A Lower Bound for Union Volume Estimation and Approximating Klee's Measure Problem

Karl Bringmann Kasper Green Larsen **André Nusser**
Eva Rotenberg Yanheng Wang

Introduction

The Problems

Union Volume Estimation

Klee's Measure Problem

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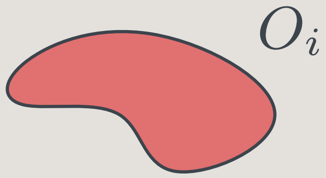
Union Volume Estimation

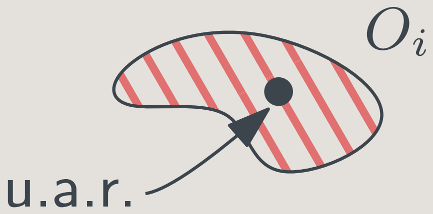
Given:

Objects $O_1, \ldots, O_n$ with access:

1) $\mathrm{Volume}(i)$:



2) $\mathrm{Sample}(i)$



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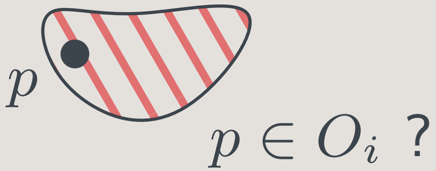

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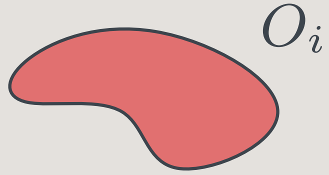

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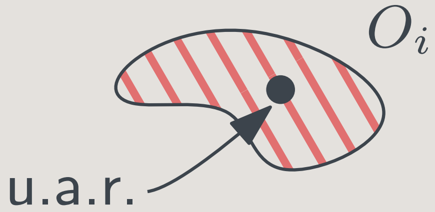

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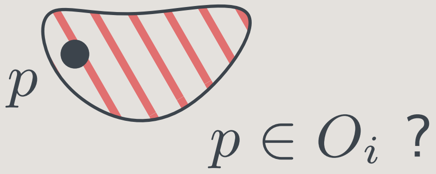

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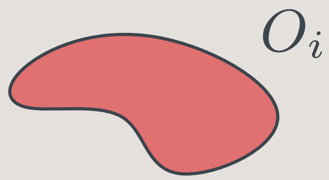

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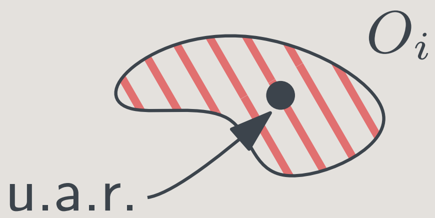

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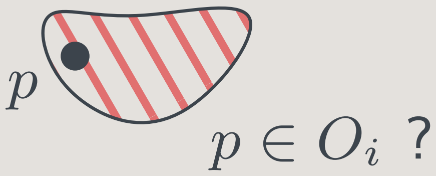

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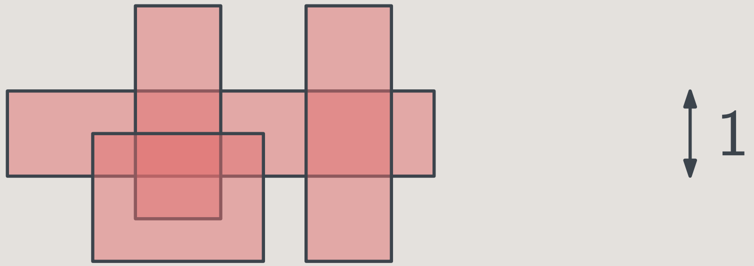

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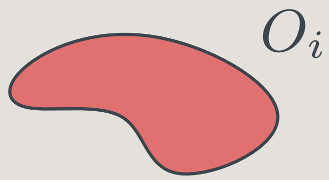

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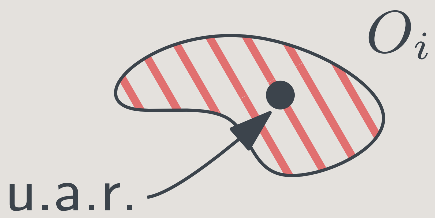

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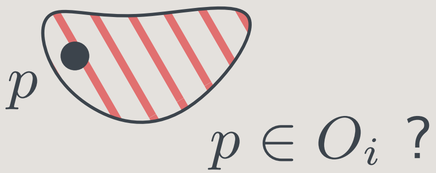

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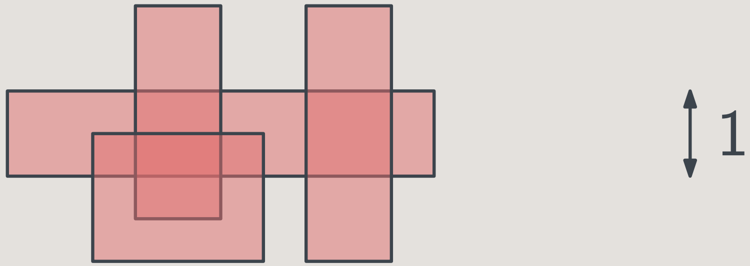

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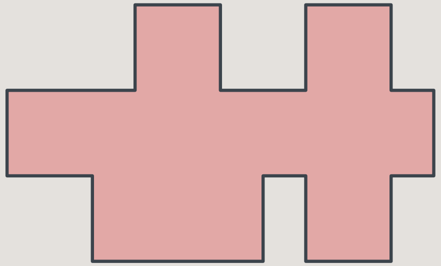
Vol() = 10

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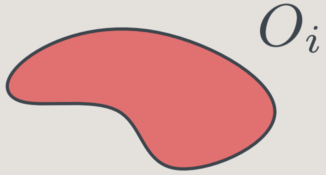

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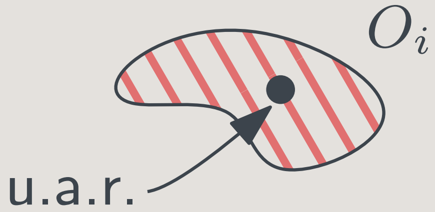

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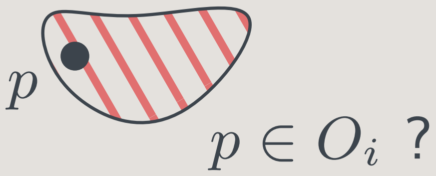

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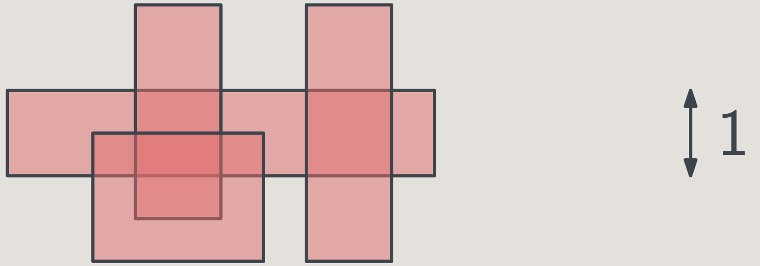

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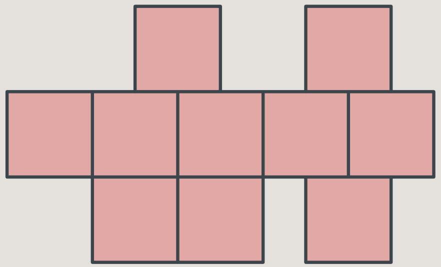
$\mathrm{Vol}($  $) = 10$

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Continuous Graph k-Center

Given a graph G, find k centers (potentially also on the edges) such that each vertex is in distance r to some center.

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Lower Bound
for
Union Volume Estimation

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$\Longrightarrow$

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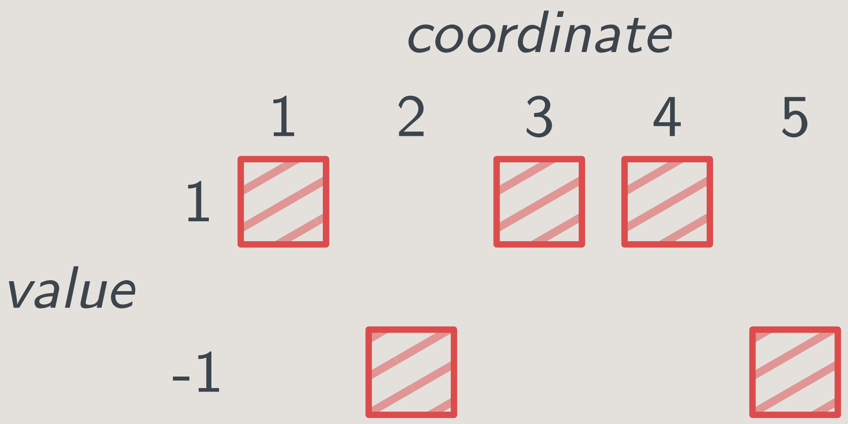

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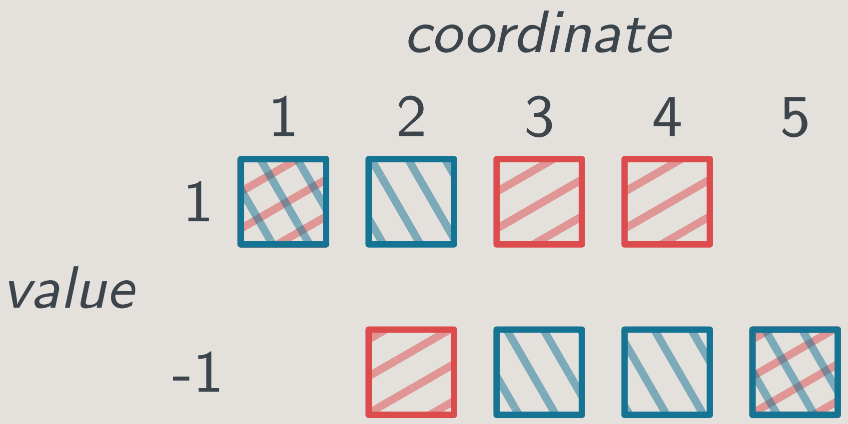

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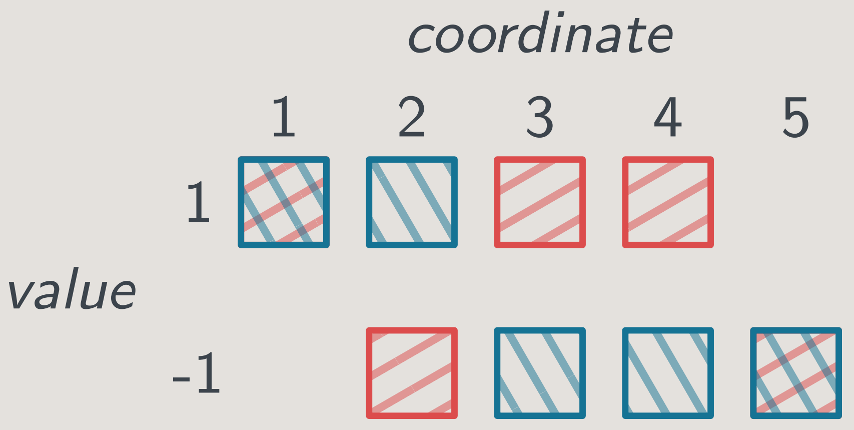

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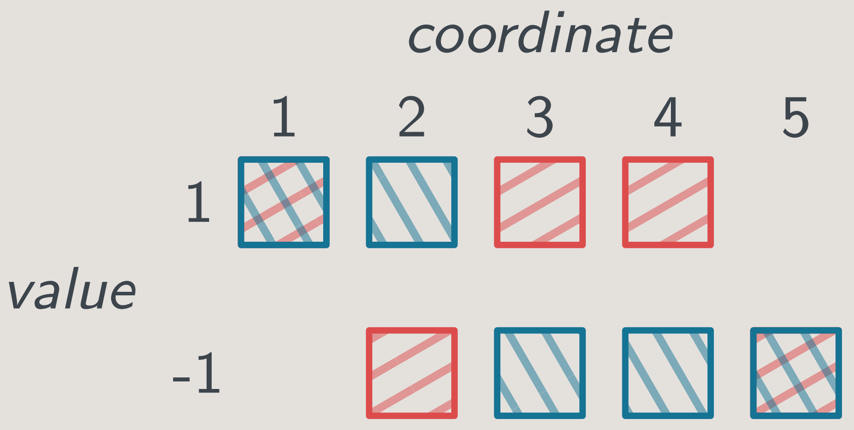

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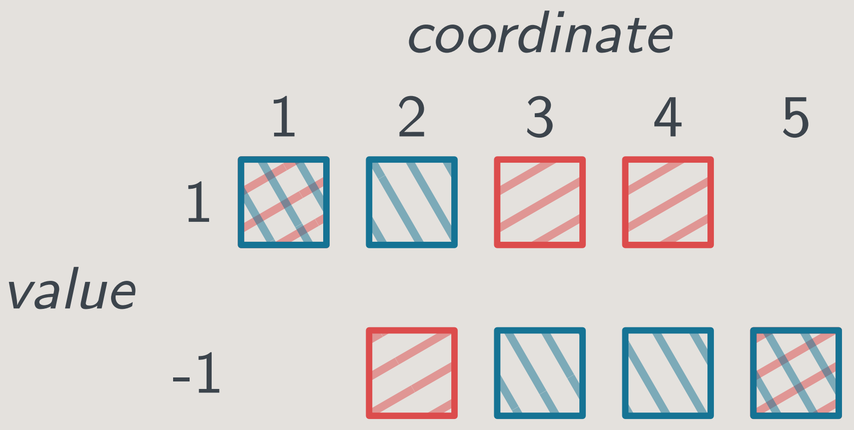

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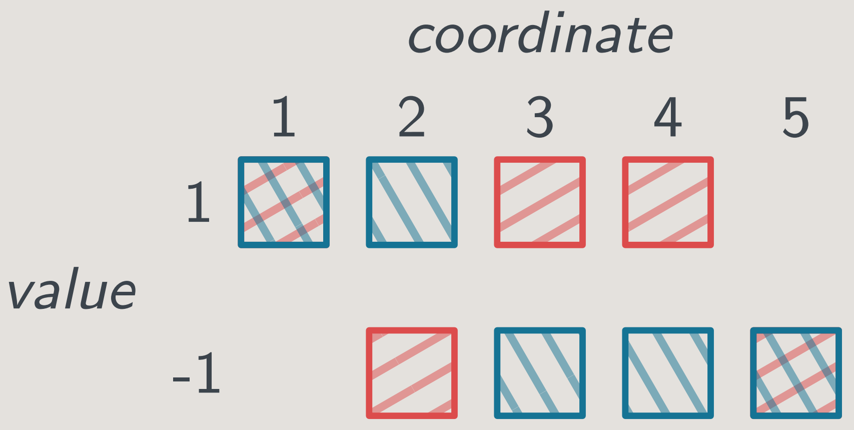

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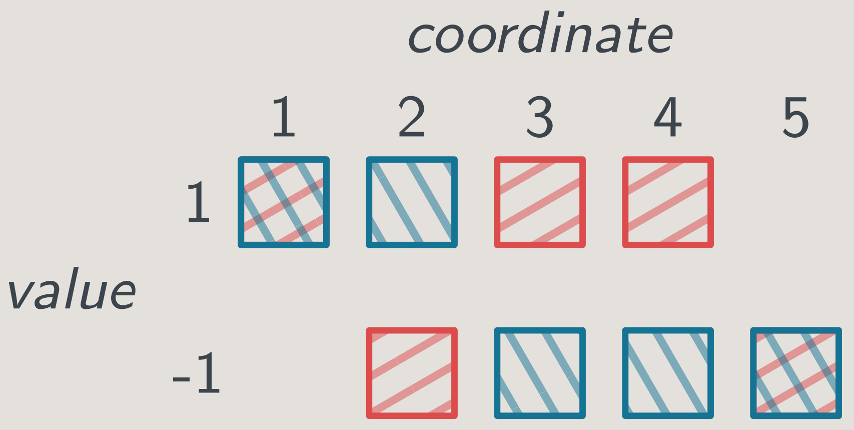

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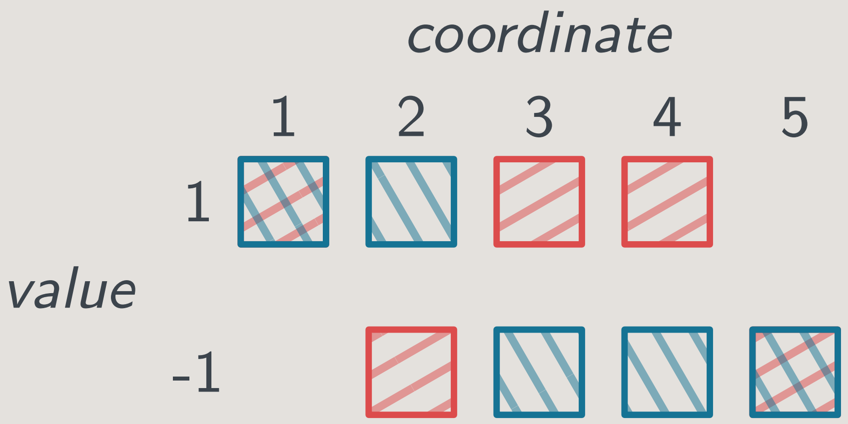

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Warm up: $\Omega(\frac{1}{\varepsilon^2})$ lower bound

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Given: $x, y \in \{-1, 1\}^T$

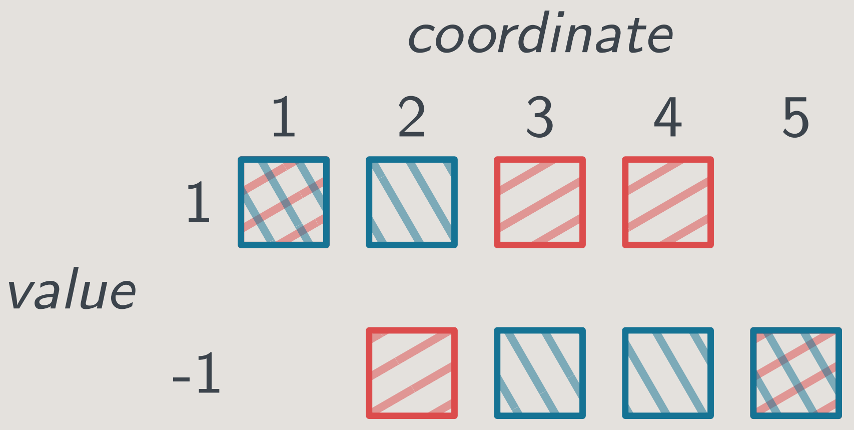
Goal: decide whether
$\langle x, y \rangle \leq -\sqrt{T}$, or
$\langle x, y \rangle \geq \sqrt{T}$

Input:

$x = (1, -1, 1, 1, -1)$

$y = (1, 1, -1, -1, -1)$

Reduction:



We have: $\langle x, y \rangle = 3T - 2 \cdot \mathrm{Vol}(\square \cup \square)$

Given a $(1+\varepsilon)$-approximation:

$\in [T, 2T]$

$$\langle x, y \rangle = 3T - 2(1 \pm \varepsilon) \cdot \mathrm{Vol}(\square \cup \square)$$

$\Longrightarrow$ approximates $\langle x, y \rangle$ within $\pm \varepsilon T = \pm \sqrt{T}$

$\frac{1}{\sqrt{T}}$

$\Longrightarrow$ we need $\Omega(T) = \Omega(\frac{1}{\varepsilon^2})$ queries!

Lower Bound

$\Omega(n/\varepsilon^2)$ lower bound

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Idea: “hide coordindates” using random permutations

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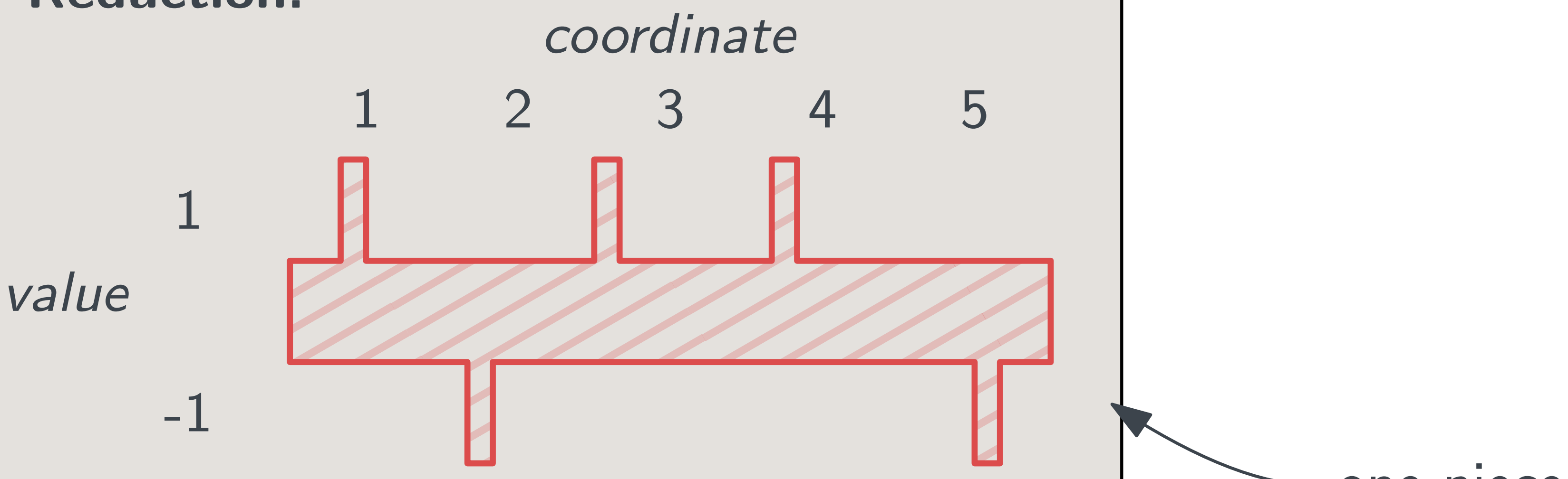

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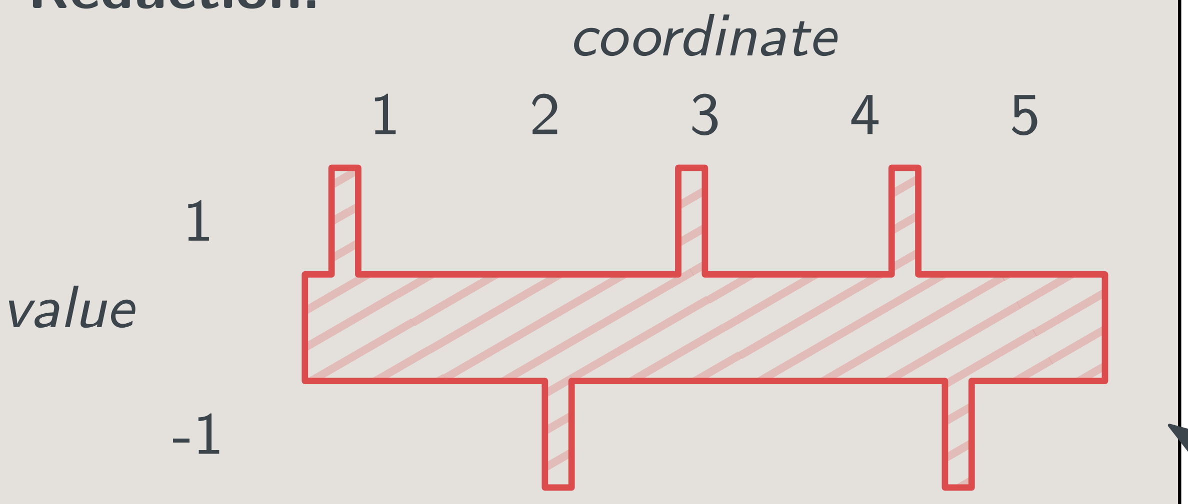

one piece

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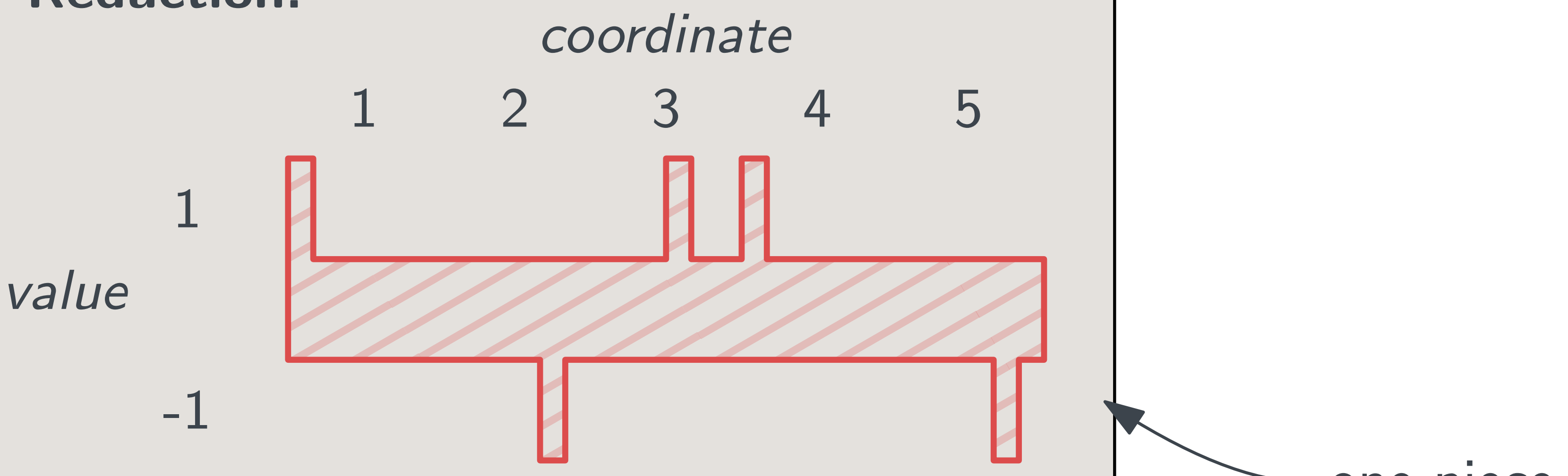

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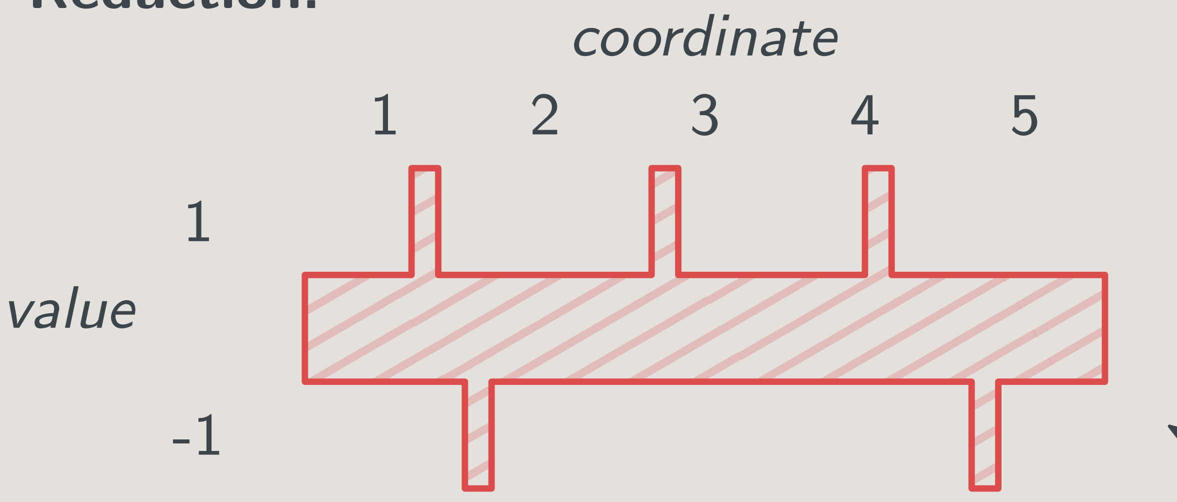

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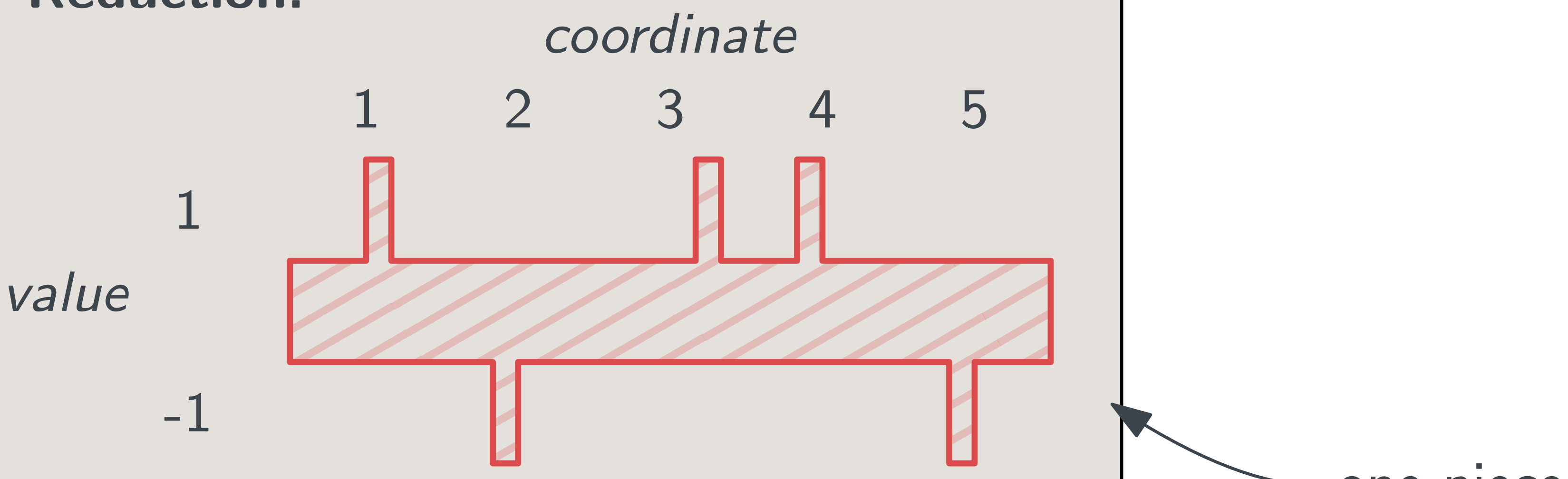

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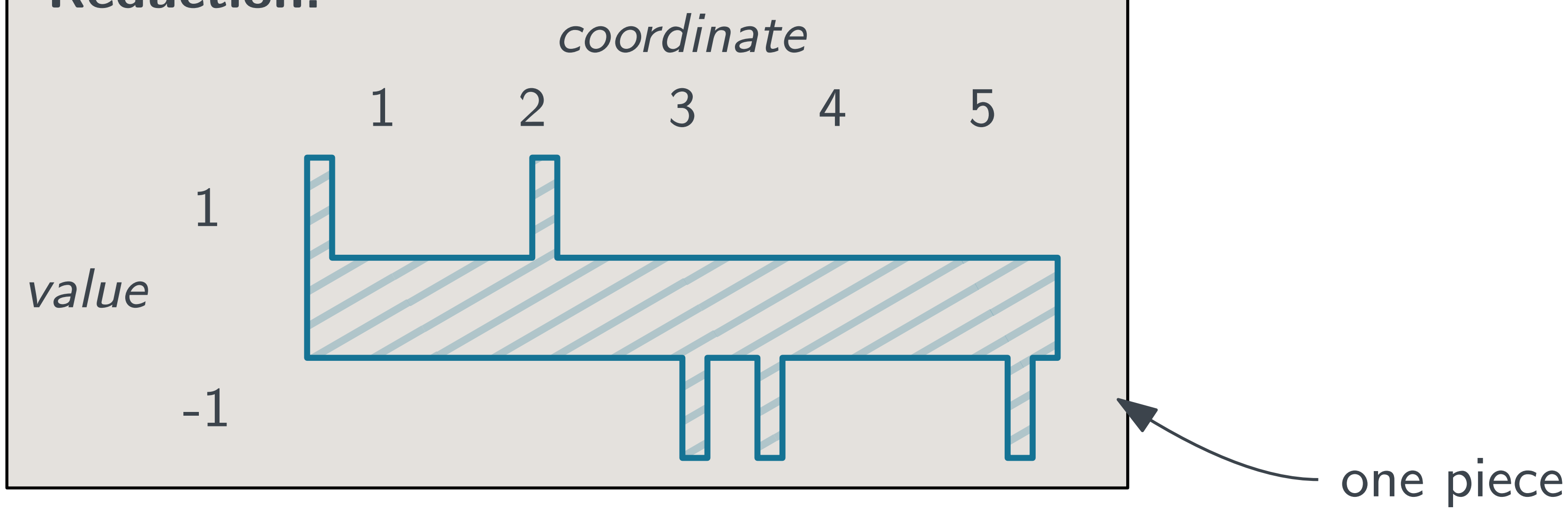

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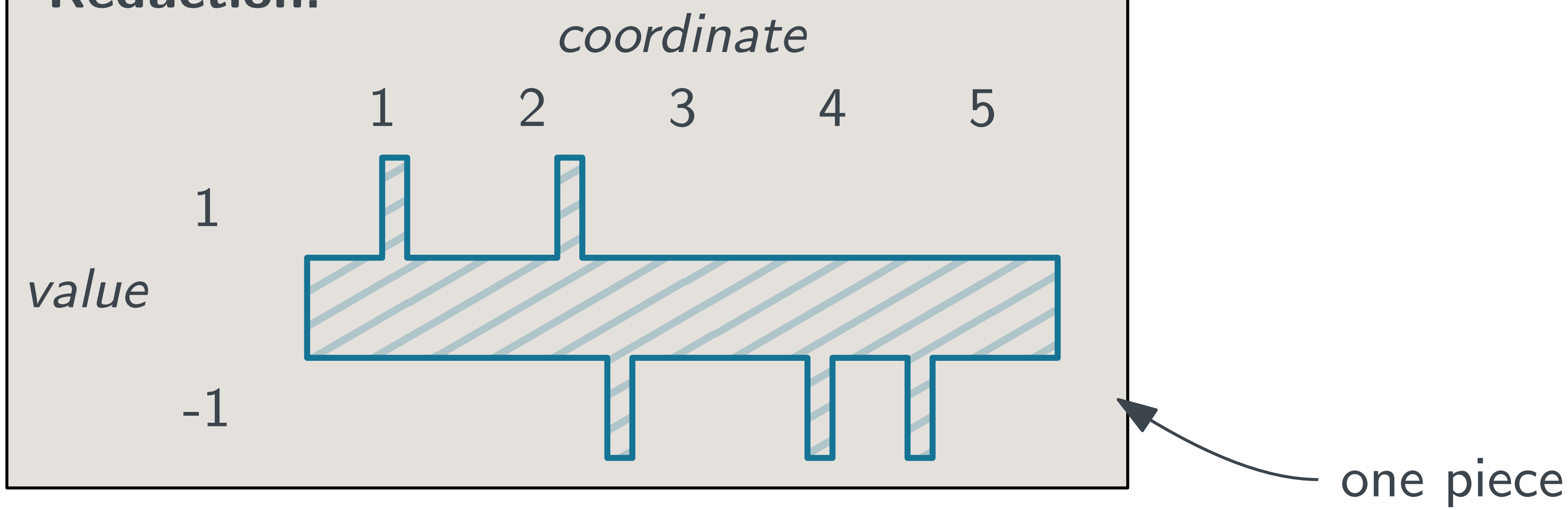

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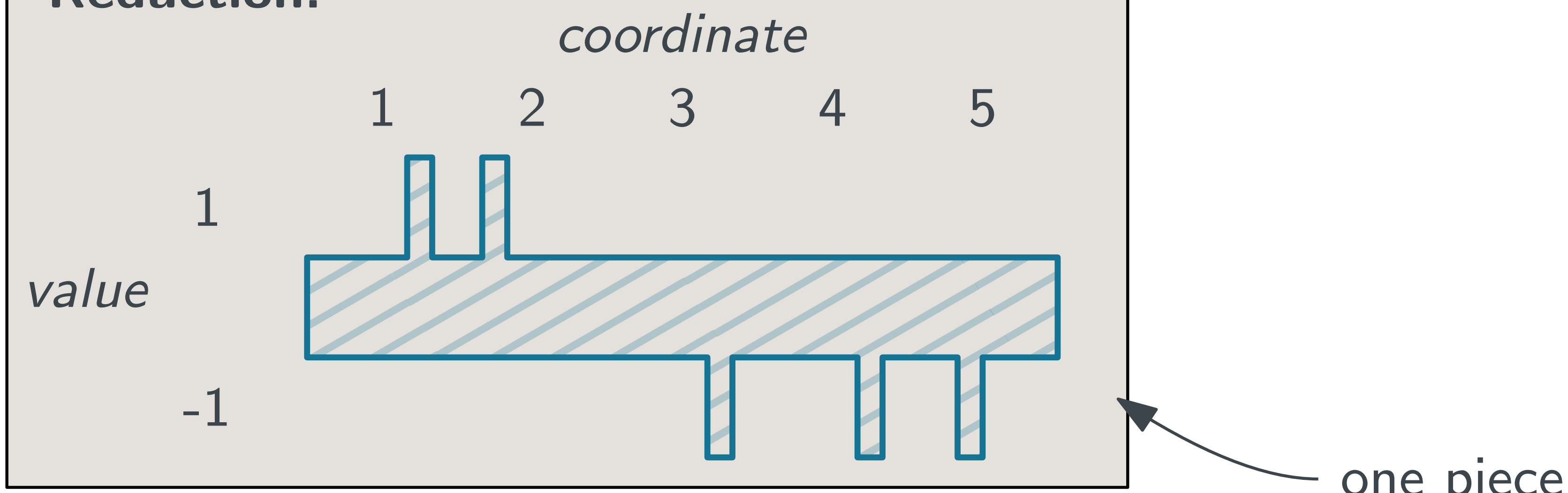

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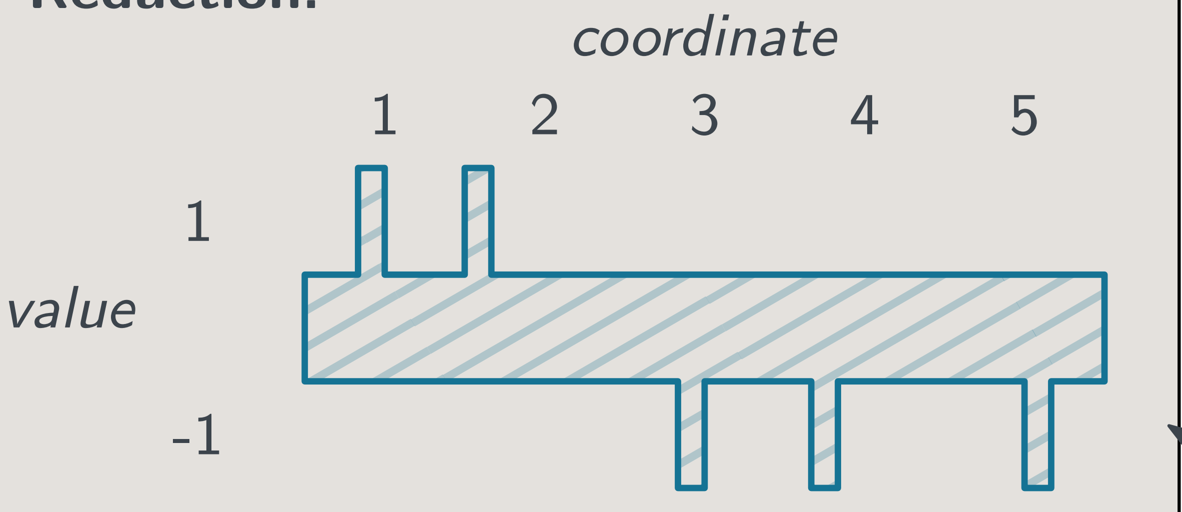

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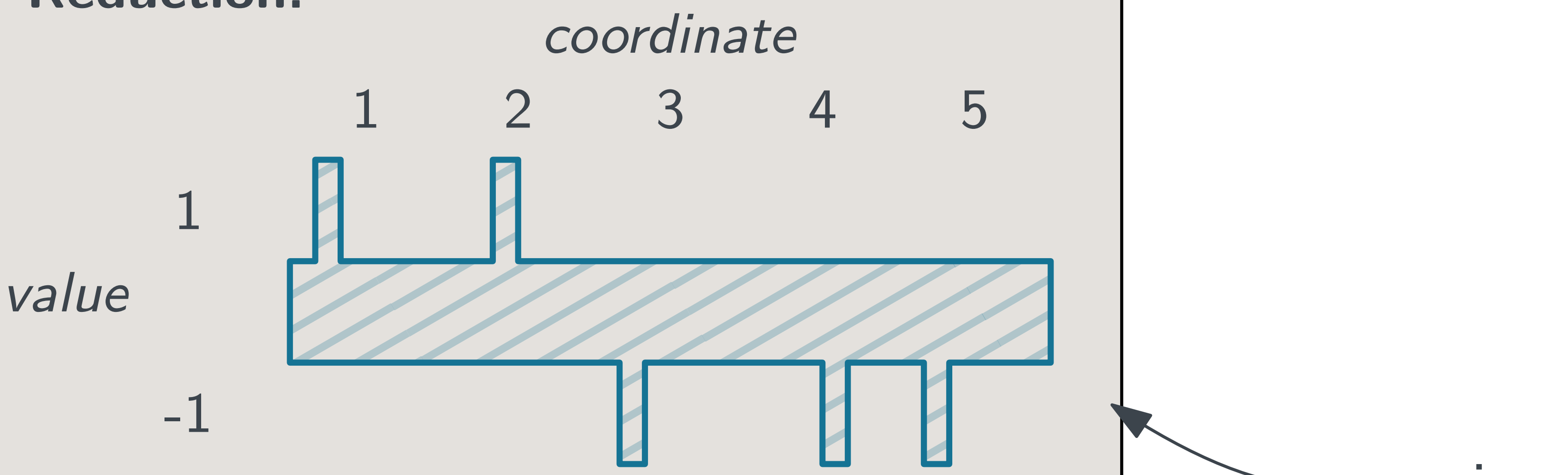

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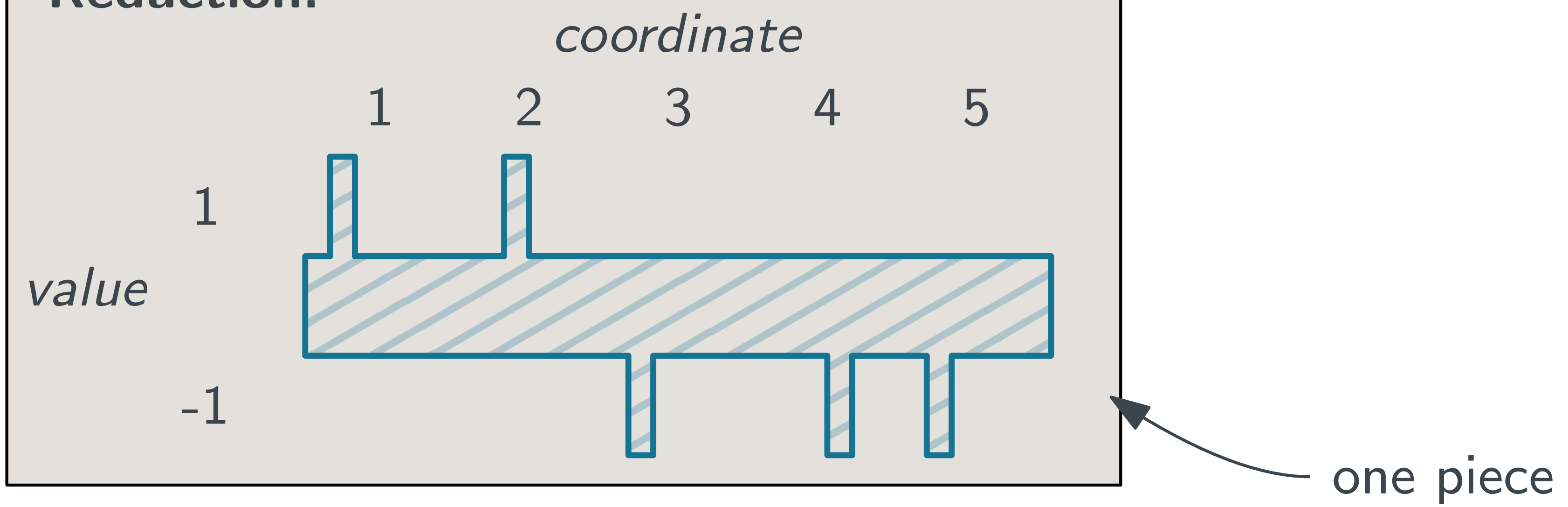

$P(\text{sample a coordinate point}) = \frac{1}{n}$

$P(\text{containment check succeeds}) = \frac{1}{n}$

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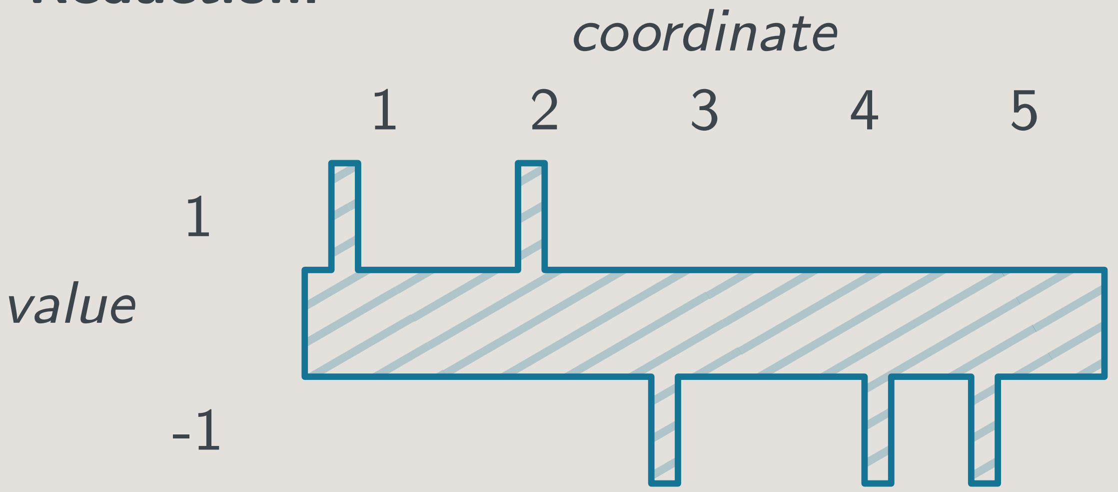

one piece

$P(\text{sample a coordinate point}) = \frac{1}{n}$

$P(\text{containment check succeeds}) = \frac{1}{n}$

↳ $O(n)$ queries for checking single bit

Lower Bound

$\Omega(n/\varepsilon^2)$ lower bound

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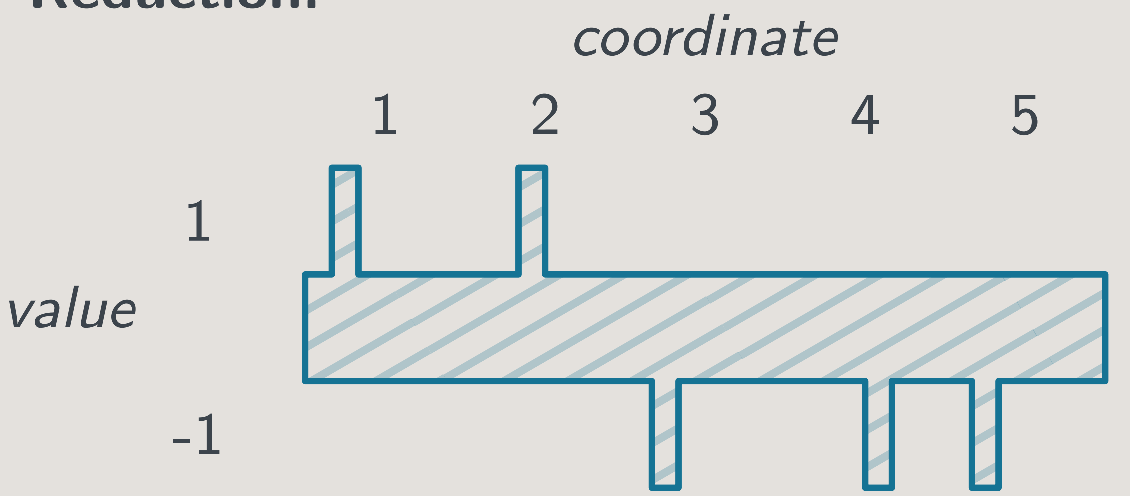

one piece

$P(\text{sample a coordinate point}) = \frac{1}{n}$
$P(\text{containment check succeeds}) = \frac{1}{n}$

↳ $O(n)$ queries for checking single bit

↳ $\Omega(n \cdot \frac{1}{\varepsilon^2})$ lower bound

Lower Bound

Restrictions

Lower Bound

Restrictions

can:

- use 3 queries of the model (volume, sample, contains)
- inspect coordinates of sampled points
- query arbitrary (not sampled!) points

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[Karp, Luby, Madras '89] only uses this!

Lower Bound

Restrictions

can:

- use 3 queries of the model (volume, sample, contains)
- inspect coordinates of sampled points
- query arbitrary (not sampled!) points

cannot:

- inspect coordinates of objects

[Karp, Luby, Madras '89] only uses this!

Improved **Upper Bound**

for

Klee's Measure Problem

Upper Bound

Insight from Lower Bound:

Have to use geometry of objects!

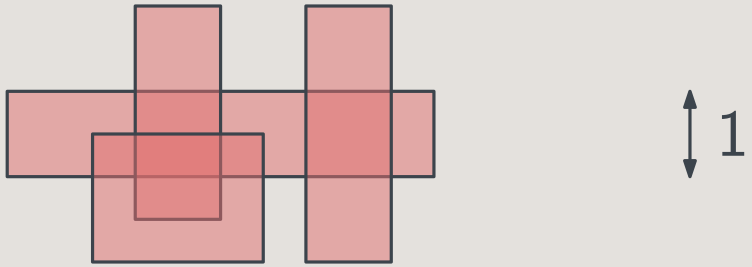
Upper Bound

Reminder:

Klee's Measure Problem

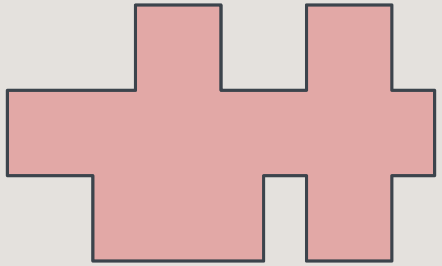
Given:

Boxes $O_1, \ldots, O_n \subset \mathbb{R}^d$



Goal:

$(1+\varepsilon)$-approximate $\mathrm{Vol}(\bigcup_i O_i)$



Upper Bound

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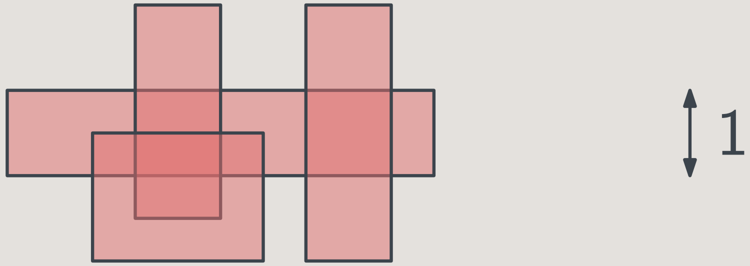

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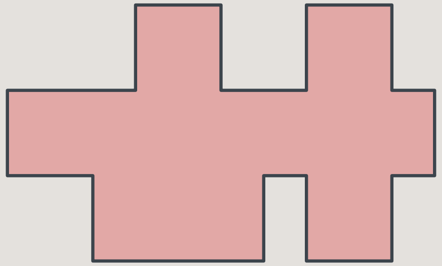

Algorithm:

$\tilde{V} \leftarrow O(1)$-approx. of $\mathrm{Vol}(\bigcup_i O_i)$

$S \leftarrow p$-Sample with density $p \approx \frac{1}{\tilde{V}\varepsilon^2}$

return $|S|/p$

Upper Bound

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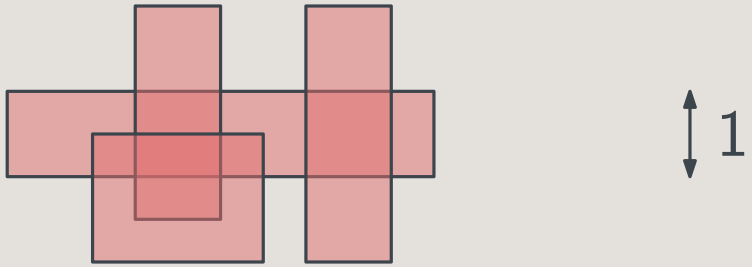
Boxes $O_1, \ldots, O_n \subset \mathbb{R}^d$



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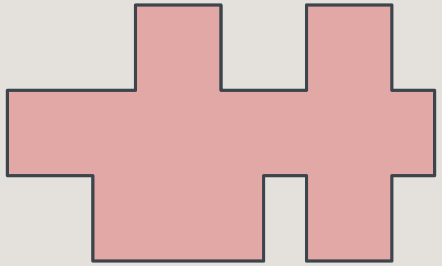

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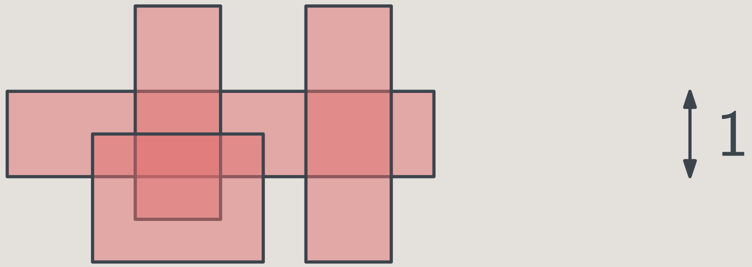

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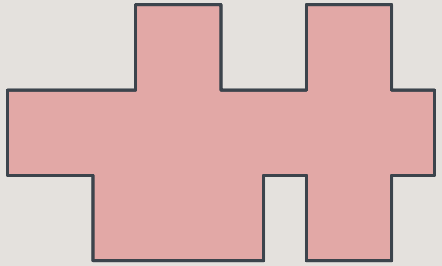

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$(1+\varepsilon)$-approximation with good probability

Upper Bound

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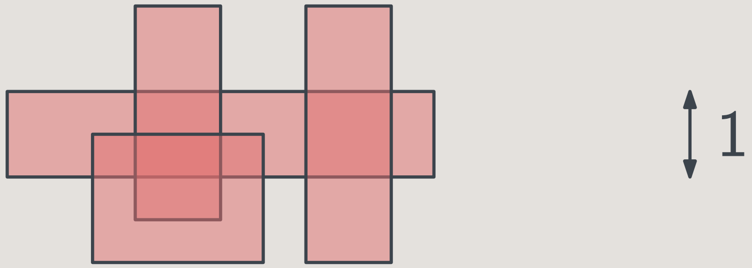

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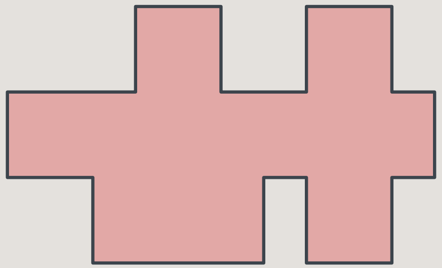

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return $|S|/p$

$(1+\varepsilon)$-approximation with good probability

Goal:
Draw p-Sample from union of boxes.

Upper Bound

p-Sample: Similar Shapes

Upper Bound

p-Sample: Similar Shapes

Class: All boxes have sizes in
$[s_1, 2s_1] \times [s_2, 2s_2] \times \cdots \times [s_d, 2s_d]$.

Upper Bound

p-Sample: Similar Shapes

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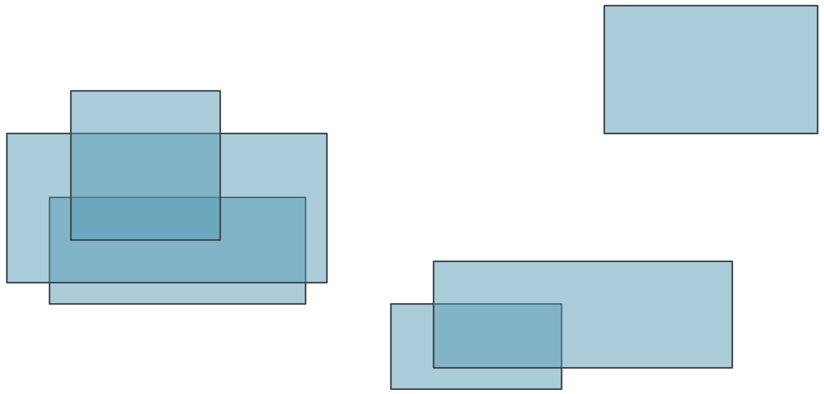

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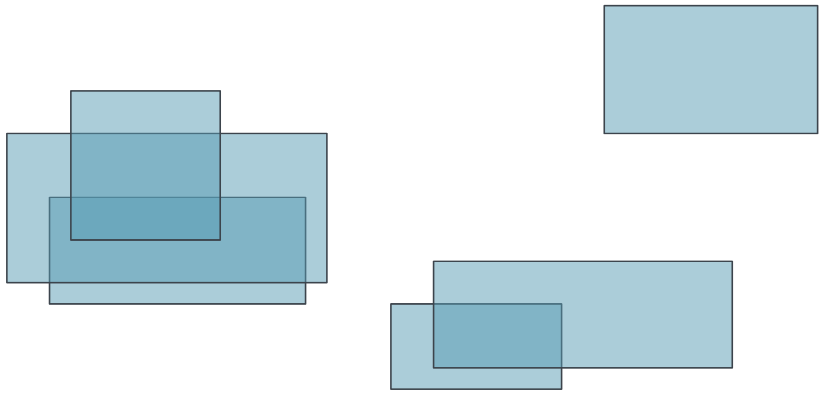

Algorithm:

build orthogonal range data structure

Upper Bound

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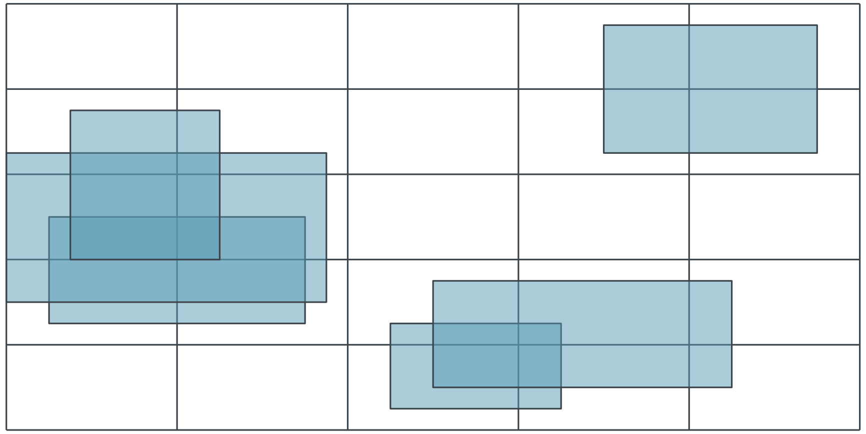

Algorithm:

build orthogonal range data structure

create grid, mark active cells $G_1, \ldots G_T$

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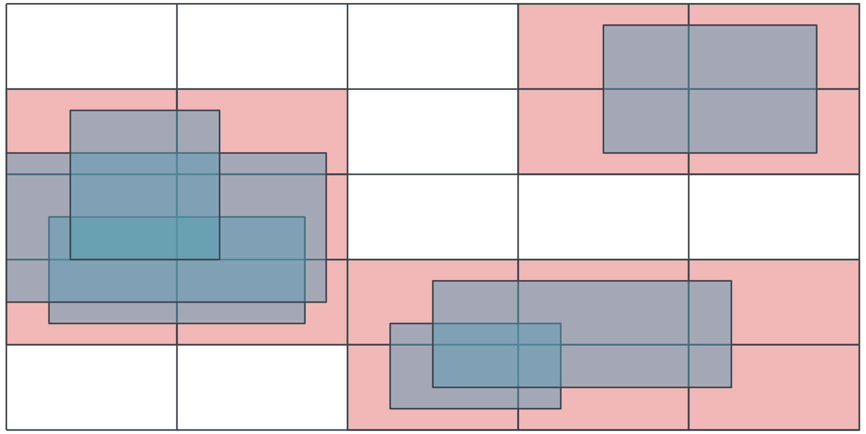

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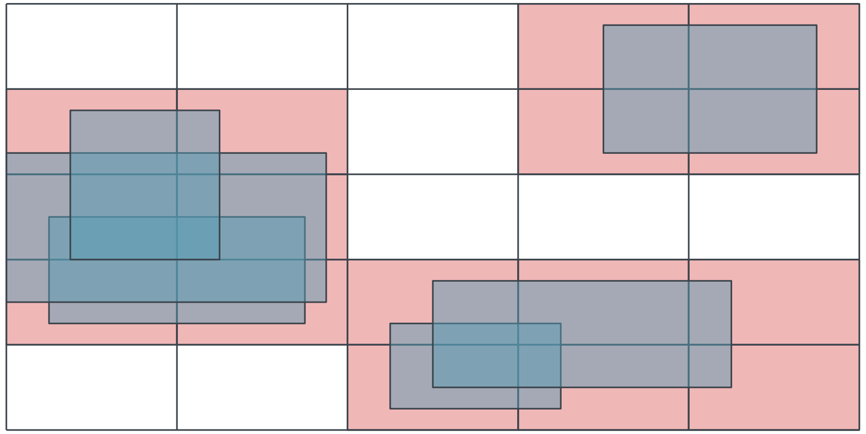

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$K \sim \text{Pois}(p \cdot \text{Volume}(\bigcup_i G_i))$

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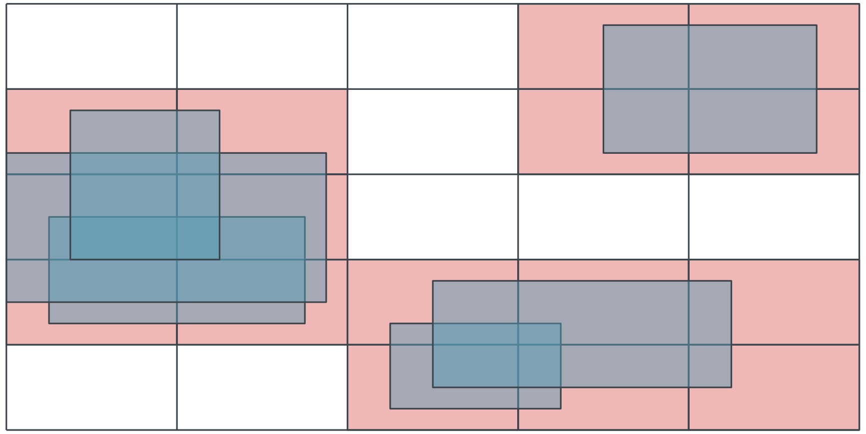

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for $i \in [K]$
  $G \leftarrow$ random cell
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  if q contained in box: add q to samples

Upper Bound

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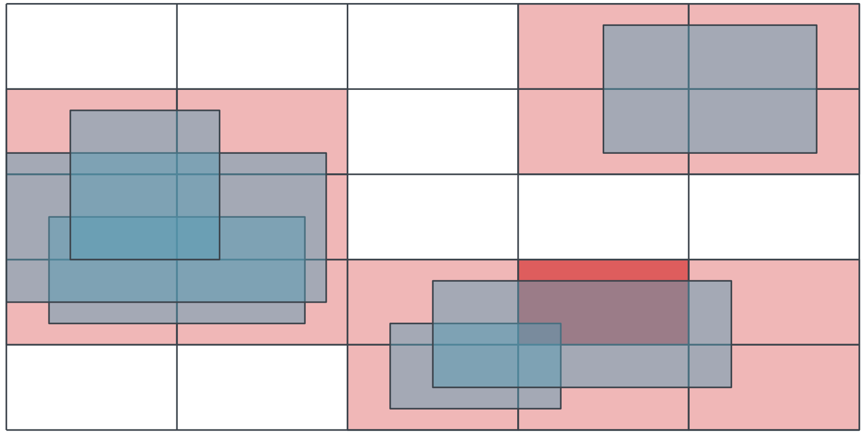

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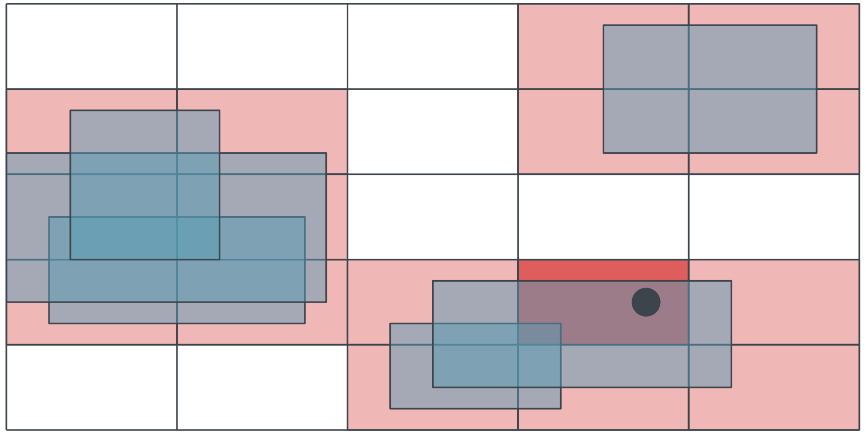

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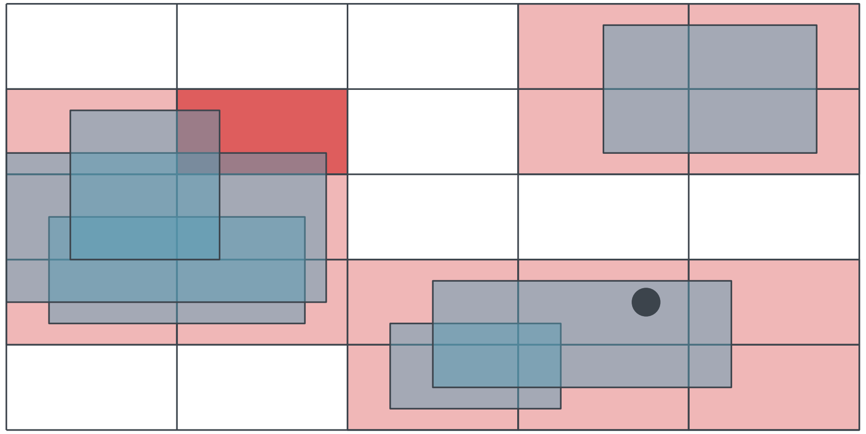

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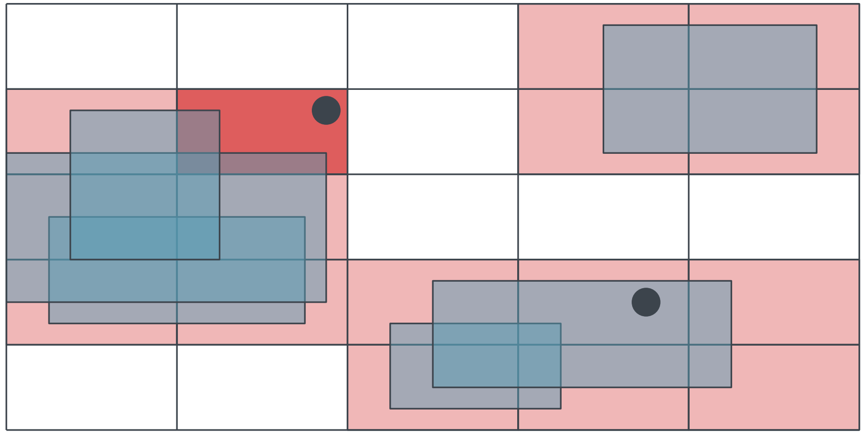
Class: All boxes have sizes in $[s_1, 2s_1] \times [s_2, 2s_2] \times \cdots \times [s_d, 2s_d]$.



Algorithm:

build orthogonal range data structure

create grid, mark active cells $G_1, \ldots G_T$

$K \sim \text{Pois}(p \cdot \text{Volume}(\bigcup_i G_i))$

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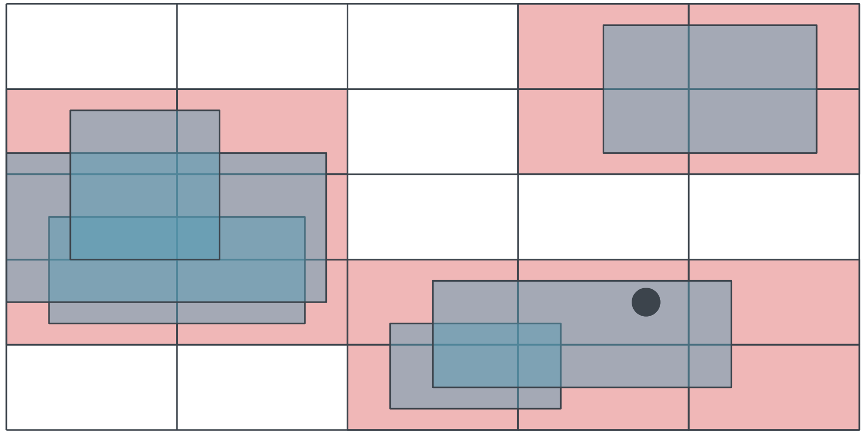

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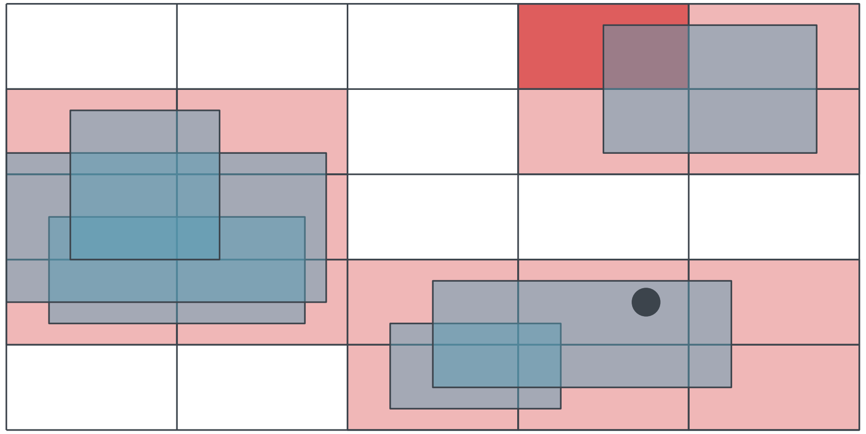

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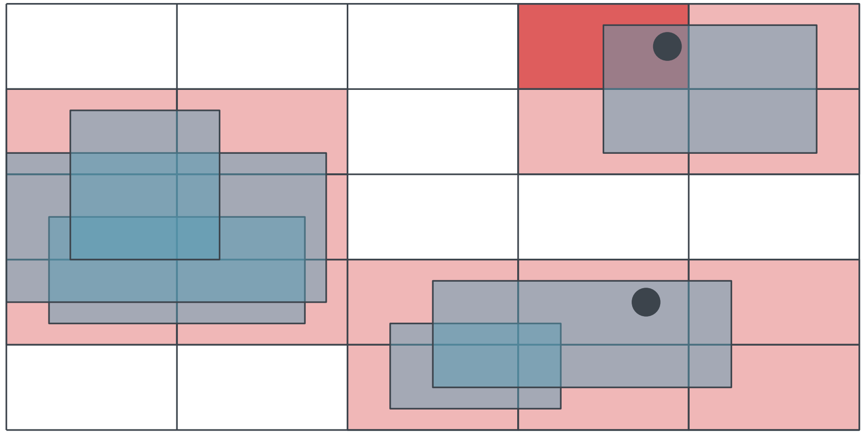

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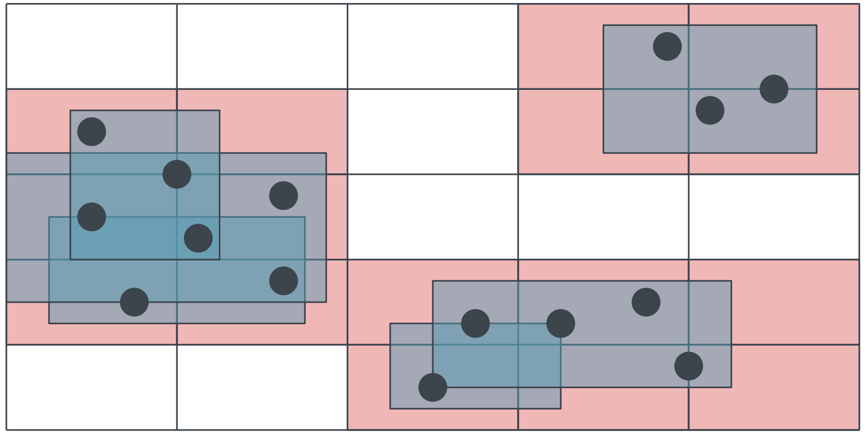

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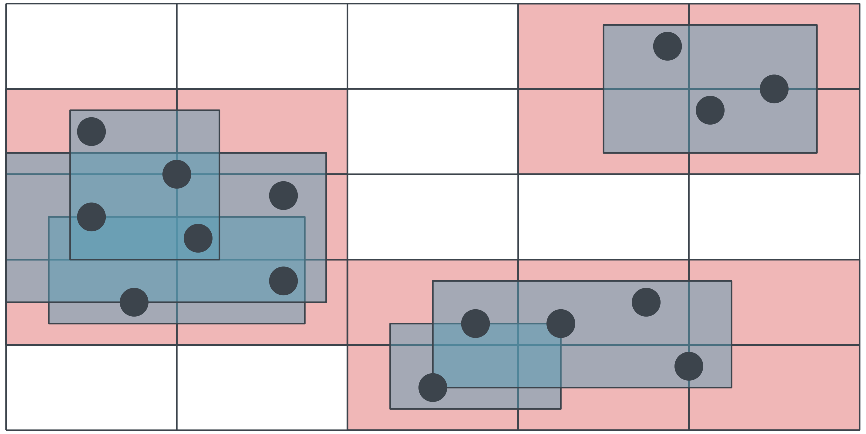

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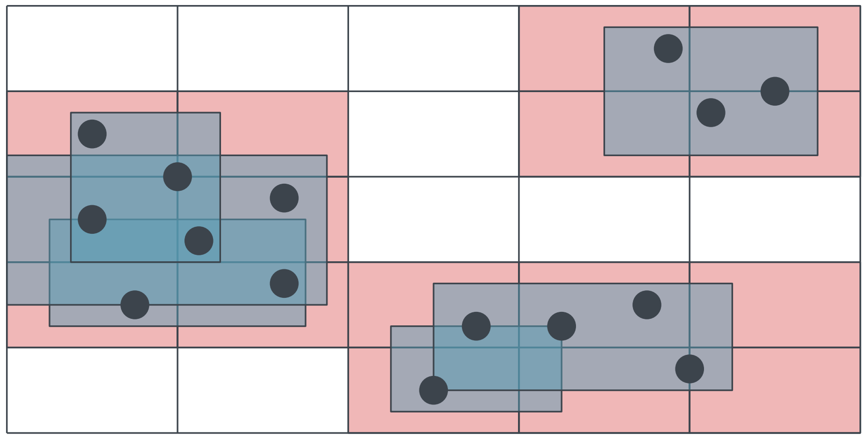

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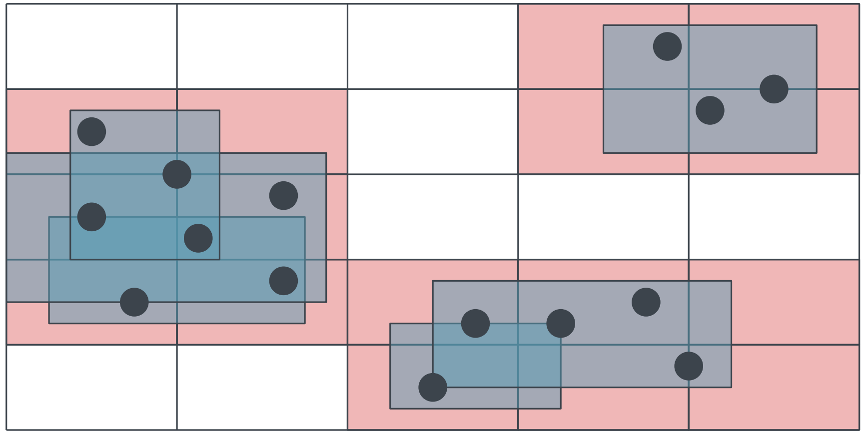

Running Time?

$$O((n+K) \cdot \log^{O(d)} n)$$

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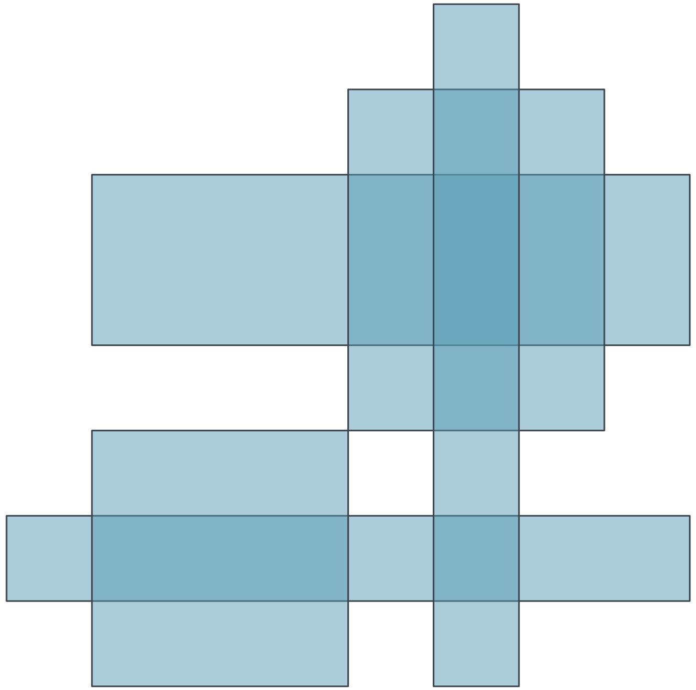
Correctness?

Upper Bound

p-Sample: General Case



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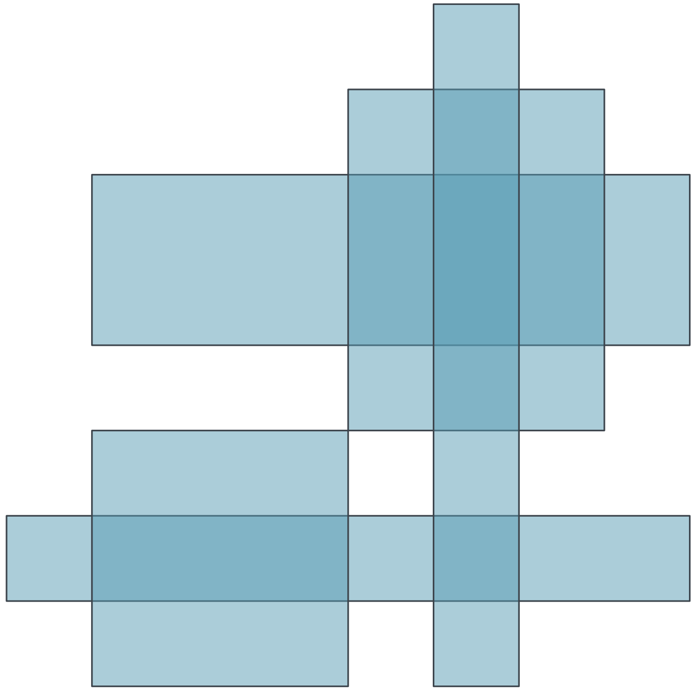

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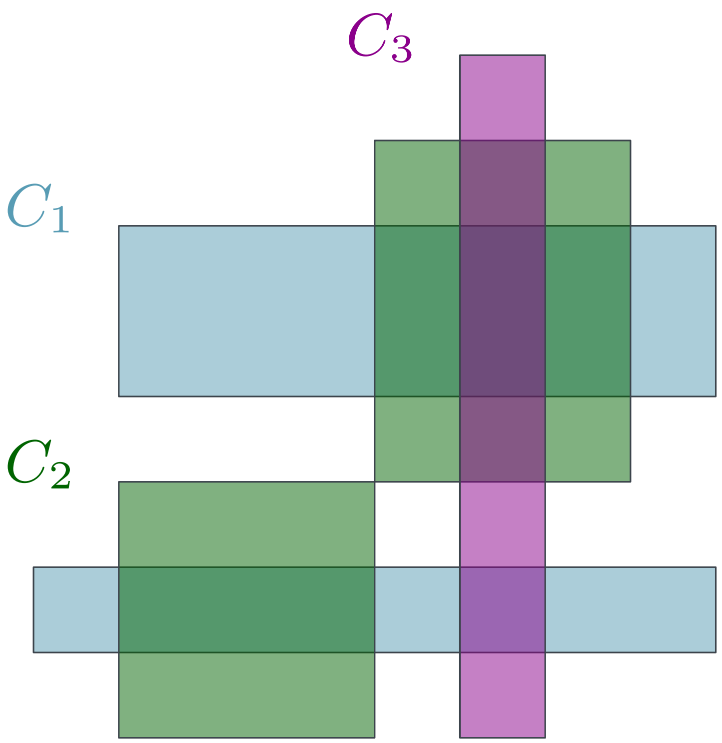

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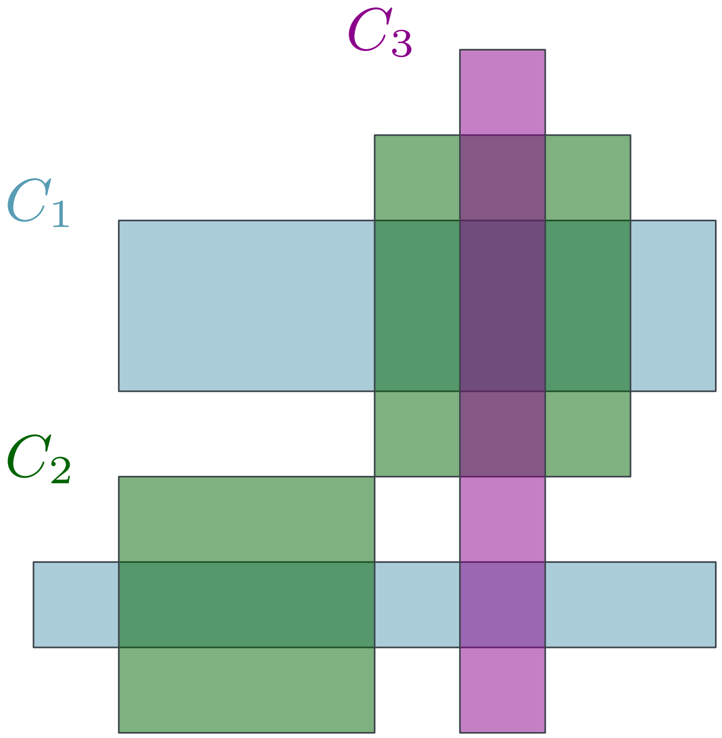

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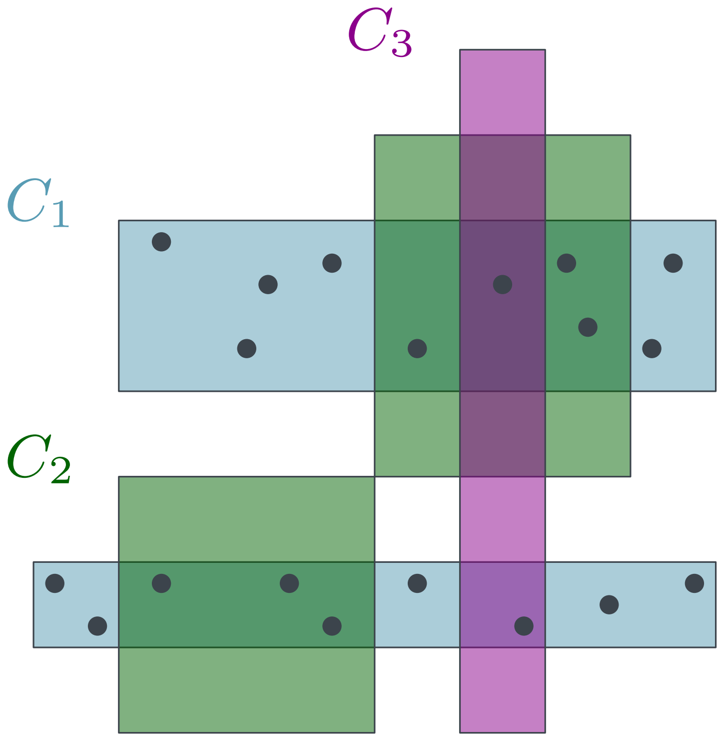

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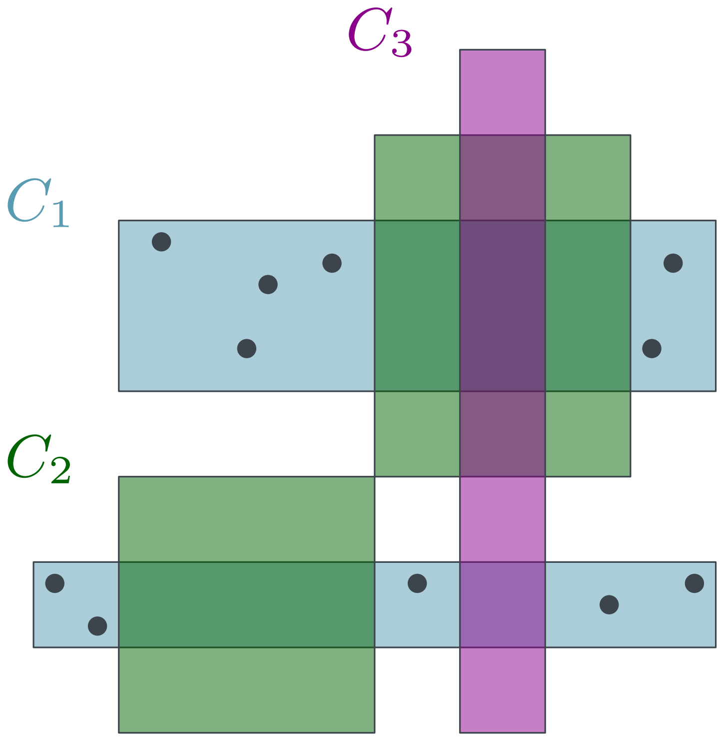

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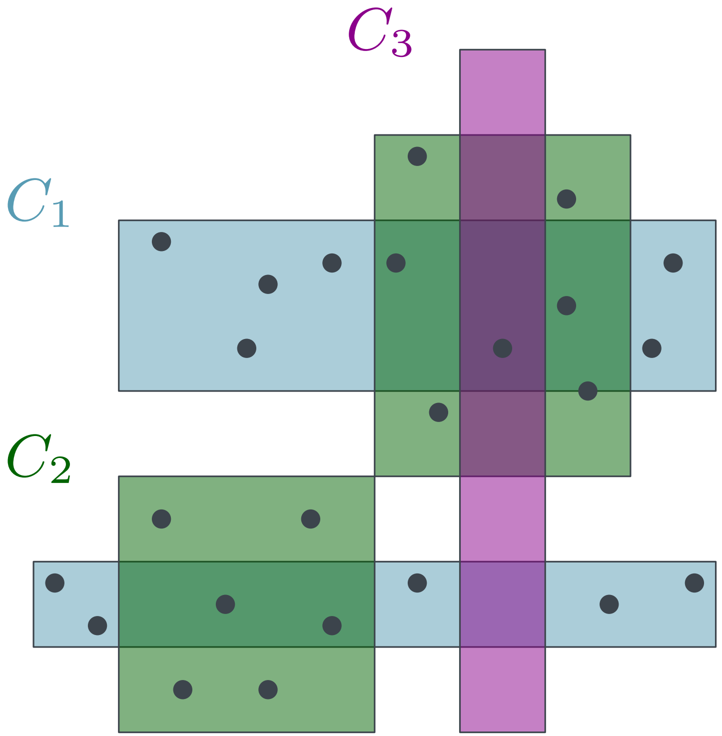

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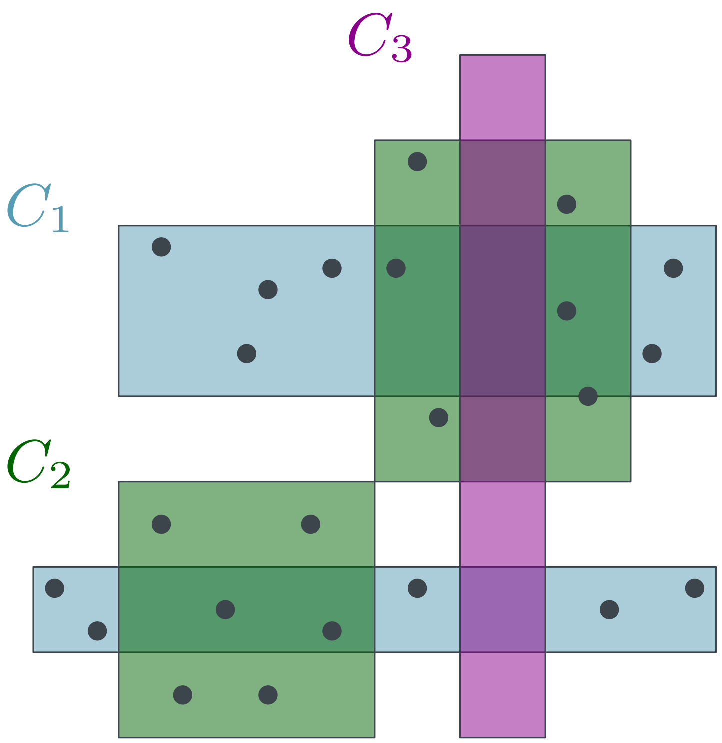

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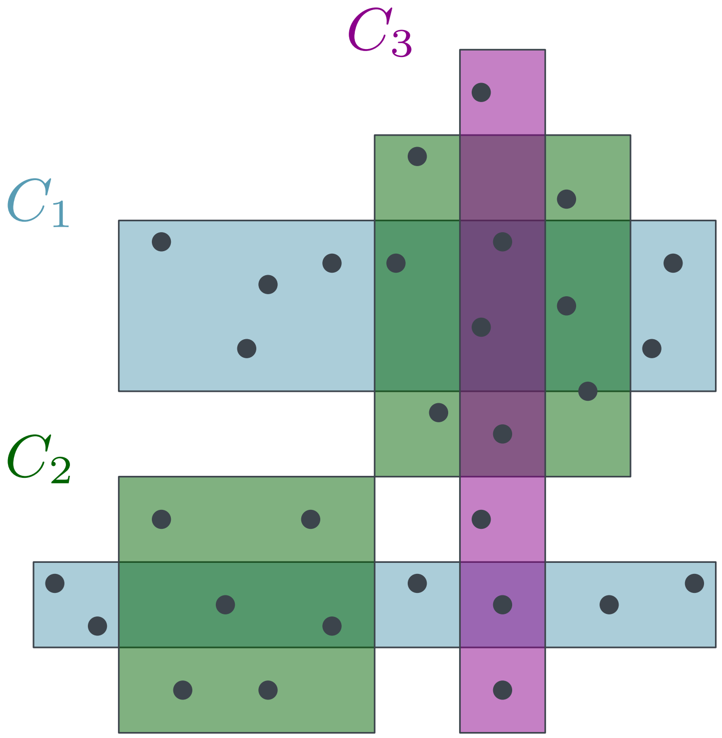

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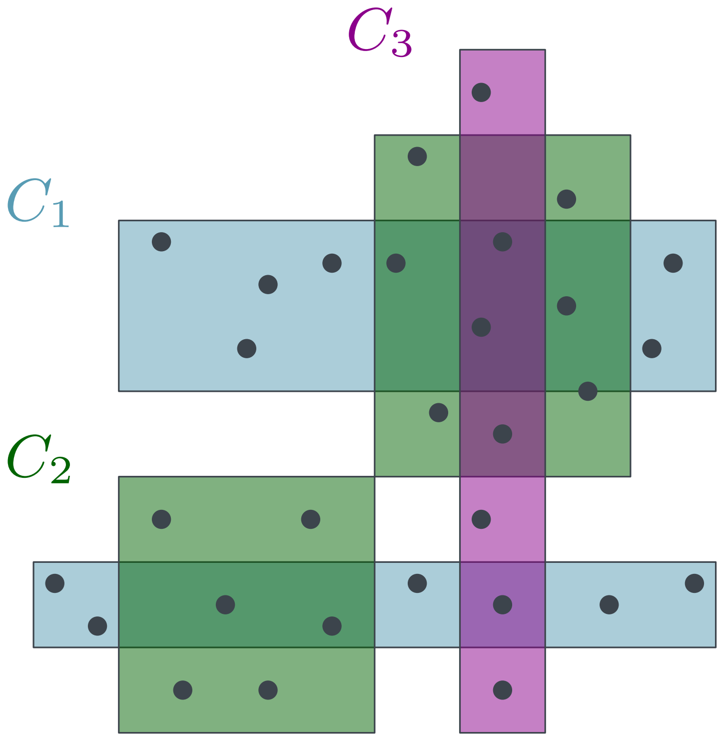

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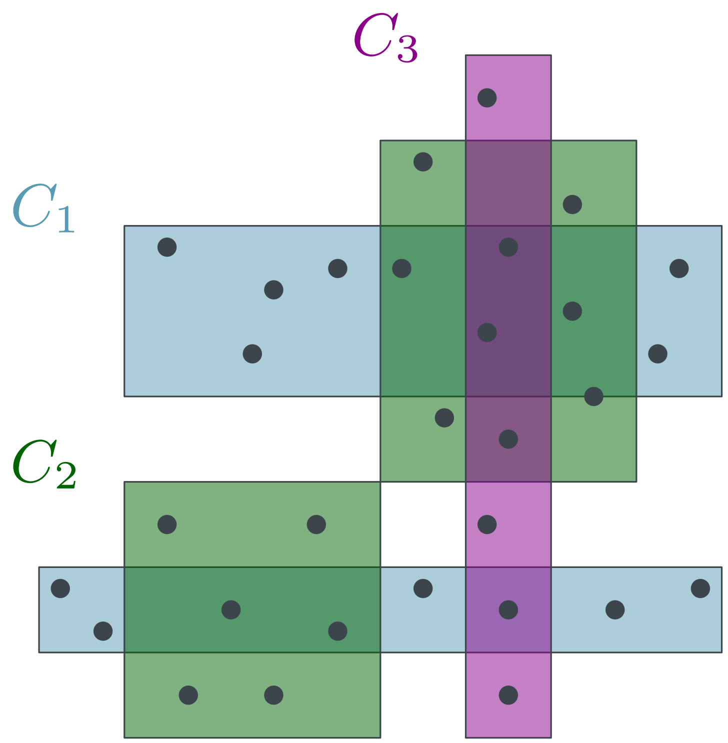

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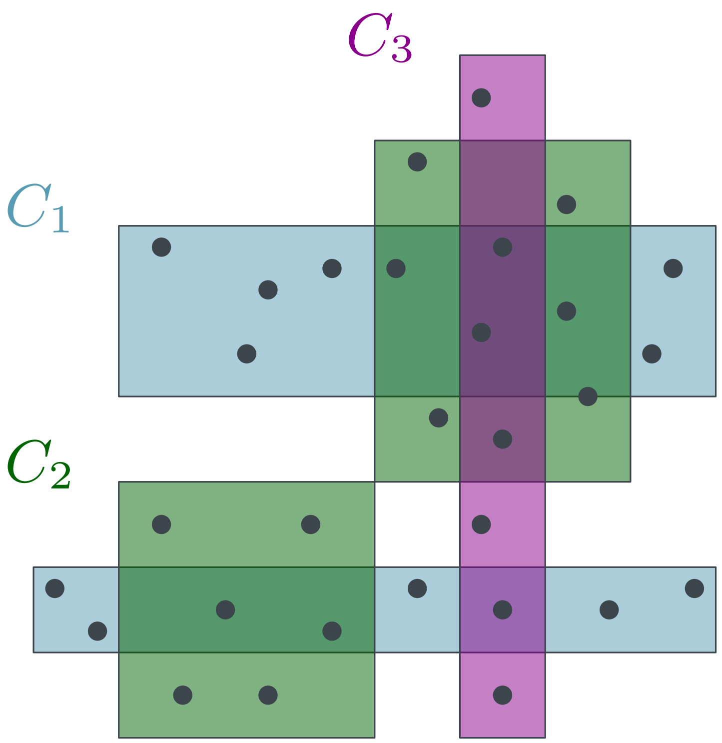

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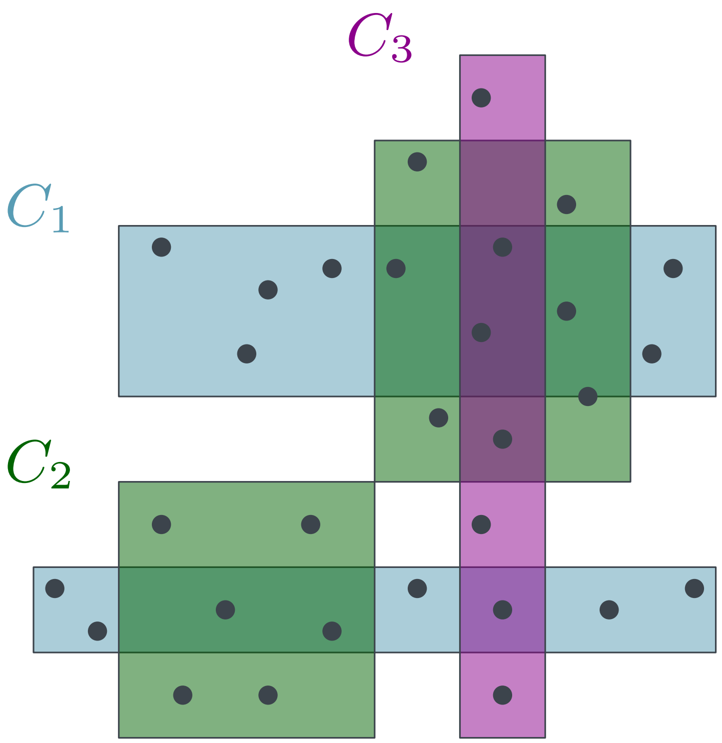

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A random $q \in \bigcup_i C_i$ is contained in at most $\log^{O(d)} n$ classes in expectation.

Upper Bound

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A random $q \in \bigcup_i C_i$ is contained in at most $\log^{O(d)} n$ classes in expectation.

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Running Time?

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Conclusion

Union Volume Estimation

"requires $\Omega(n/\varepsilon^2)$ time"

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Questions?