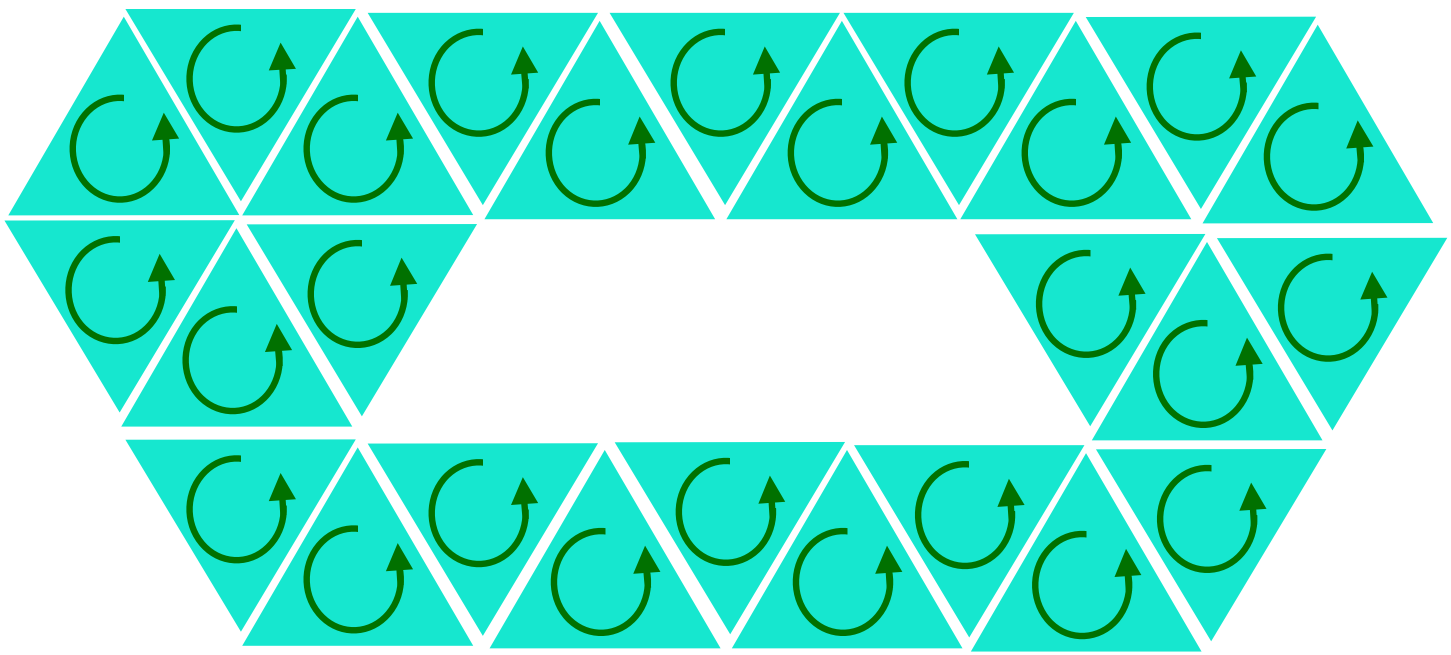

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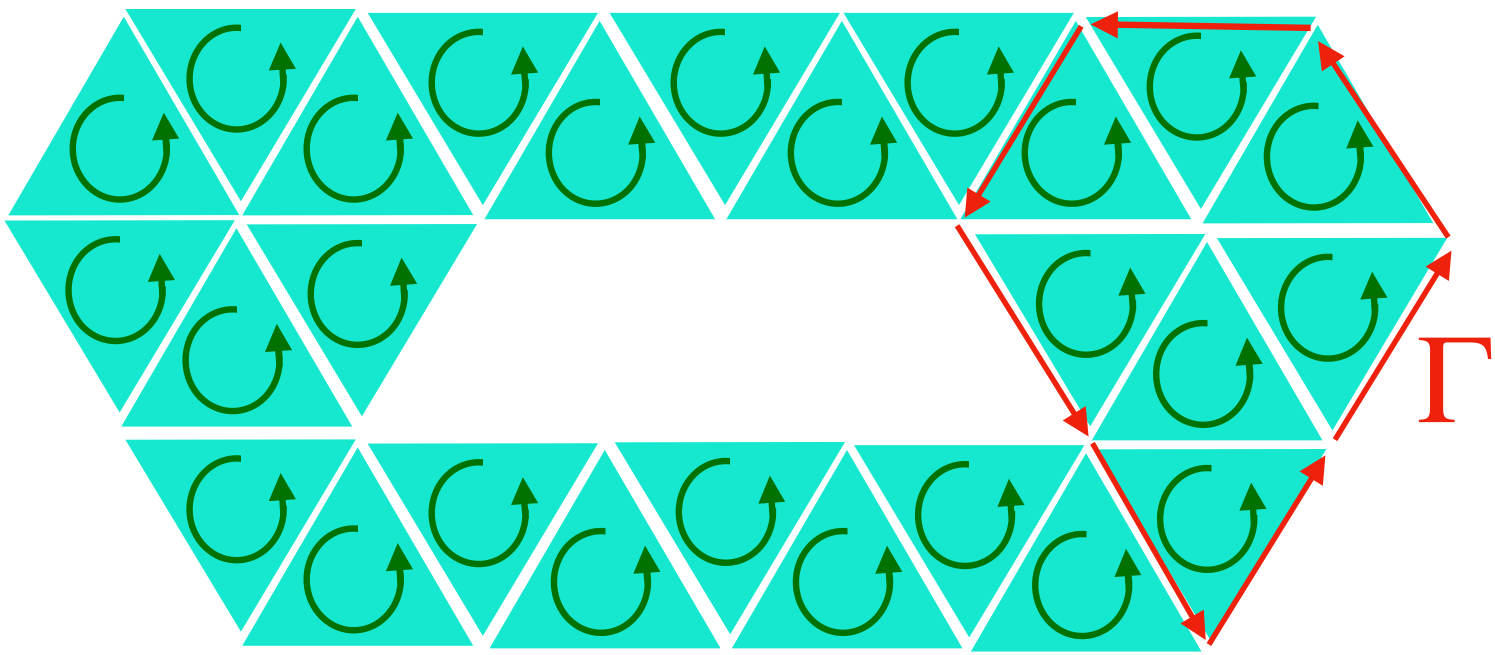
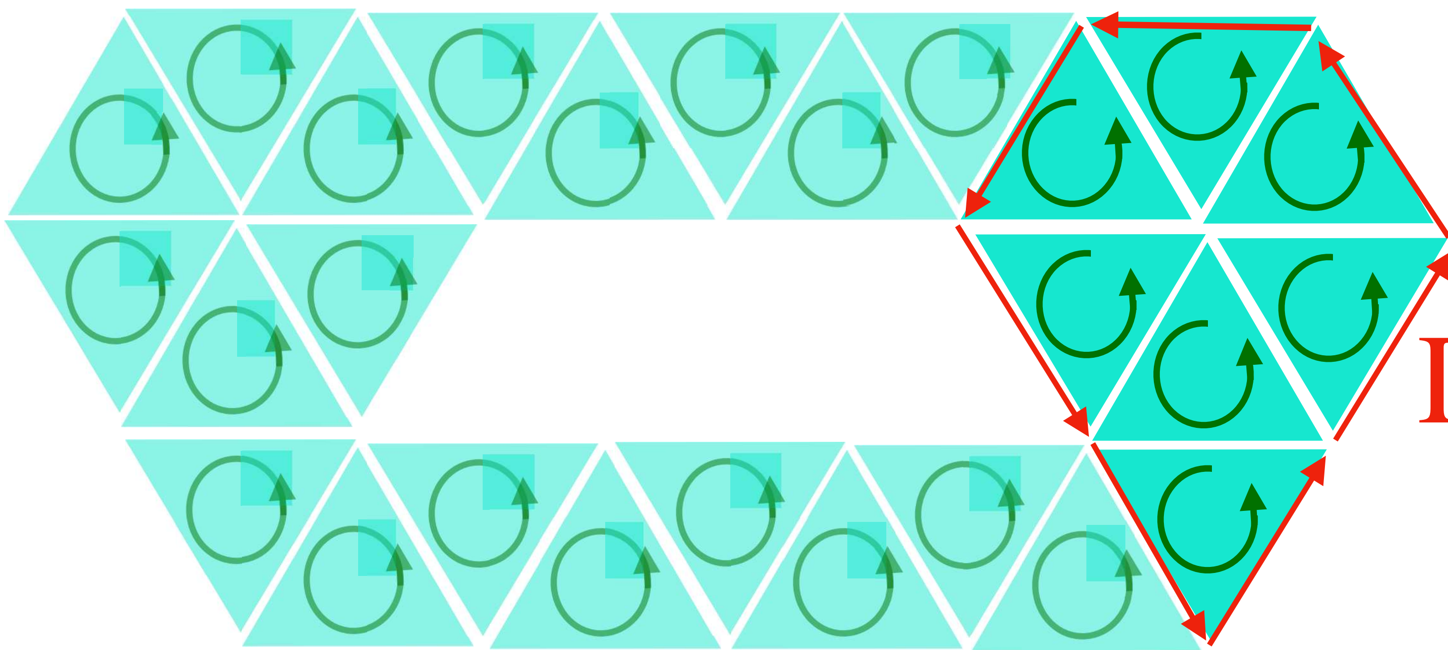


$$\partial_1 \Gamma = 0 \quad (\Rightarrow \Gamma \in \ker \partial_1)$$

Manifold fundamental homology class



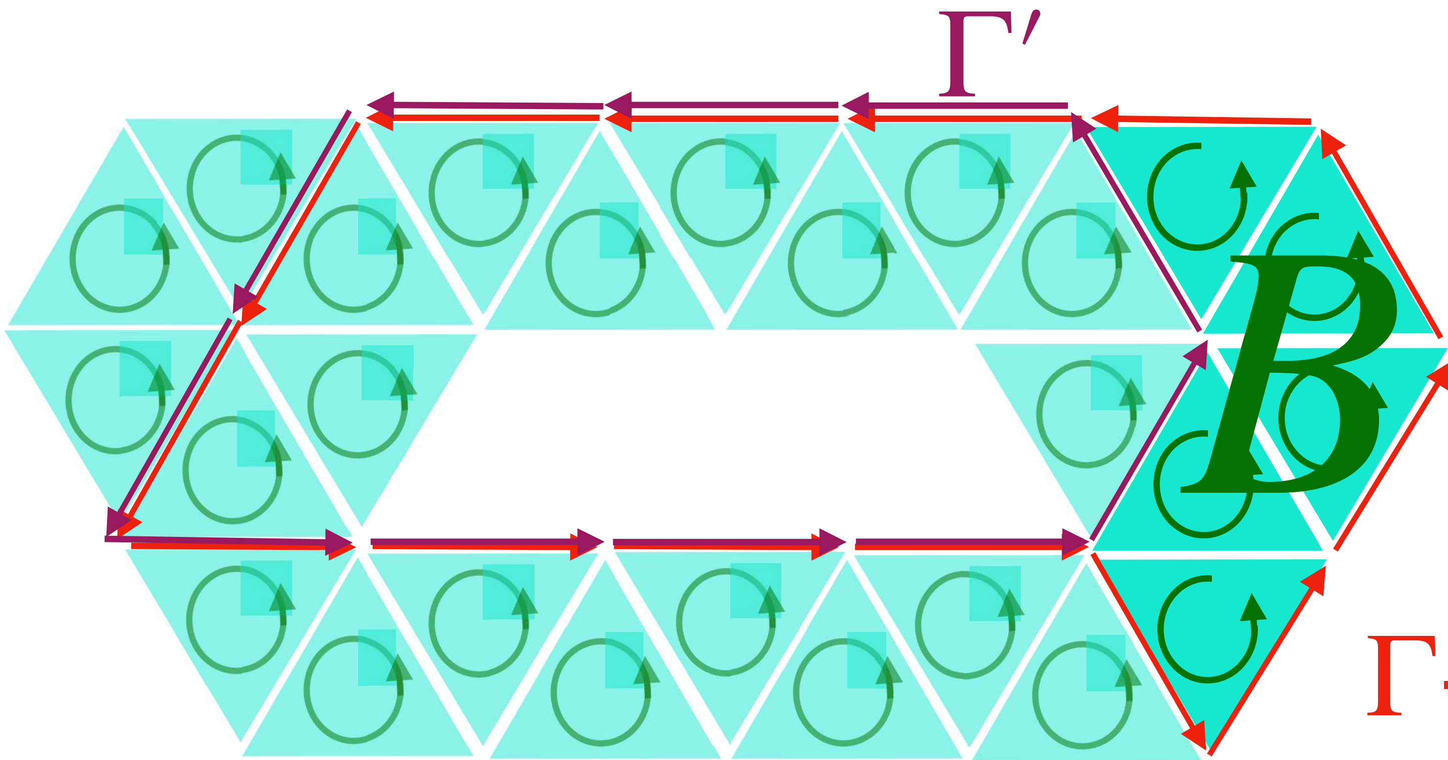
$$H_1 = \frac{\ker \partial_1}{\mathrm{Im}\, \partial_2}$$

$$\Gamma = \partial_2 B$$

$$\partial_1 \Gamma = 0 \qquad \Rightarrow \Gamma \in \ker \partial_1$$

But... $\Gamma \in \mathrm{Im}\, \partial_2 \;\Rightarrow [\Gamma]_{\mathrm{Im}\, \partial_2} = 0$

Manifold fundamental homology class

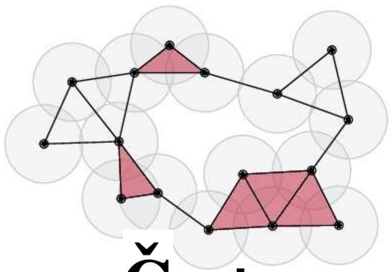


$$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2}$$

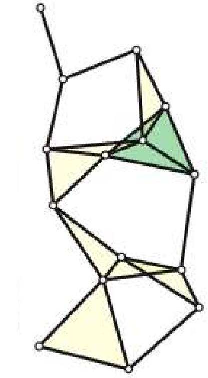
$$\Gamma - \Gamma' = \partial_2 B$$

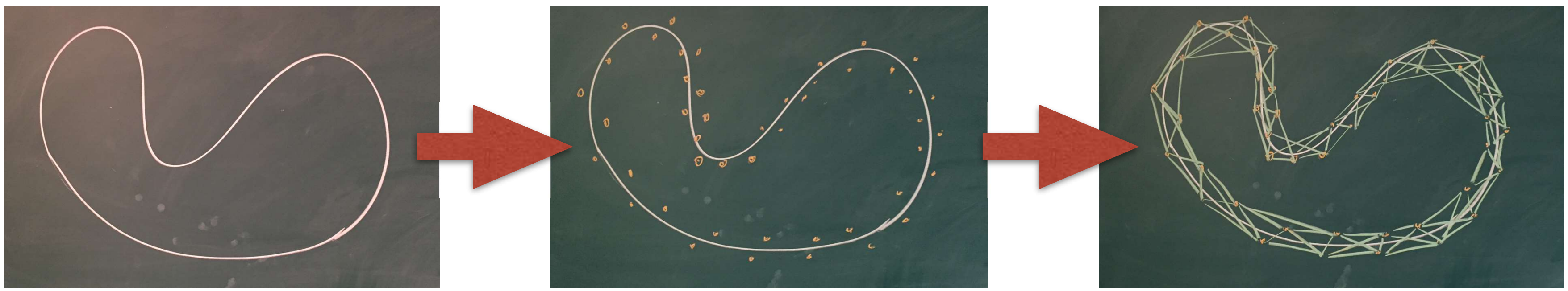
$$\Gamma - \Gamma' \in \operatorname{Im} \partial_2 \Longleftrightarrow [\Gamma]_{\operatorname{Im} \partial_2} = [\Gamma']_{\operatorname{Im} \partial_2}$$

$\Longleftrightarrow$ Γ and Γ' are homologous cycles



Fundamental class



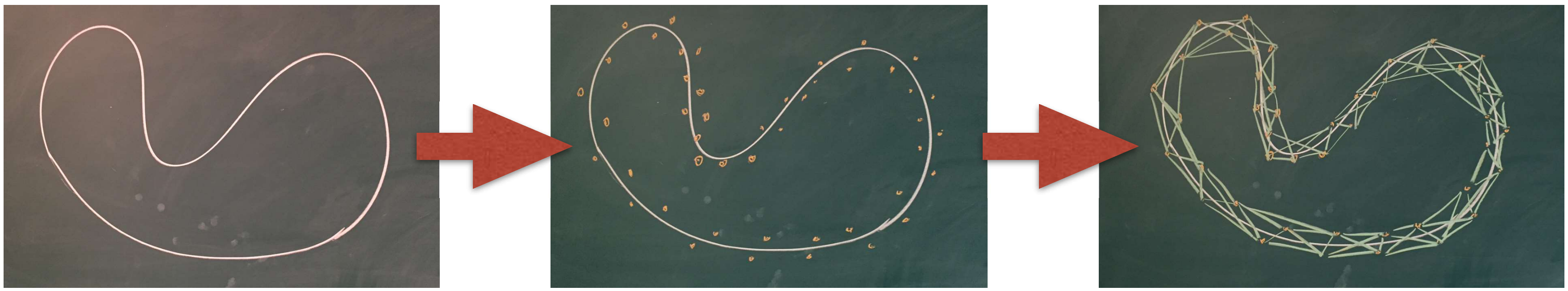


In particular, under adequate sampling conditions and parameters, **Čech** or **Vietoris-Rips** complexes ***K*** share **the homotopy type** and therefore the **d-homology of the complex.**
Which is then is one dimensional and **reproduces the fundamental class of the manifold**.

$$\Rightarrow \boxed{\mathbf{H}_d(K, \mathbb{Z}_2) \simeq \mathbb{Z}_2}$$

$\Rightarrow \mathbf{H}_d(K)$ contains a single non zero element.

Fundamental class



$$\Rightarrow \boxed{\mathbf{H}_d(K, \mathbb{Z}_2) \simeq \mathbb{Z}_2}$$

$\Rightarrow \mathbf{H}_d(K)$ contains a single non zero element.

But **Homology classes are not geometric**: we look for **a particular simplicial chain representative of the homology class** whose **support** could be **homeomorphic** to the sampled manifold:

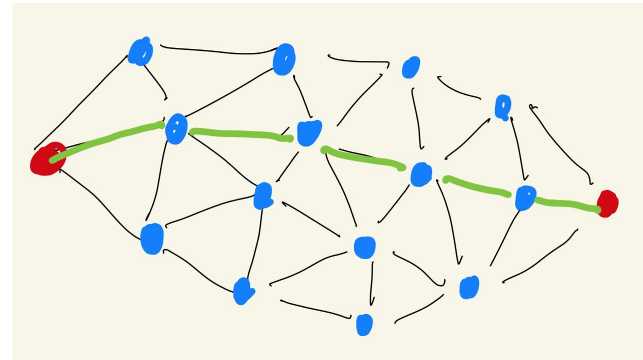
We search for it as the **minimum representative** chain in the fundamental class

Two canonical problems

Minimal chain for a given boundary β

Given $\beta \in C_{d-1}(K, \mathbb{F})$ **find:**

$$\Gamma_{\min} = \min\{\Gamma \in C_d(K, \mathbb{F}), \partial\Gamma = \beta\}$$

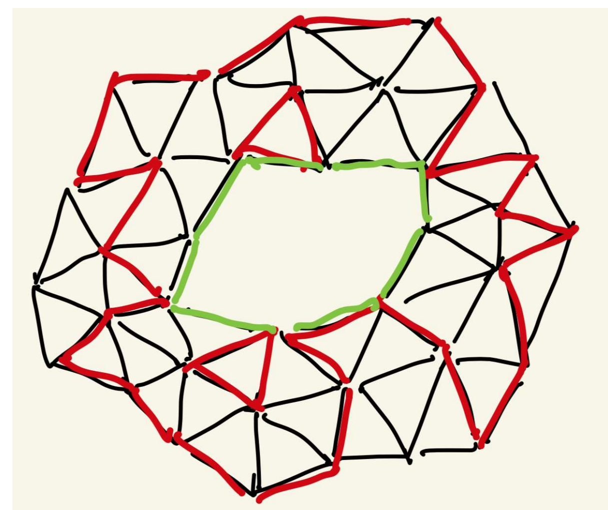


$\mathbf{dim}(K) = 1$

Minimal chain homologous to α

Given $\alpha \in C_d(K, \mathbb{F})$ **find:**

$$\Gamma_{\min} = \min\{\alpha + \partial\omega, \omega \in C_{d+1}(K, \mathbb{F})\}$$



$\mathbf{dim}(K) = 2$

Two canonical problems ($\mathbb{Z}_2$ coefficients)

Minimal chain for a given boundary β

Given $\beta \in C_{d-1}(K, \mathbb{Z}_2)$ **find:**

$$\Gamma_{\min} = \min\{\Gamma \in C_d(K, \mathbb{Z}_2), \partial\Gamma = \beta\}$$

Minimal chain homologous to α

Given $\alpha \in C_d(K, \mathbb{Z}_2)$ **find:**

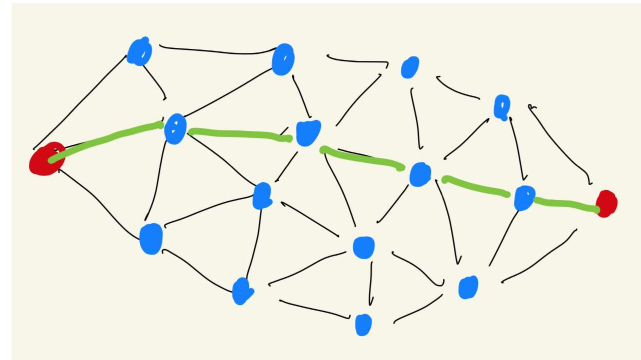
$$\Gamma_{\min} = \min\{\alpha + \partial\omega, \omega \in C_{d+1}(K, \mathbb{Z}_2)\}$$

$\min$ according to:

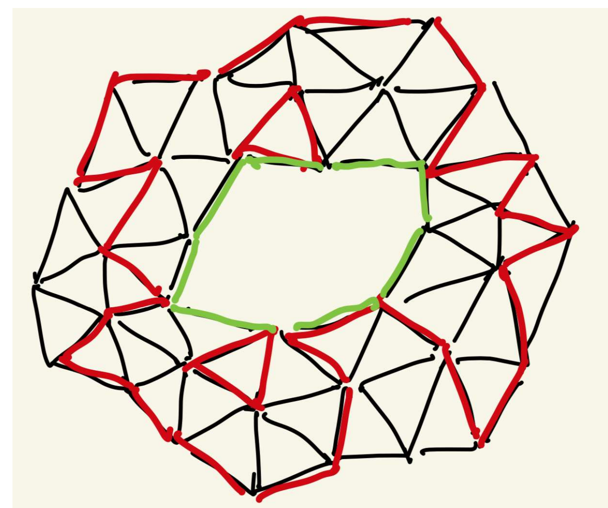
- * L^1 **norm,**
- * **lexicographic order.**

NP-hard in general *(Chen, Freedman, 2011)*

$\mathcal{O}(n^3)$ *(Cohen-Steiner, L, Vuillamy, 2019)*



$\mathbf{dim}(K) = 1$



$\mathbf{dim}(K) = 2$

$\mathcal{O}(n^3)$ general algorithm

same as total reduction in homological persistence

$\partial_{d+1} = \mathbf{R} \, . \, \mathbf{V}$

$+$

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \Gamma_0 = \alpha = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \Gamma_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \Gamma_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In $\mathbf{R}$, there is exactly one column with a lowest 1 for each reducible simplex 1

Total reduction of Γ using the reduced boundary operator $\mathbf{R}$

Algorithm 2: Total reduction algorithm

Inputs: A d-chain Γ, the reduction matrix R from Algorithm 1

for $i \leftarrow m$ **to** 1 **do**

  if $\Gamma[i] \neq 0$ **and** $\exists j \in [1, n]$ *with* $\text{low}(j) = i$ *in* R **then**

    $\Gamma \leftarrow \Gamma + R_j$

  end

end

Minimal homology representative cycle

Some related works on L^1 minimal homologous chain…

Erin W Chambers, Jeff Erickson, and Amir Nayyeri. Minimum cuts and shortest homologous cycles. In *Proceedings of the twenty-fifth annual symposium on Computational geometry*, pages 377–385. ACM, 2009.

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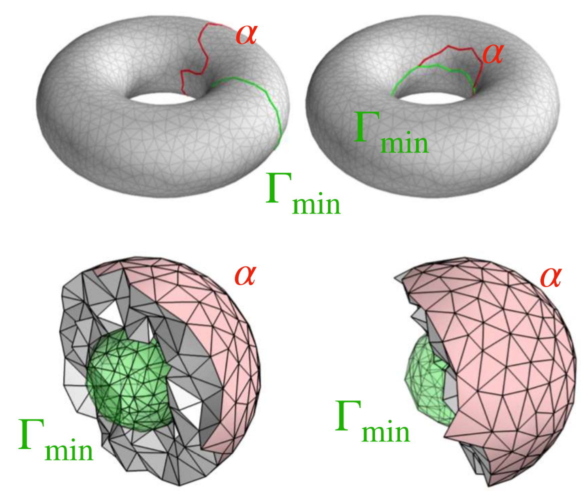
Tamal K Dey, Tao Hou, and Sayan Mandal. Computing minimal persistent cycles: Polynomial and hard cases. *arXiv preprint arXiv:1907.04889*, 2019.

Hardness results (linear programming):

NP-Hard in general for coefficients in $\mathbb{Z}_2$



(Thanks to T. Dey et Al. for the figures)

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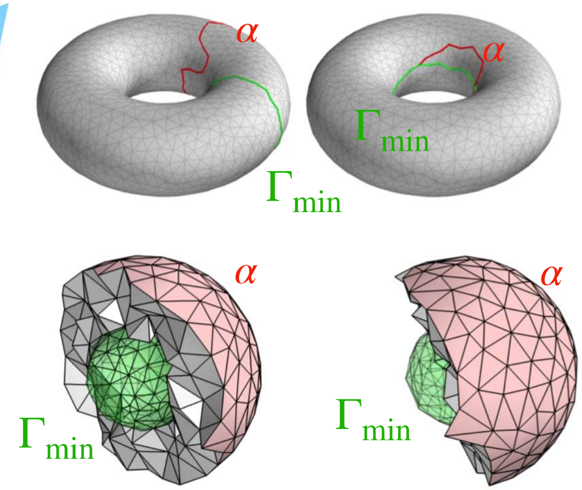
Hardness results (linear programming):

polynomial algorithm when total unimodularity of boundary operator



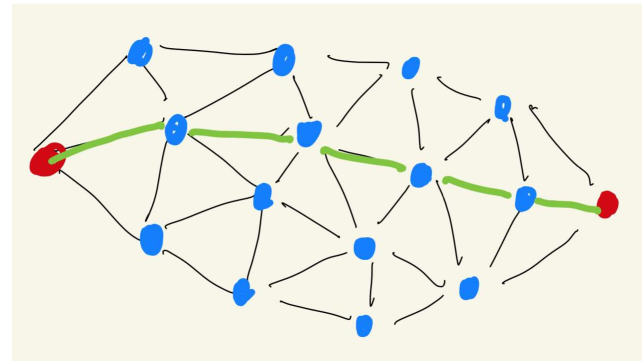
(Thanks to T. Dey et Al. for the figures)

Two canonical problems again (for lexicographic minima)

Lexicographic-minimal chain for a given boundary

Given $\beta \in C_{d-1}(K, \mathbb{Z}_2)$ **find:**

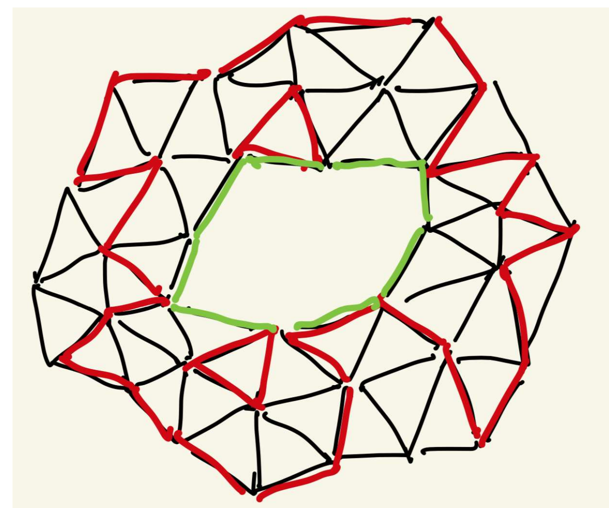
$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\Gamma \in C_d(K, \mathbb{Z}_2), \partial\Gamma = \beta\}$$



Lexicographic-minimal homologous chain:

Given $\alpha \in C_d(K, \mathbb{Z}_2)$ **find:**

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\alpha + \partial\omega, \omega \in C_{d+1}(K, \mathbb{Z}_2)\}$$



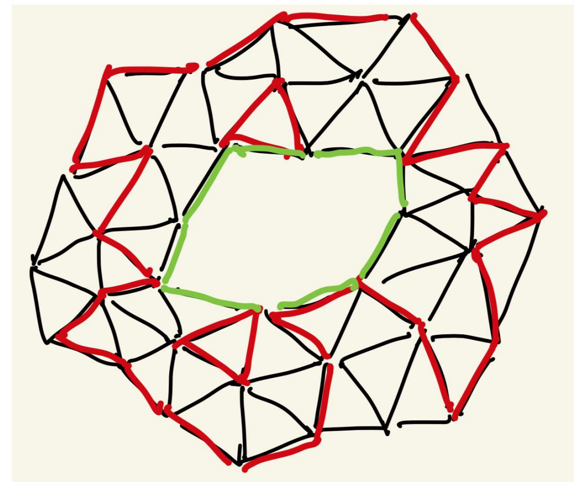
$\beta \mapsto \Gamma_{\min}$ and $\alpha \mapsto \Gamma_{\min}$ are **linear maps**, (as for L^2 minima) but minima are **sparse** (as for L^1 minima).

$\mathcal{O}(n\alpha(n))$ algorithm in co-dimension 1

Lexicographic-minimal homologous chain:

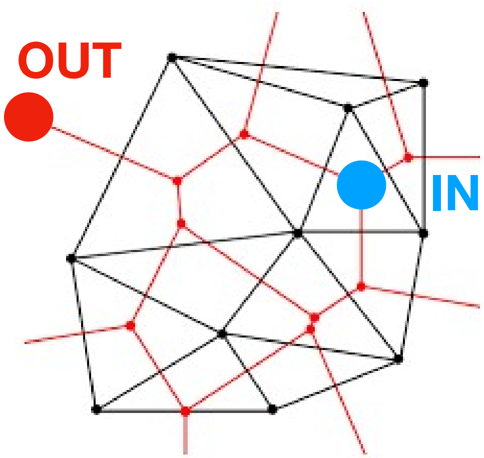
Given $\alpha \in C_d(K, \mathbb{Z}_2)$ **find:**

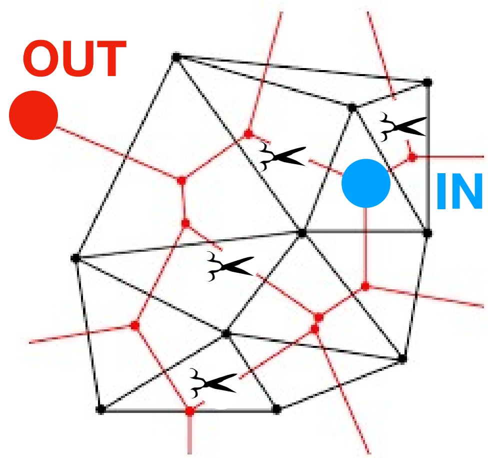
$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\alpha + \partial\omega, \omega \in C_{d+1}(K, \mathbb{Z}_2)\}$$

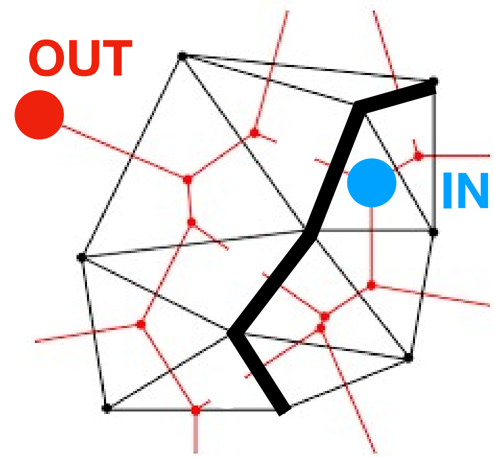


Once d-simplices are **sorted** (in time $\mathcal{O}(n \log n)$):

$\mathcal{O}(n\,\alpha(n))$ algorithm using **union-find** data structure on the **dual graph** to solve a **lexicographic MIN-CUT/MAX-FLOW** problem.

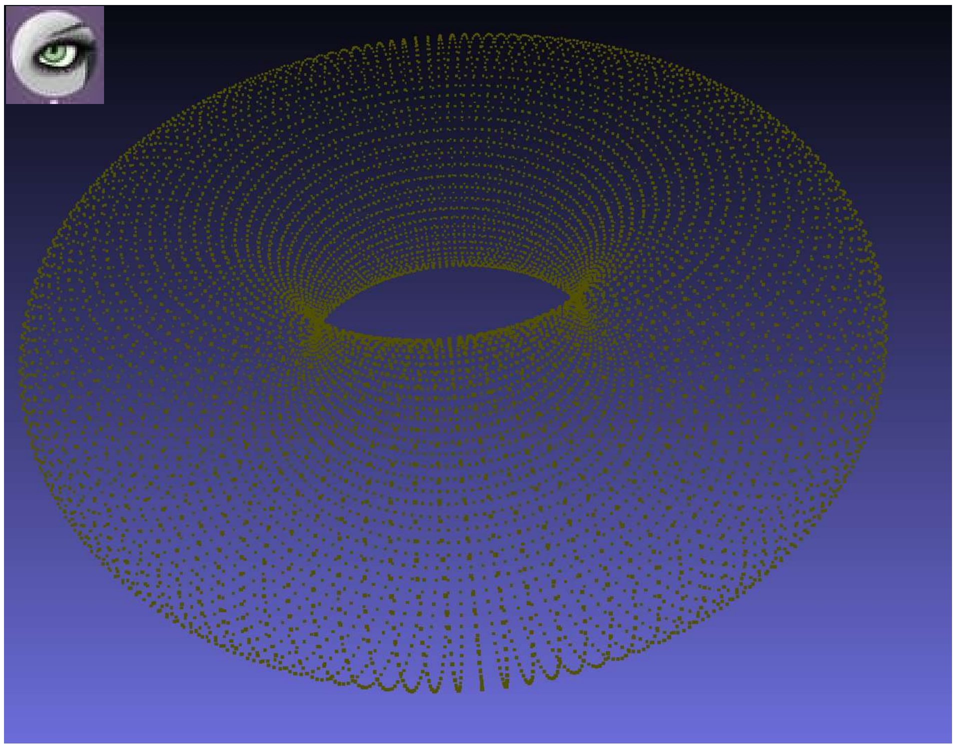


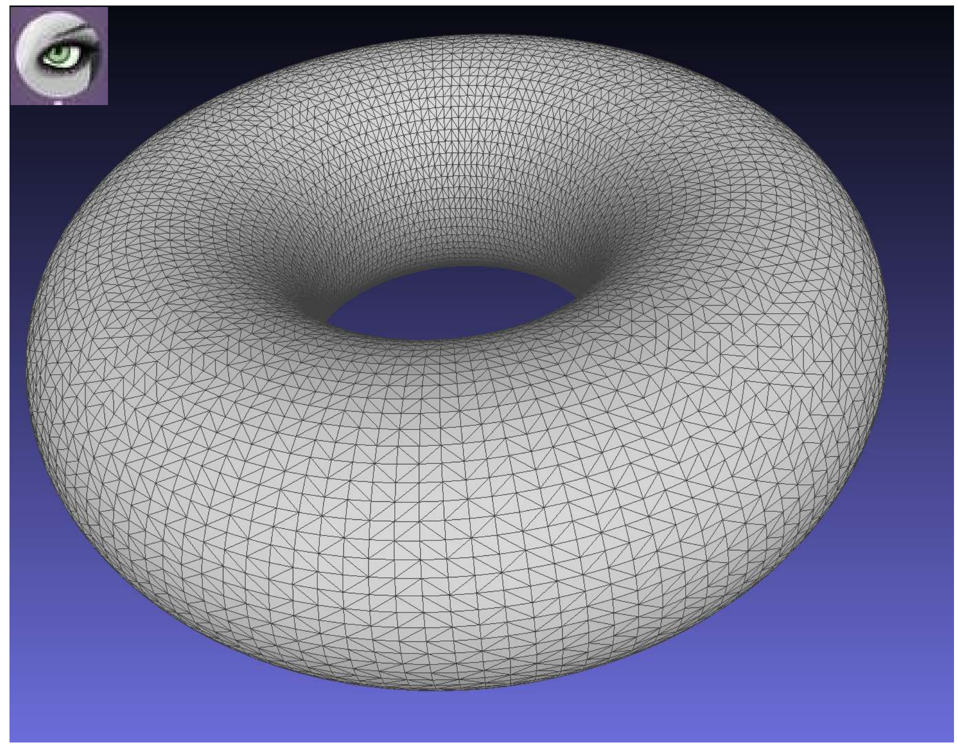




Minimal homology representative cycle

Examples of lexicographic-minimal cycle





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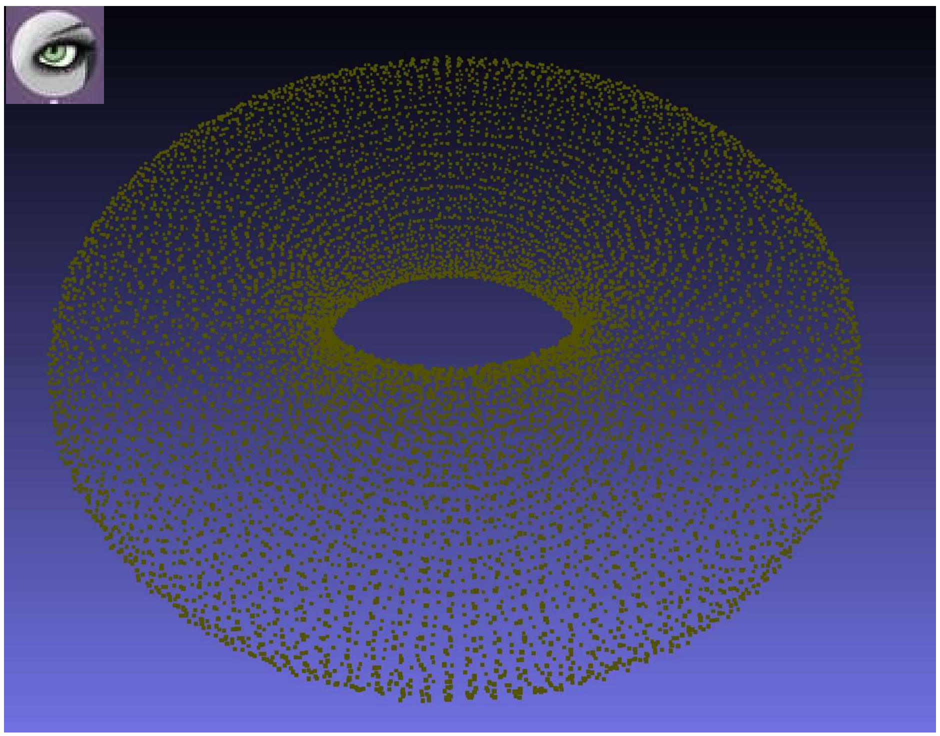

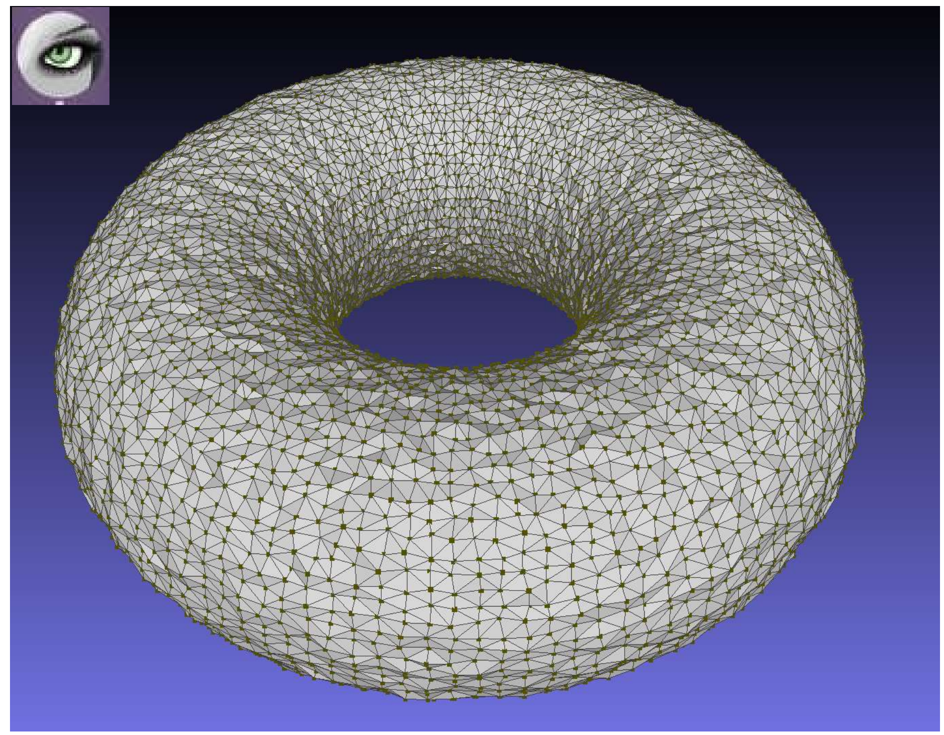

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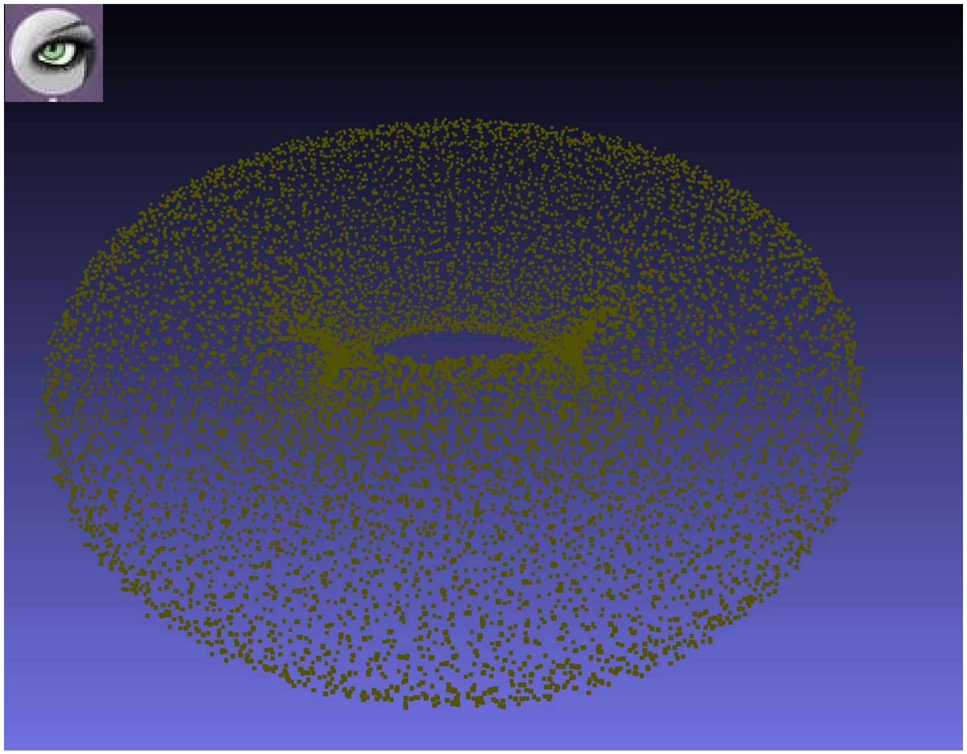

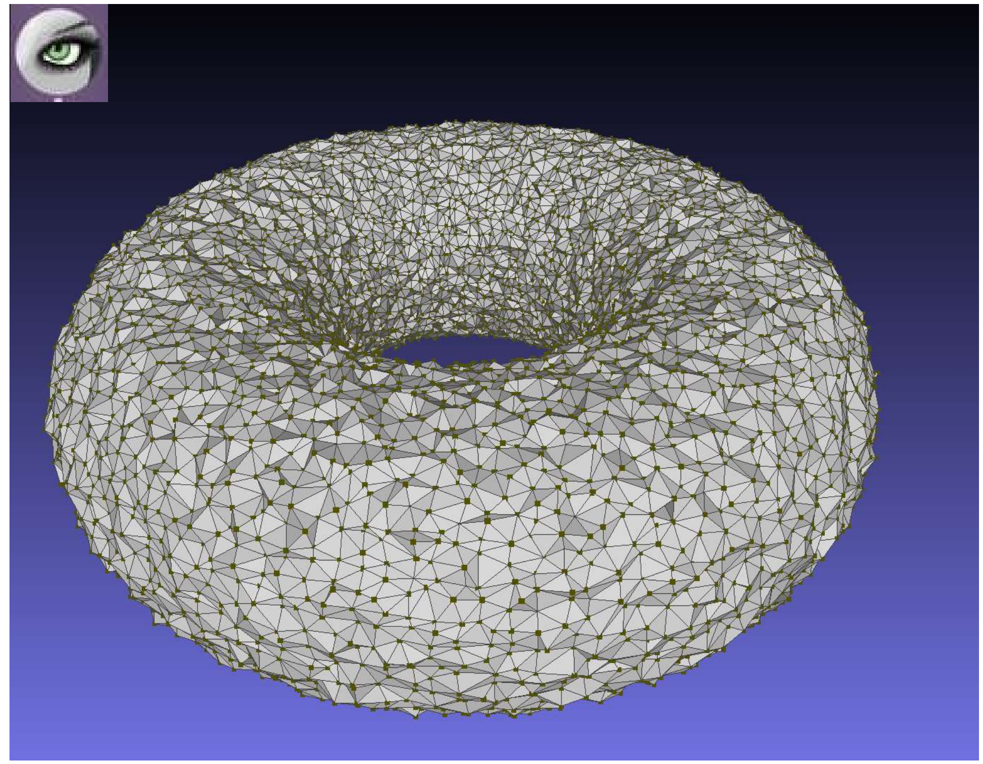

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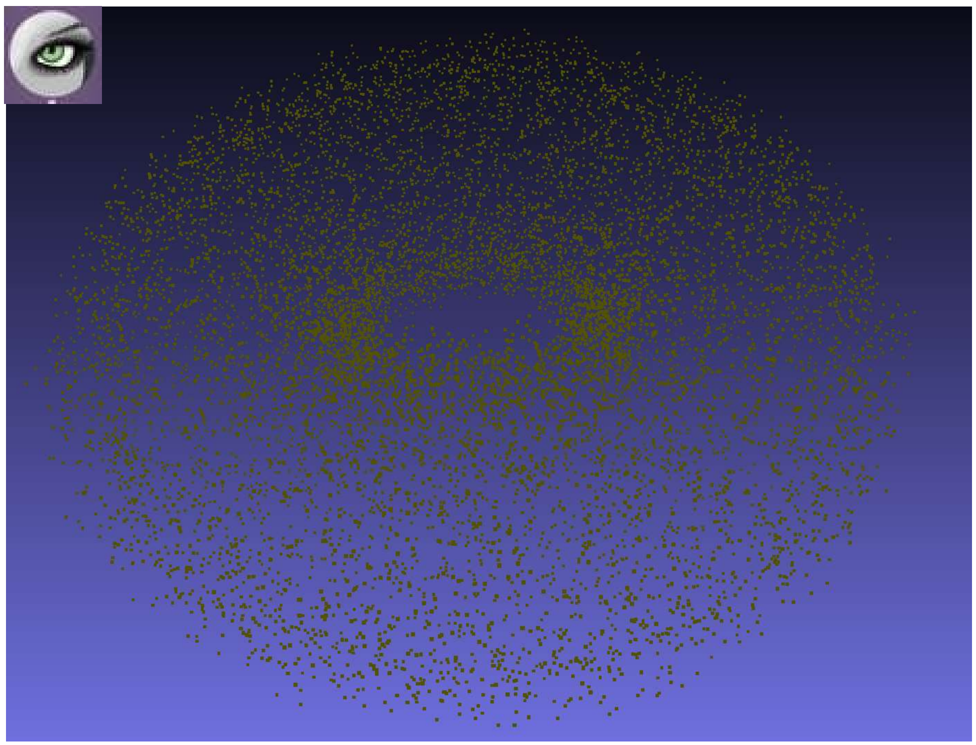

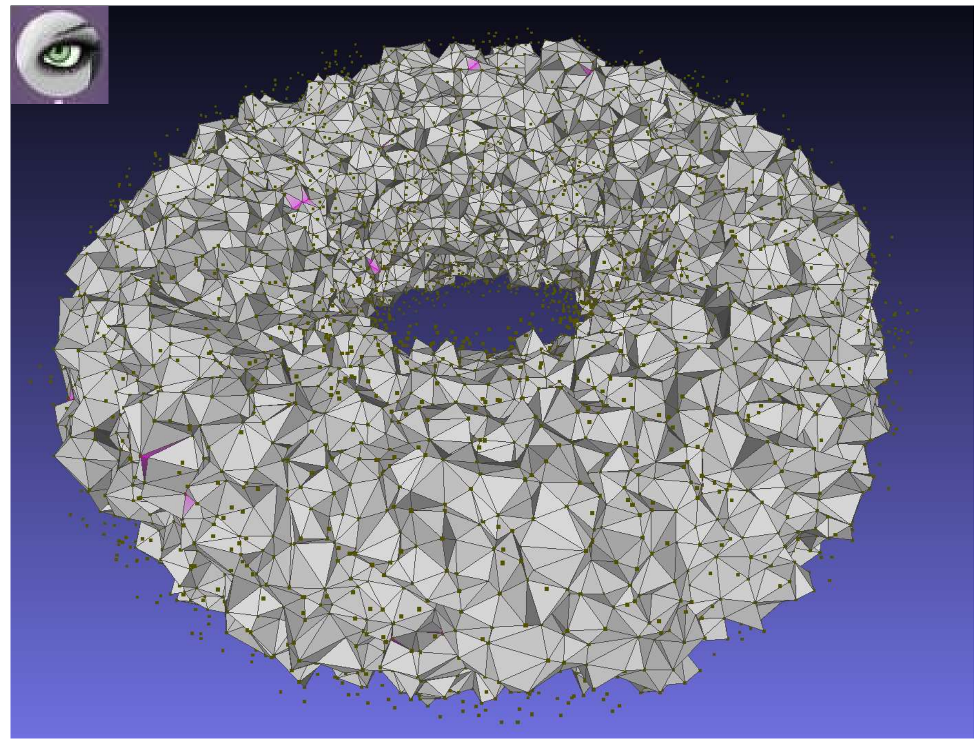

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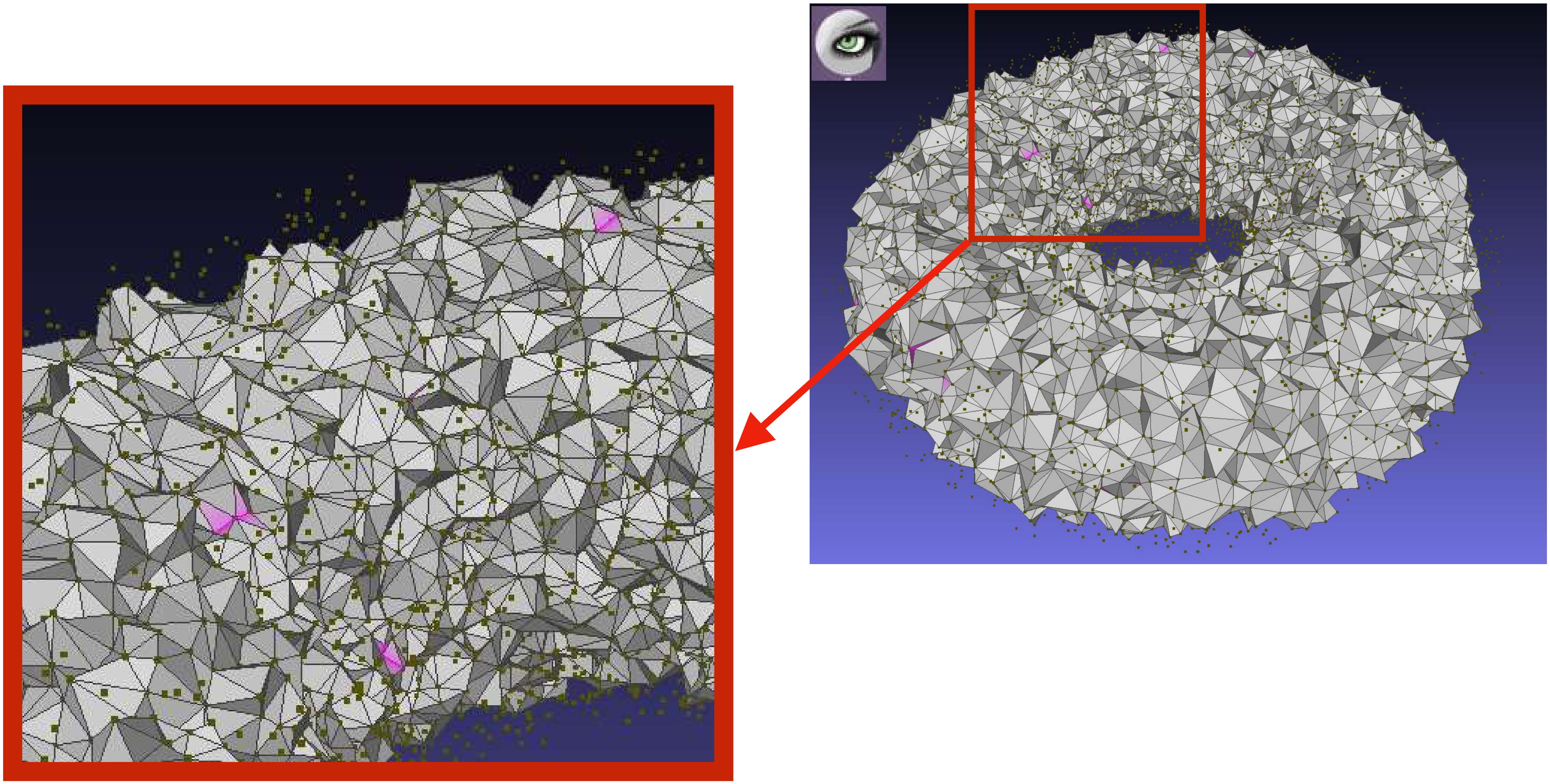

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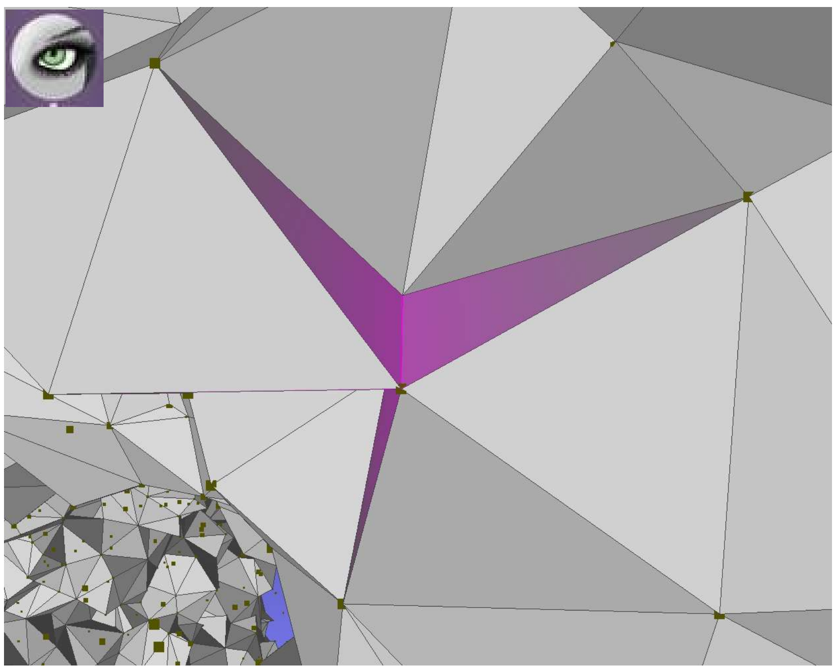
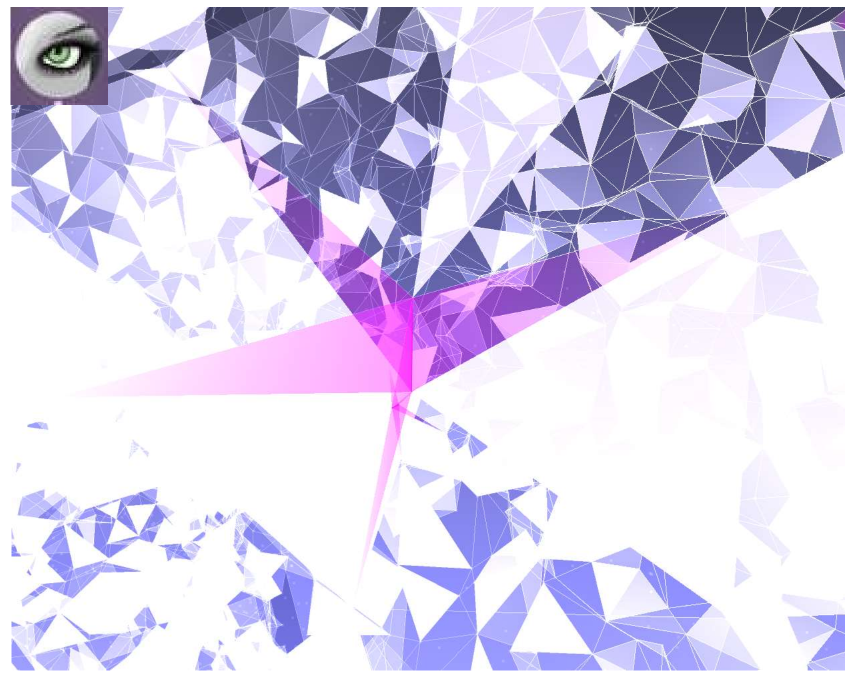
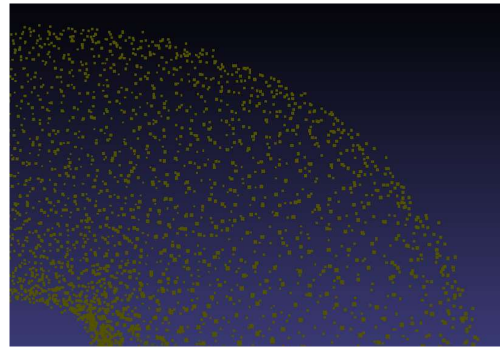
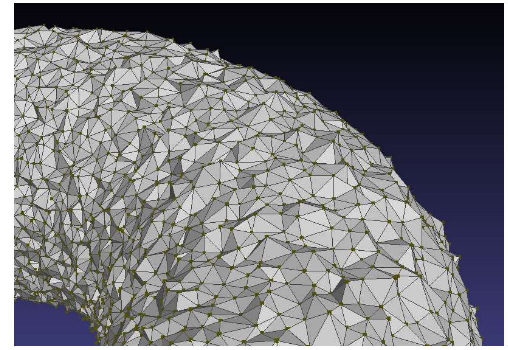




Triangulation of positive reach 2-manifolds





$\mathbf{P} \subset \mathcal{M}$ is an (ϵ, η)-sampling of $\mathcal{M}$ iff:

- $d_H(\mathbf{P}, \mathcal{M}) < \epsilon$
- $\forall p, q \in \mathbf{P}, p \neq q \Rightarrow d(p,q) > \eta$

Theorem 1. *There are constants C_1, C_2, C_3 such that:*

If $\mathcal{M}$ is a smooth 2-manifold embedded in $\mathbb{R}^n$ with reach $\mathcal{R}$, $\mathbf{P}$ an (ϵ, η)-sampling of $\mathcal{M}$ and K a Čech or Vietoris-Rips complex on K with parameter λ, such that:

$$C_1 \epsilon < \lambda < C_2 \mathcal{R}$$

and:

$$\frac{\epsilon}{\mathcal{R}} < C_3 \left(\frac{\eta}{\epsilon}\right)^{10}$$

Then if:

$$\mathcal{T} = \min_{\sqsubseteq_{lex}} \mathrm{Ker}(\partial_2) \setminus \mathrm{Im}(\partial_3)$$

The restriction of $\pi_{\mathcal{M}}$ to $|\mathcal{T}|$ is an homeomorphism on $\mathcal{M}$. It follows that $(|\mathcal{T}|, \pi_{\mathcal{M}})$ is a triangulation of $\mathcal{M}$.

K captures the homotopy type $\Rightarrow \beta_2 = 1$

Lexicographic minimal chain in $H_2(K, \mathbb{Z}_2)$ is a triangulation

Triangulation of positive reach 2-manifolds

(by linear programming)

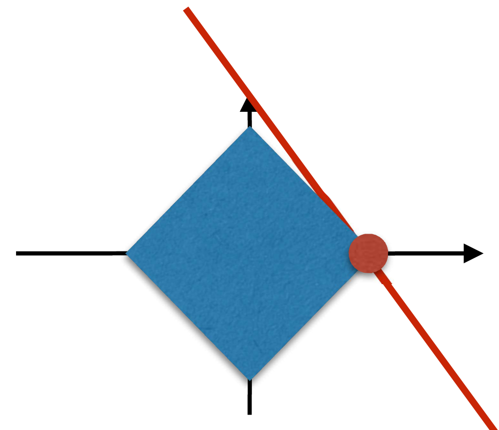
$$\|\Gamma\|_p = \sum_{\sigma \in K_d} w_p(\tau)^p \, |\Gamma(\tau)|$$

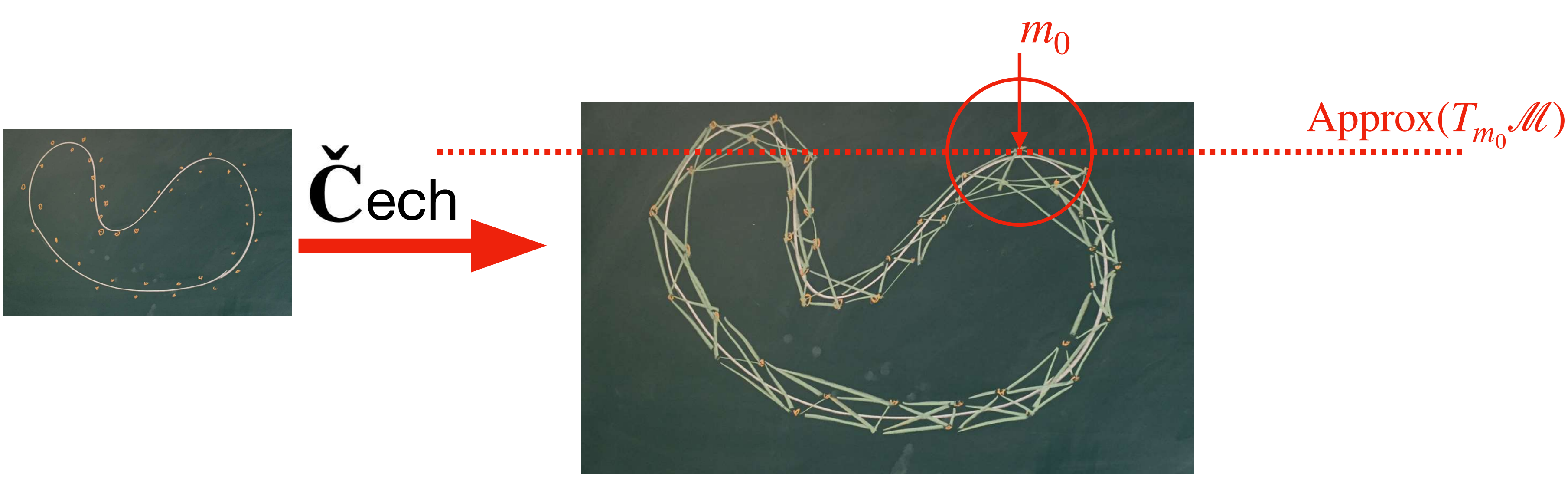
(Dominique Attali and L. "Delaunay-Like Triangulation of Smooth Orientable Submanifolds by ℓ1-Norm Minimization. » 2022)

The support of the chain that minimizes $\Gamma \mapsto \|\Gamma\|_1$

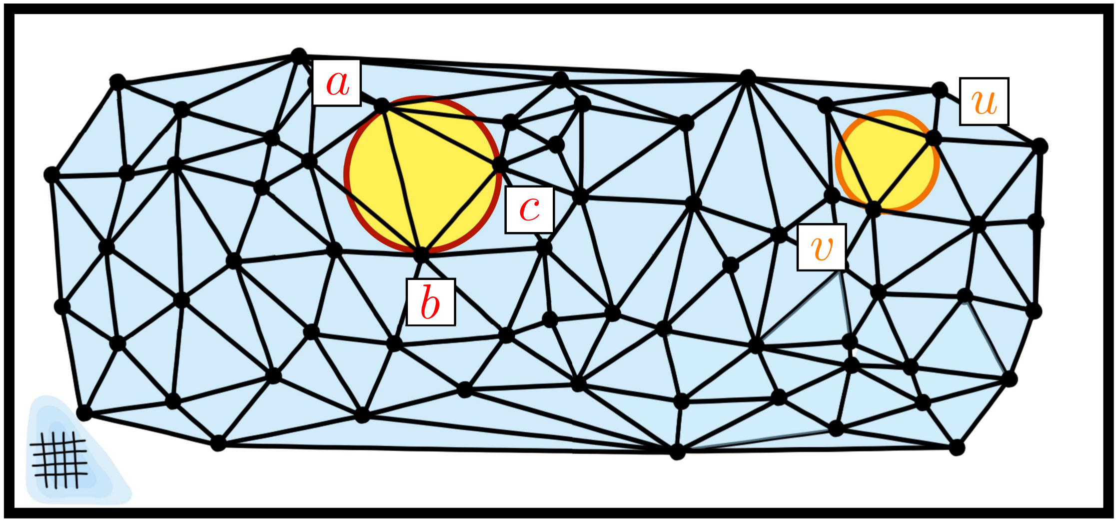
under constraint $\begin{cases} \partial\Gamma = 0 \\ \text{load}_{\,m_0,\ \text{Approx}(T_{m_0}\mathscr{M})} = 1, \end{cases}$

***triangulates** the manifold.*

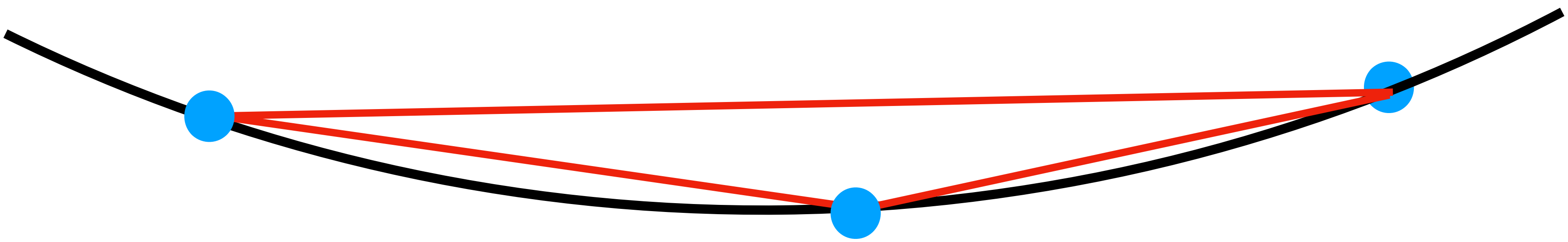




Why proofs does not extend to 3-manifolds ?



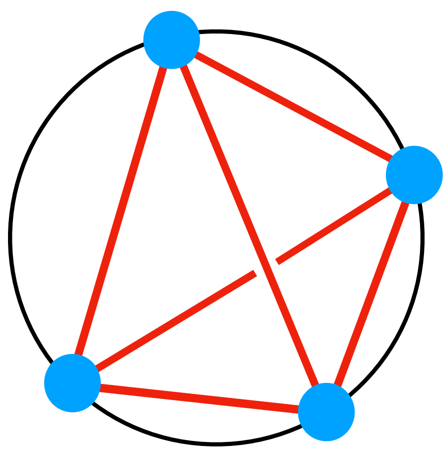
good sampling conditions
=> Delaunay triangles are smalls and cannot be too flat



Therefore, on a Manifold with large reach,
they cannot be « vertical »

Why proofs does not extend to 3-manifolds ?

But for dimension $\geq$ 3 manifolds,
simplices may be arbitrary flat !



slivers !

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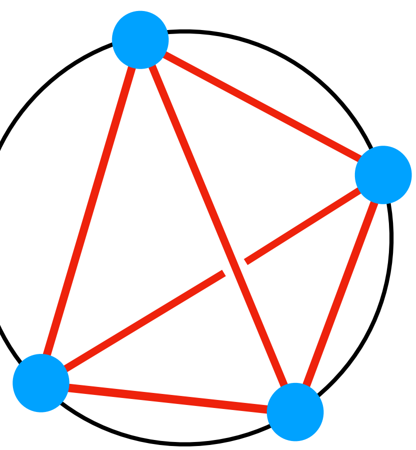
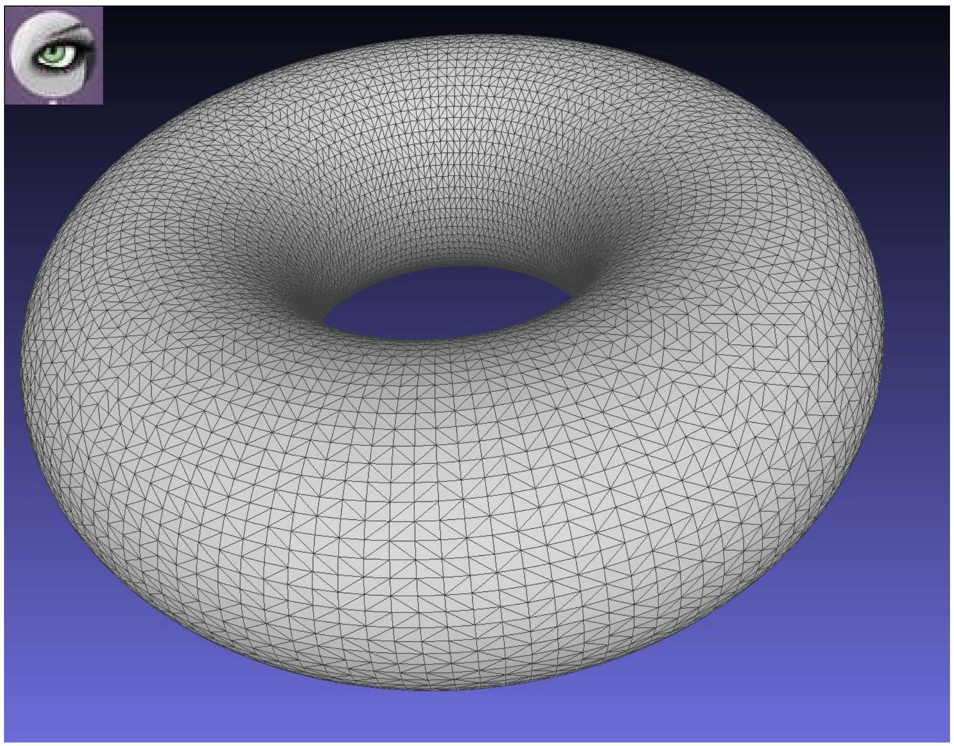


slivers !

Perturbation methods works this out … at least in theory

Minimal homology representative cycle

Examples of lexicographic-minimal cycle



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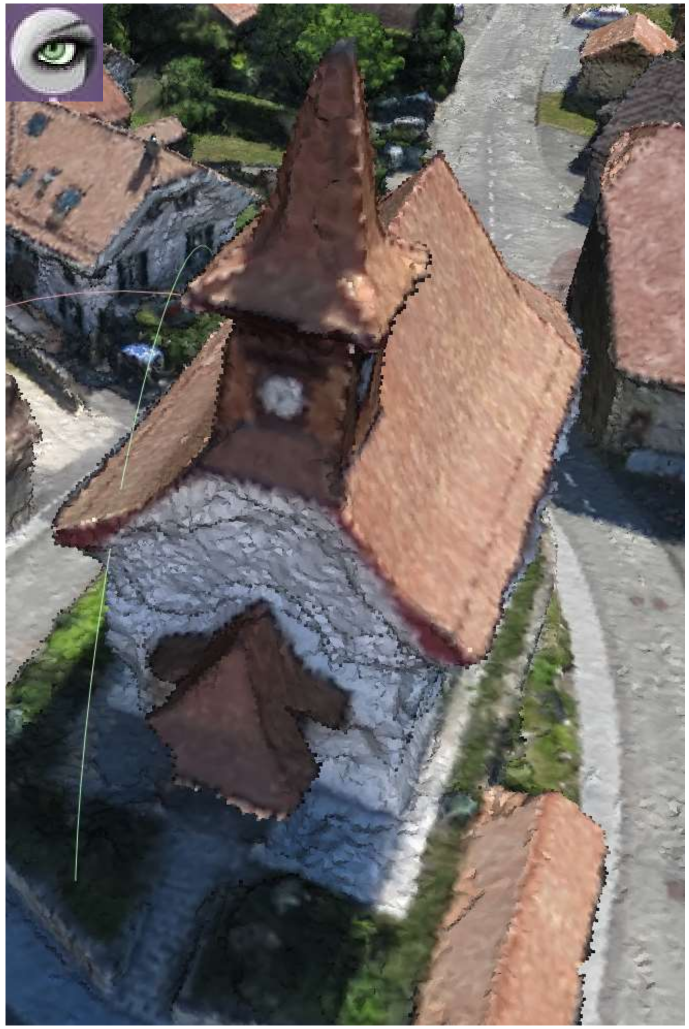

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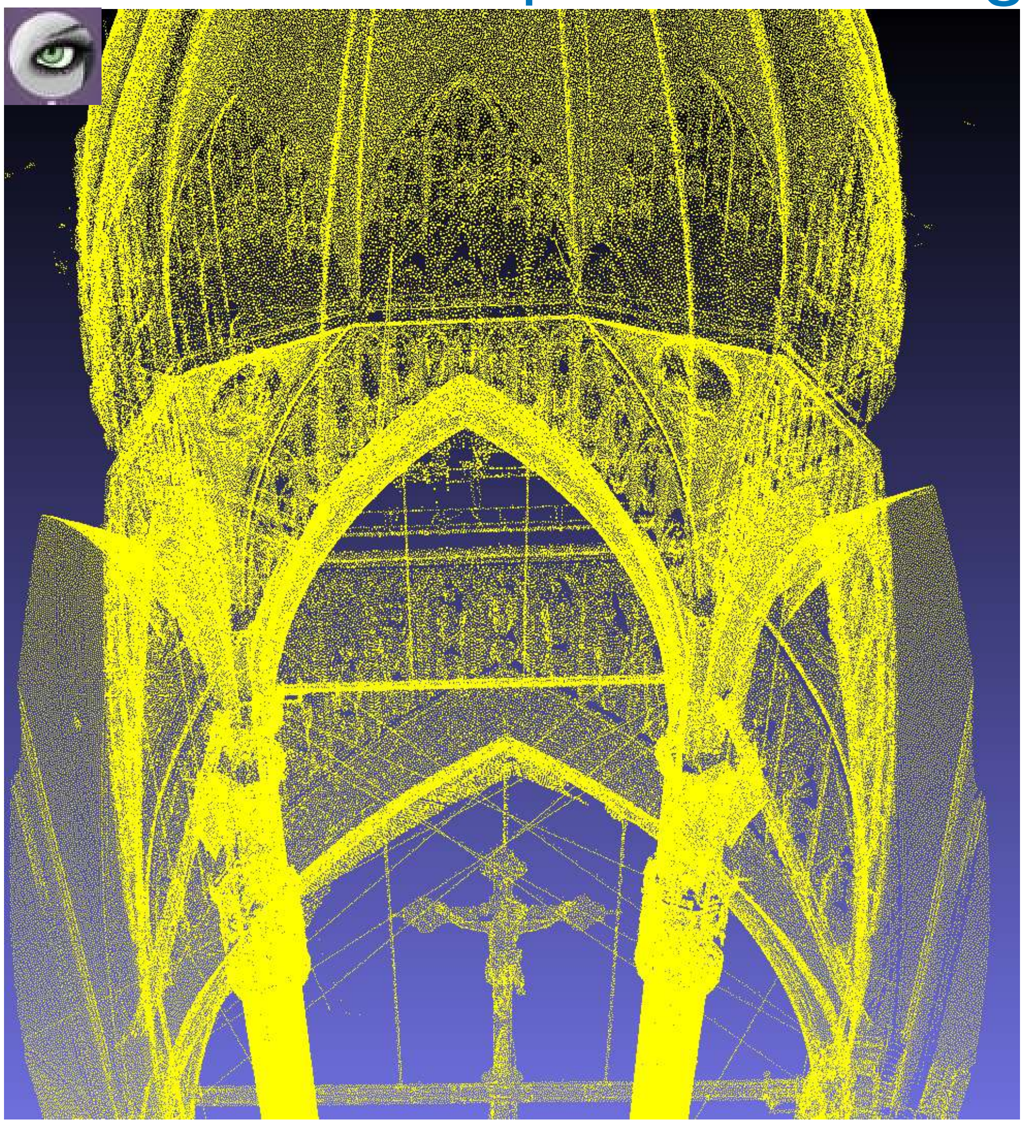

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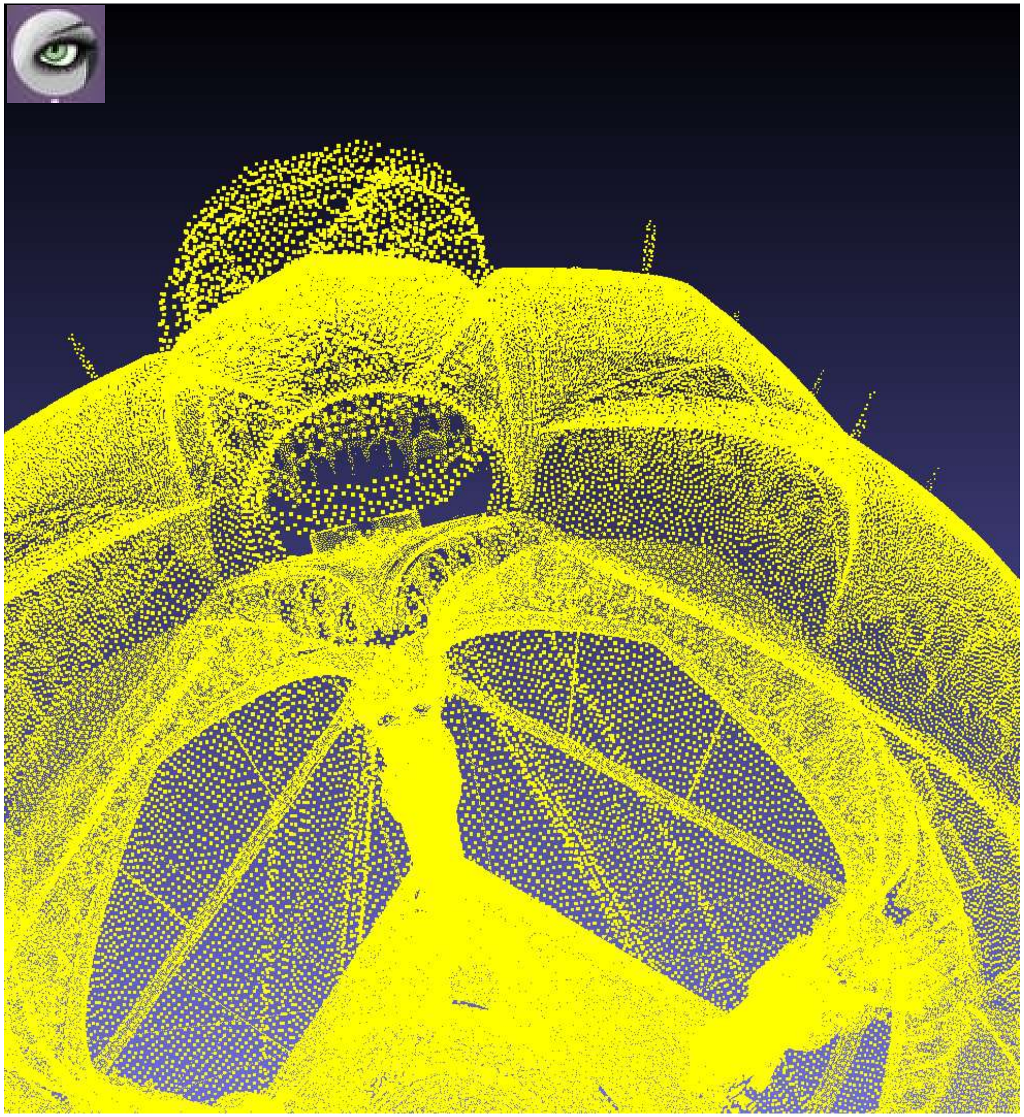

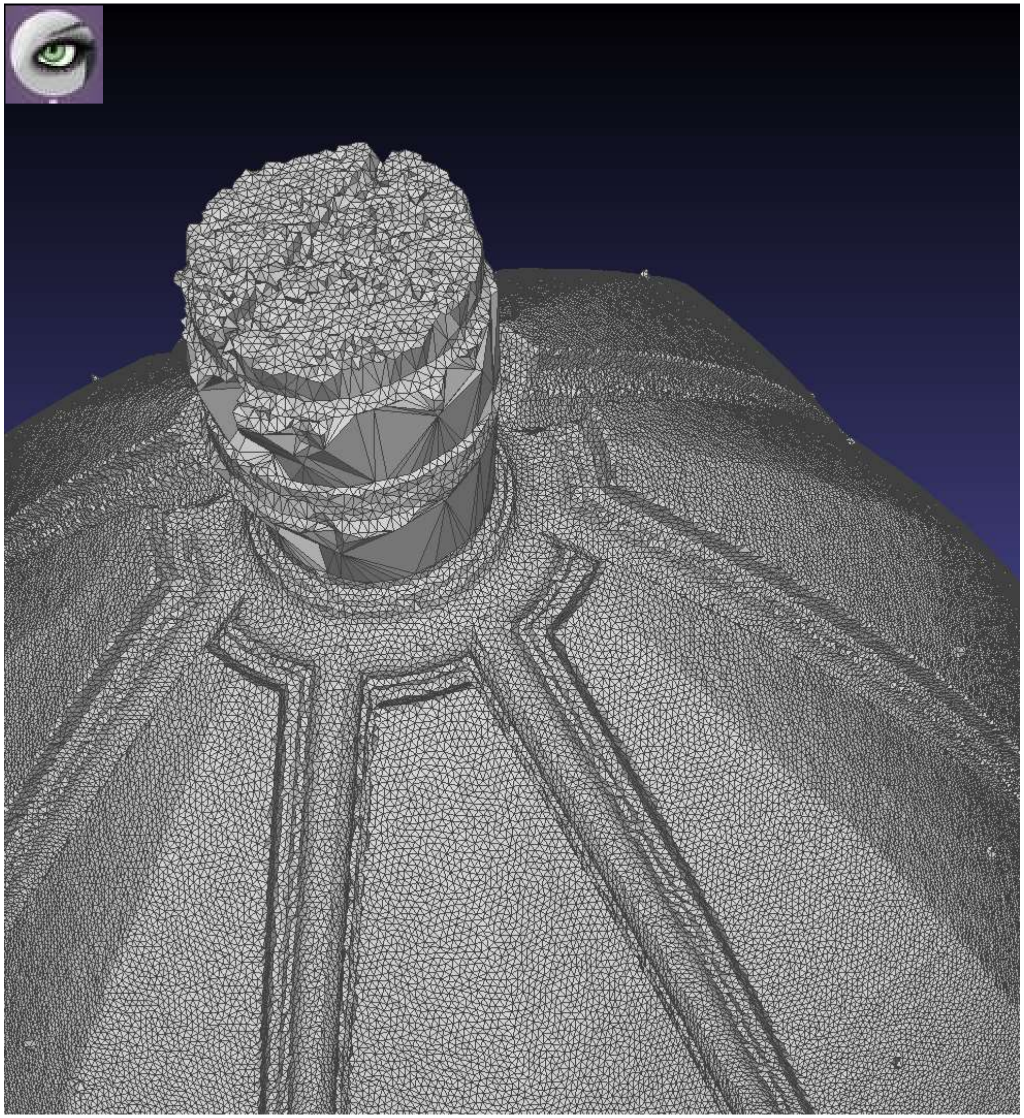

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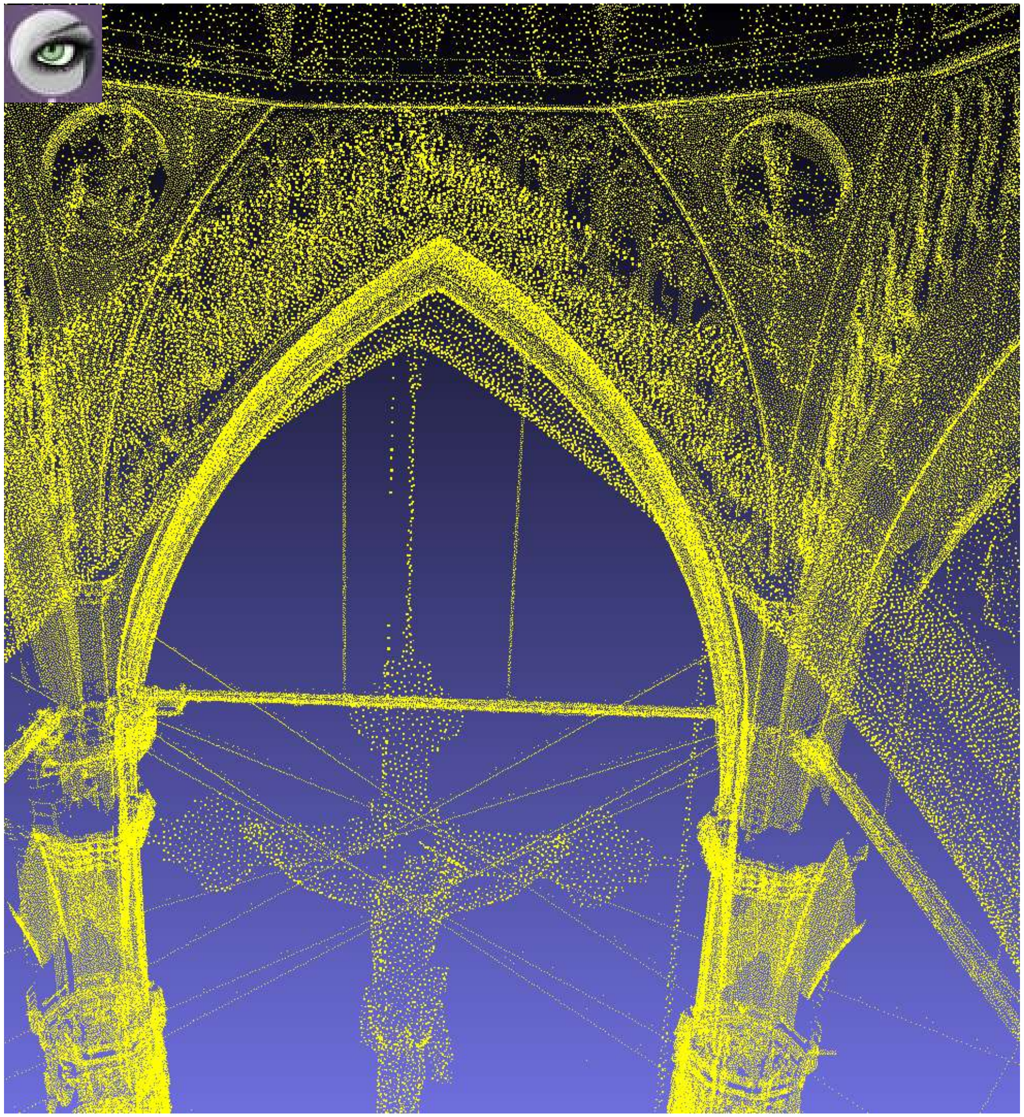

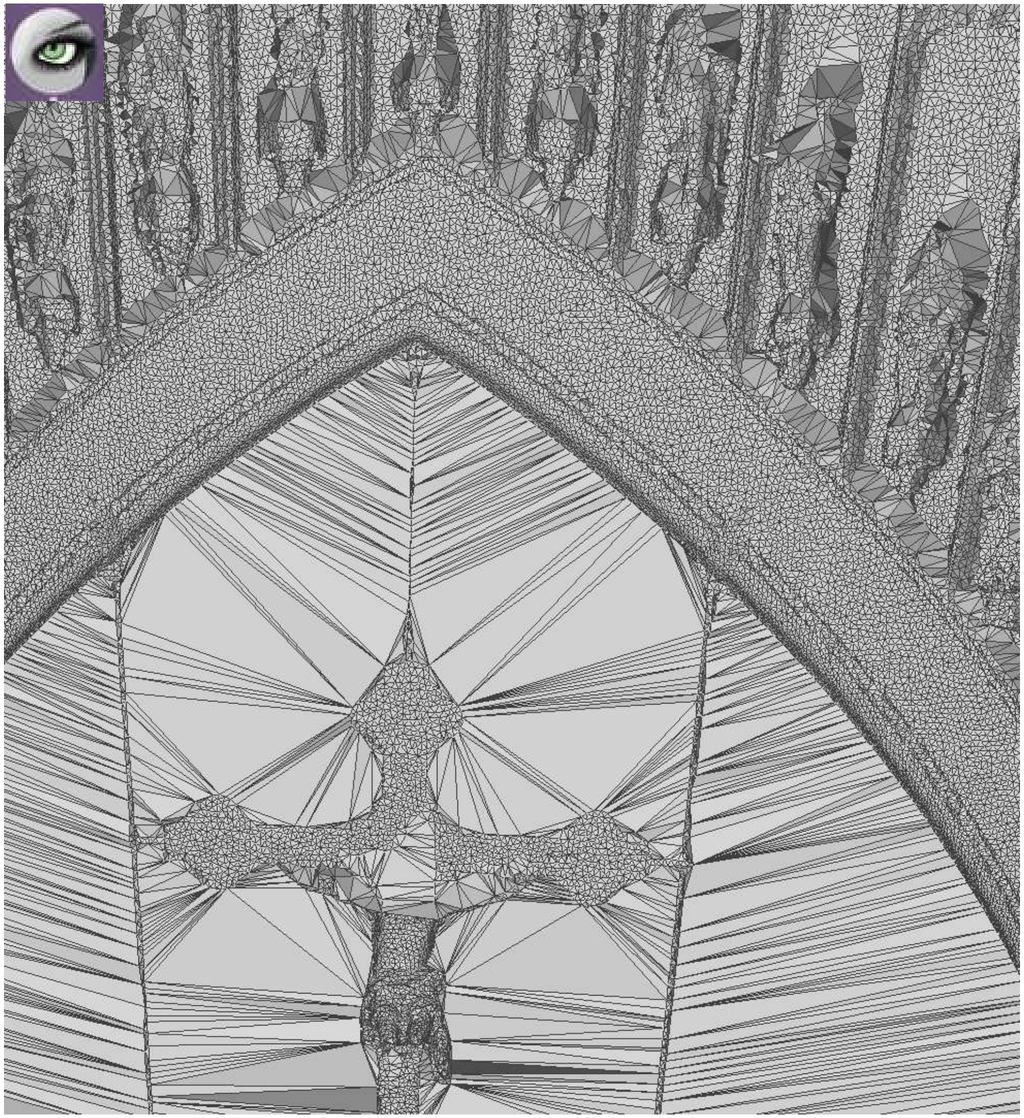

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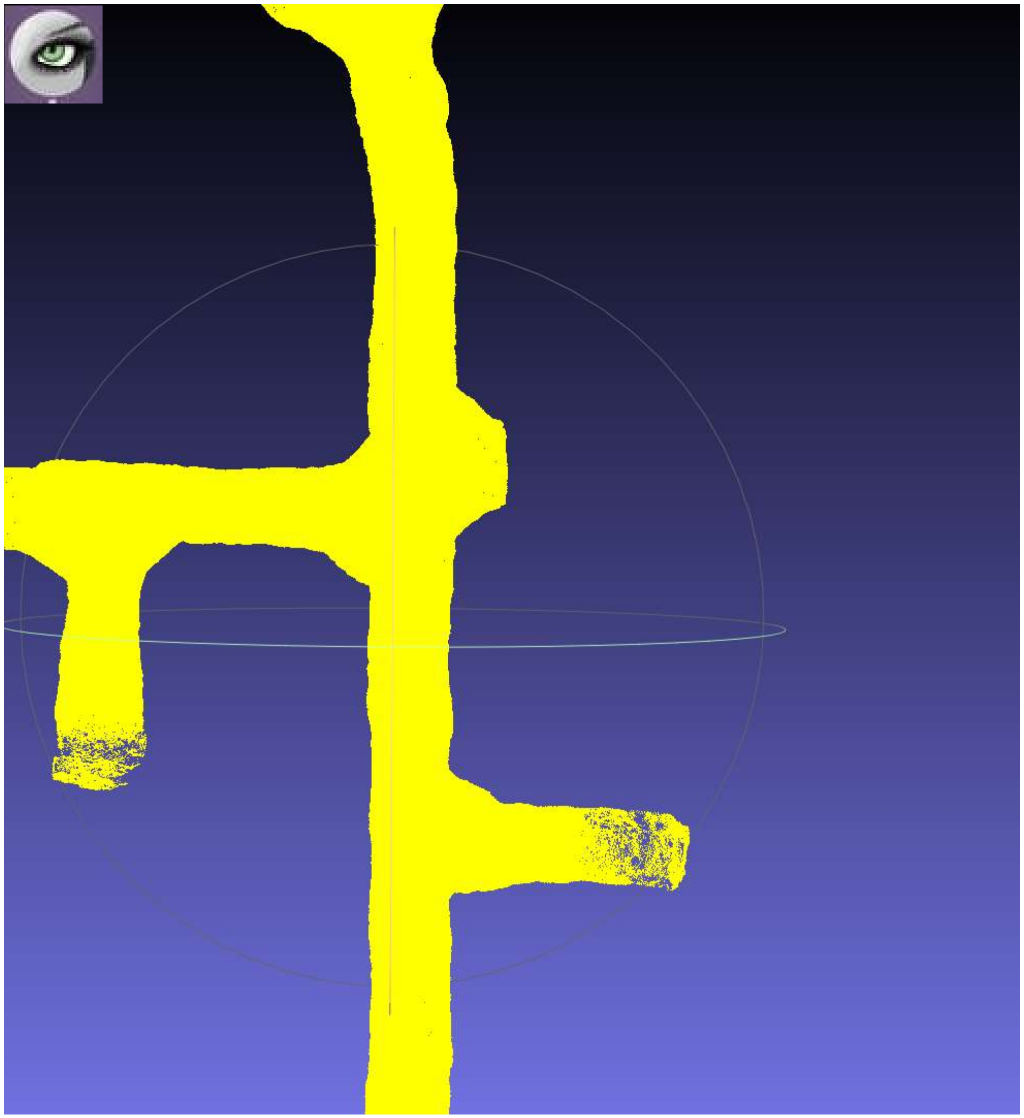

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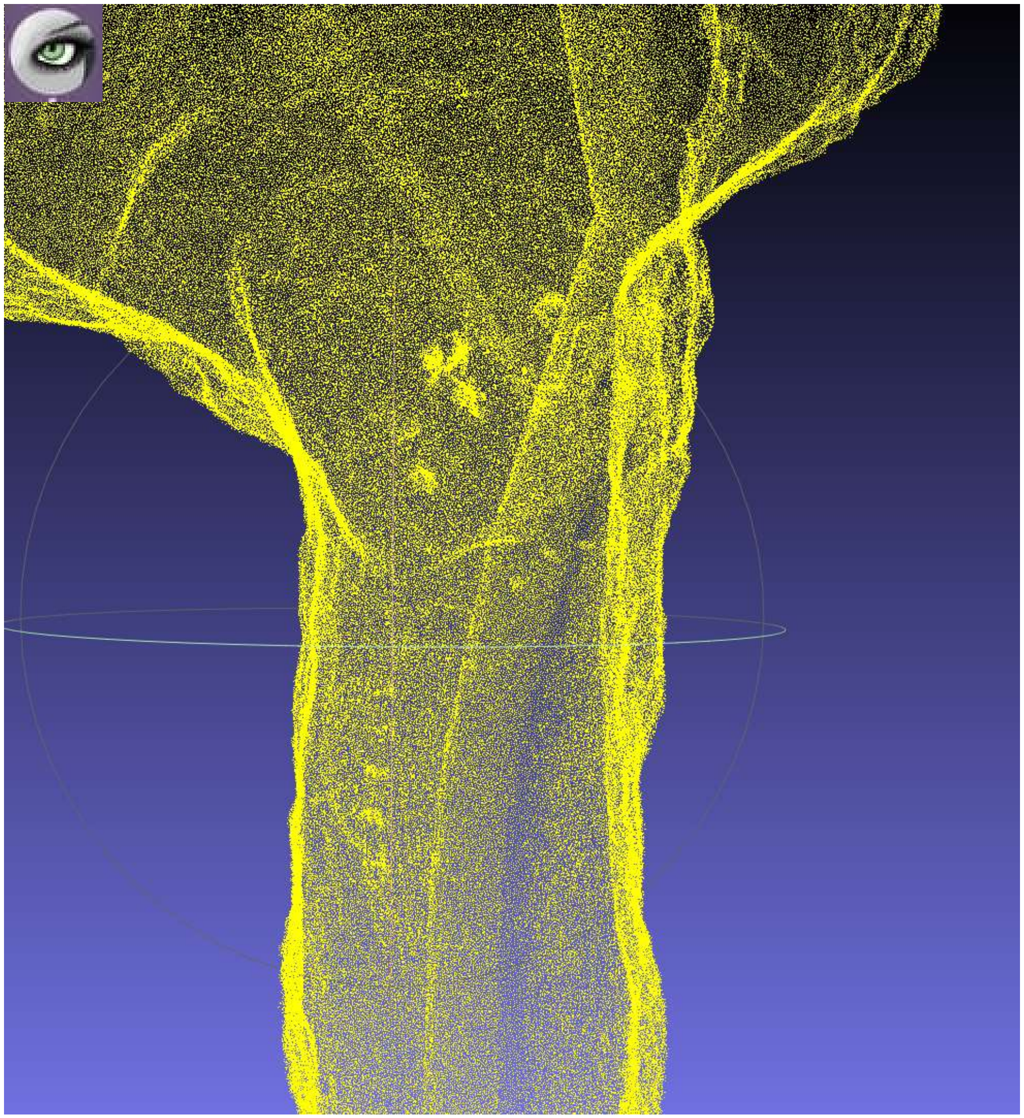

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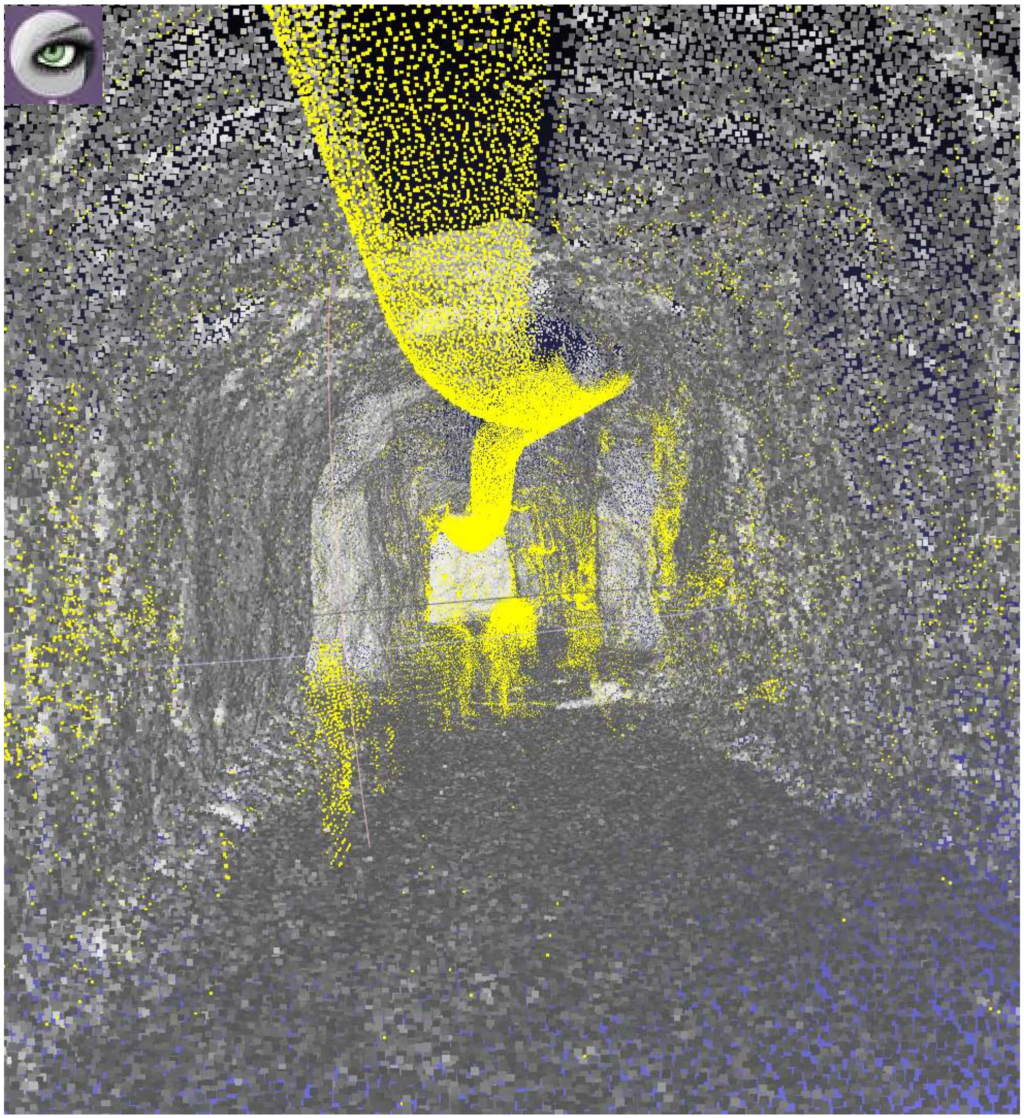

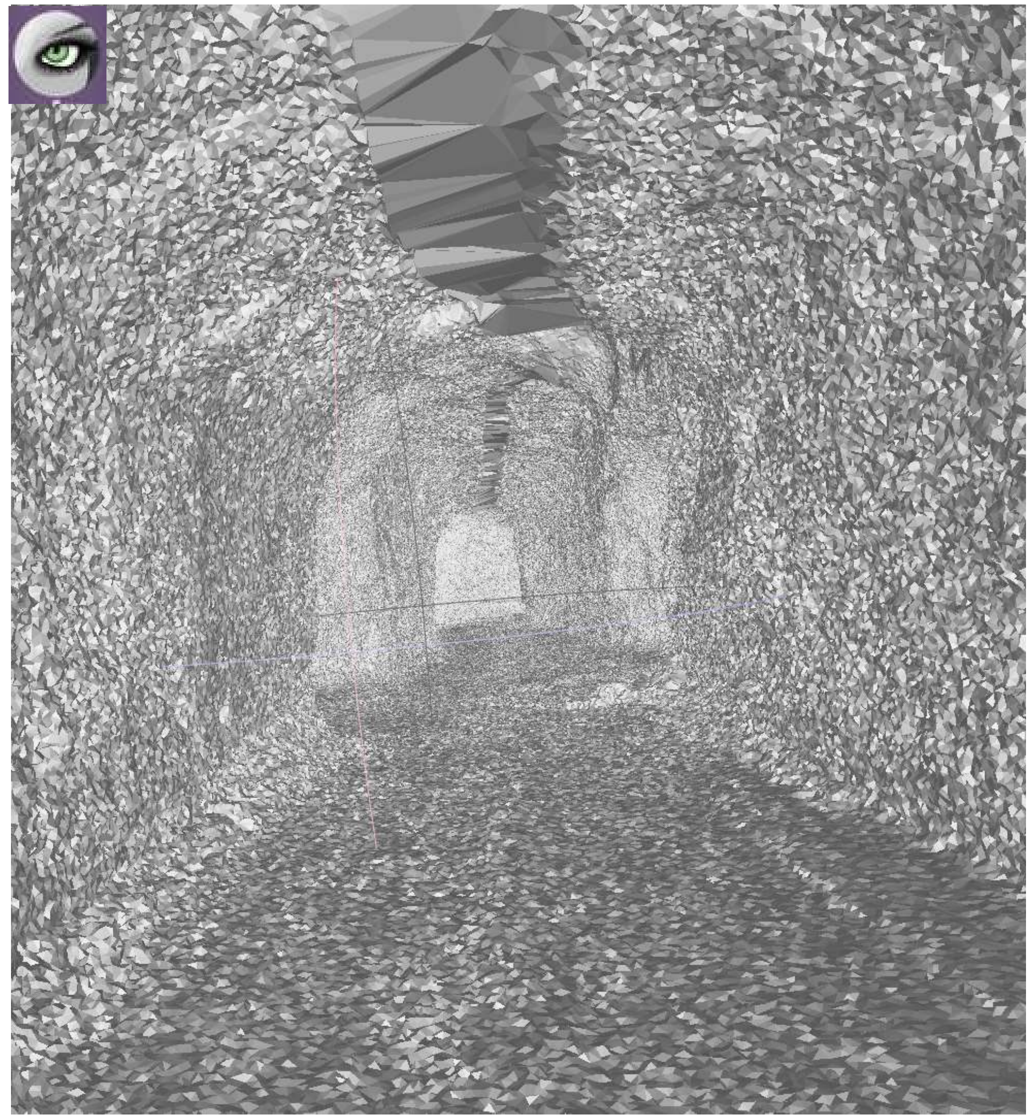

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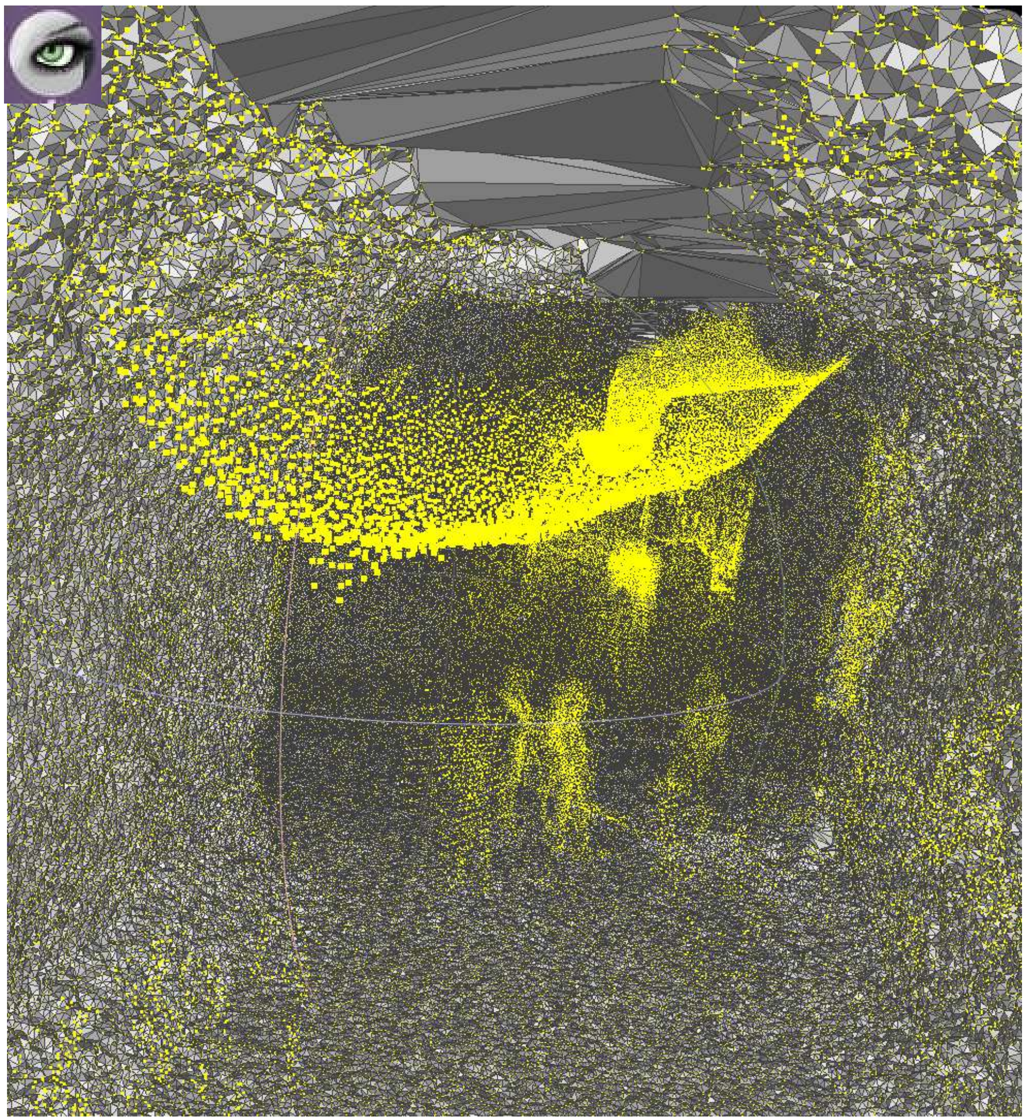
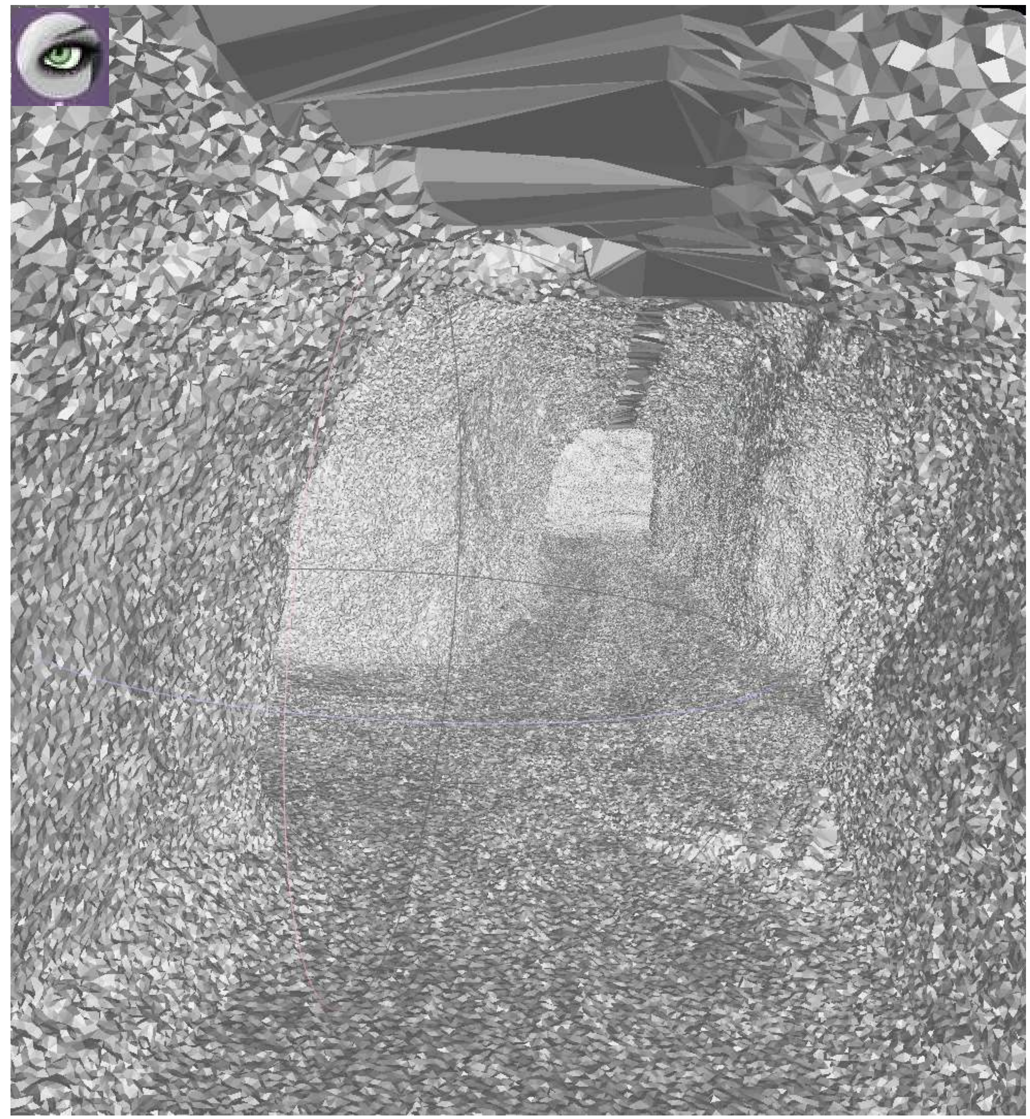




Thank you !

Metric distortion $\mathscr{D}_S$ as measure of regularity of a set S

(Boissonnat, L, Wintraecken, 2017)

Theorem 1. *If $\mathcal{S} \subset \mathbb{R}^d$ is a closed set, then*

$$\operatorname{rch} \mathcal{S} = \sup \left\{ r > 0,\ \forall a, b \in \mathcal{S},\ |a-b| < 2r \Rightarrow d_{\mathcal{S}}(a,b) \leq 2r \arcsin \frac{|a-b|}{2r} \right\},$$

where the sup *over the empty set is* 0.

Metric distortion $\mathscr{D}_S$ as measure of regularity of a set S ?

$$t \to \mathscr{D}_S(t) = \sup_{\|a-b\| \leq t} d_S(a,b)$$

Condition above can be rewritten as:

$$\mathscr{D}_S(t) \leq 2r \arcsin \frac{t}{2r}$$

According to Gromov et Al.*:

$$\mathscr{D}_S(t) \leq \frac{\pi}{2} t \Rightarrow S \text{ ***is simply connected***}$$

$$\mathscr{D}_S(t) \leq \frac{2\sqrt{2}}{\pi} t \Rightarrow S \text{ ***is contractible***}$$

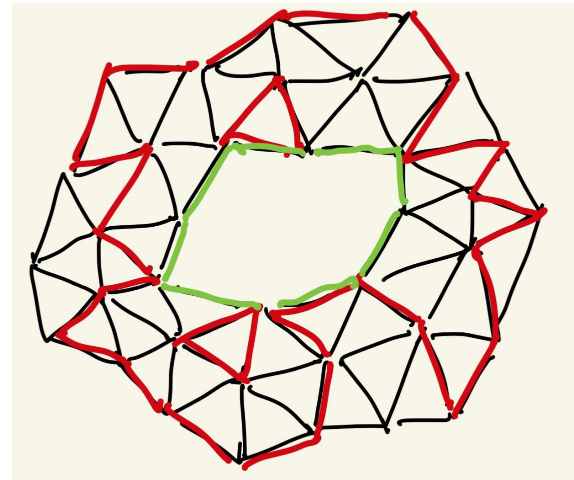
**Metric Structures for Riemannian and Non-Riemannian Spaces, M. Gromov, M. Katz, P. Pansu, S.Semmes*

$\mathcal{O}(n^3)$ general algorithm

Lexicographic-minimal homologous chain:

Given $\alpha \in C_d(K, \mathbb{Z}_2)$ **find:**

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}}\{\alpha + \partial\omega, \omega \in C_{d+1}(K, \mathbb{Z}_2)\}$$



A chain Γ' is said to be a **reduction** of a chain Γ if:

Γ' is homologous to Γ and $\Gamma' <_{lex} \Gamma$

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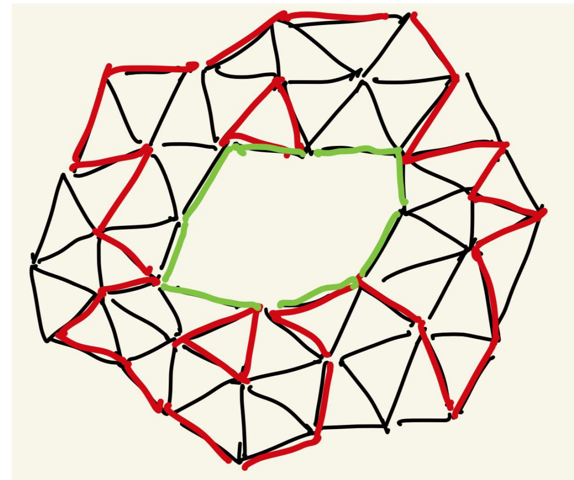

$(d+1)$-simplices

$$\partial_{d+1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

d-simplices ordered along increasing $\leq$

$\mathcal{O}(n^3)$ general algorithm

$$\partial_{d+1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} = \mathbf{R}\,.\,\mathbf{V}$$

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In $\mathbf{R}$, there is exactly one column with a lowest 1 for each reducible simplex 1

Same as **Homological persistence**

Algorithm 1: Reduction algorithm for the ∂_{d+1} matrix

R $= \partial_{d+1}$
for $j \leftarrow 1$ **to** n **do**
  while $R_j \neq 0$ **and** $\exists j_0 < j$ *with* $\mathrm{low}(j_0) = low(j)$ **do**
    $R_j \leftarrow R_j + R_{j_0}$
  end
end

$\mathcal{O}(n^3)$ general algorithm

$\partial_{d+1} = \mathbf{R}\,.\,\mathbf{V}$

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \Gamma_0 = \alpha = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

In $\mathbf{R}$, there is exactly one column with a lowest 1 for each reducible simplex 1

Total reduction of Γ using the reduced boundary operator $\mathbf{R}$

Algorithm 2: Total reduction algorithm

Inputs: A d-chain Γ, the reduction matrix R from Algorithm 1

for $i \leftarrow m$ **to** 1 **do**
  if $\Gamma[i] \neq 0$ **and** $\exists j \in [1, n]$ *with* $\text{low}(j) = i$ *in* R **then**
    $\Gamma \leftarrow \Gamma + R_j$
  end
end

$\mathcal{O}(n^3)$ general algorithm

$\partial_{d+1} = \mathbf{R} \, . \, \mathbf{V}$

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \Gamma_0 = \alpha = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \Gamma_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

+

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