Polytopal sphere packings for Tubular Surfaces, Lacunary Structures and Subdivisions

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Based on joint works with

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Sphere packings

A **sphere packing** (or **circle packing** in 2D) is a collection of spheres with pairwise disjoint interiors in Euclidean space

Sphere packings and medical imaging

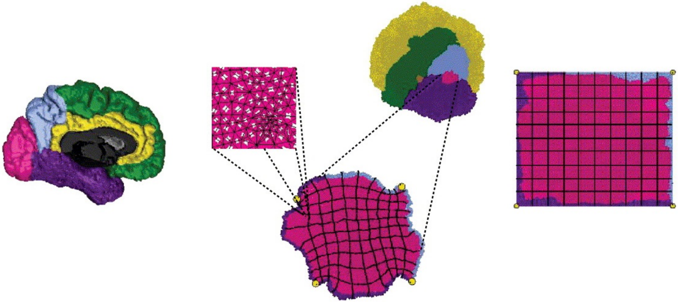


Figure from Hurdal and Stephenson, "*Cortical cartography using the discrete conformal approach of circle packings*", NeuroImage (2004)

Sphere packings and physics of soft materials

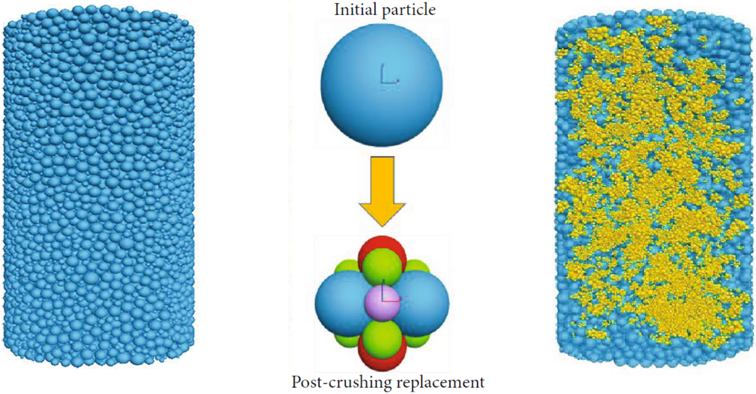
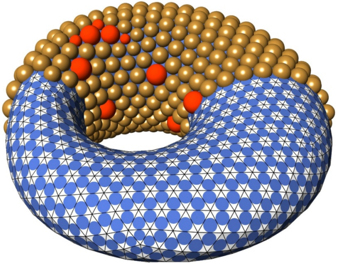
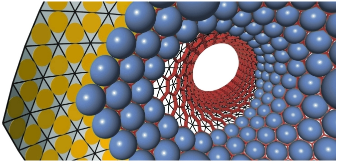


Figure from Lu et al. *Three-Dimensional Discrete Element Analysis of Crushing Characteristics of Calcareous Sand Particles*, Geofluids (2022)

Sphere packings in geometric modelling





Figures from Schiftner, Höbinger, Wallner and Pottmann, *Packing circles and spheres on surfaces*, ACM SIGGRAPH conference proceedings (2009)

Sphere packings in geometric modelling

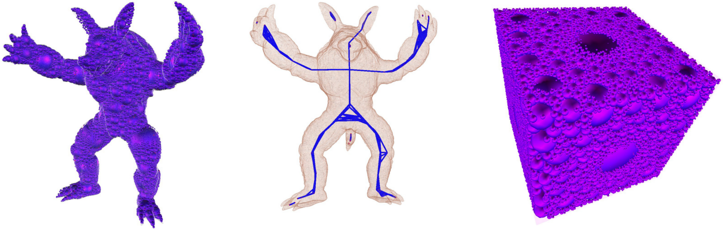


Figure from R. Weller and G. Zachmann, *ProtoSphere: A GPU-Assisted Prototype Guided Sphere Packing Algorithm for Arbitrary Objects*, ACM SIGGRAPH ASIA 2010 conference proceedings

1. The Geometry of Sphere Packings
2. Polytopal Sphere Packings
3. Tubular Surfaces and Knots
4. Lacunary Structures and Subdivisions

1. **The Geometry of Sphere Packings**
2. Polytopal Sphere Packings
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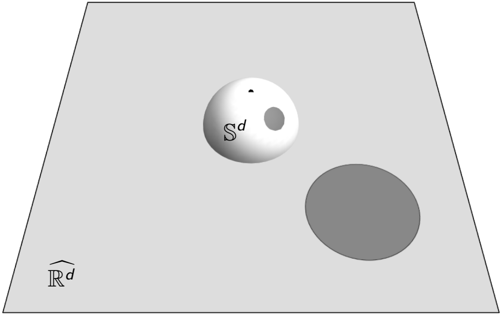
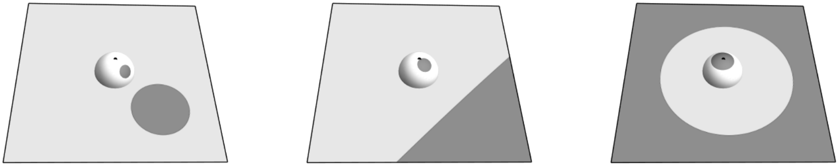
The geometry of sphere packings



An **(oriented) sphere** is the image of a spherical cap in $\mathbb{S}^d$ under stereographic projection

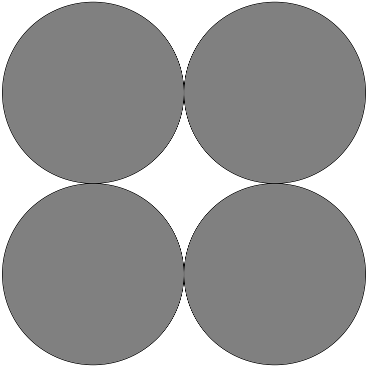
The geometry of sphere packings

Depending on the relative position between the cap and the North Pole, there are three types of spheres



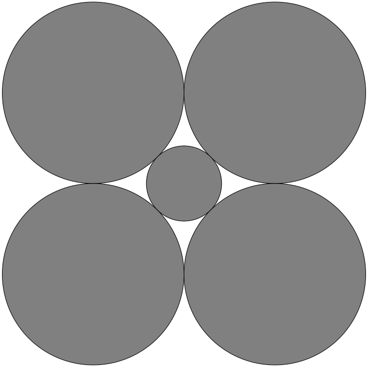
Solid sphere ($r > 0$) | Half-space ($r = \infty$) | Hollow sphere ($r < 0$)

The geometry of sphere packings



A sphere packing is **dense** if it fills almost of all the space

The geometry of sphere packings



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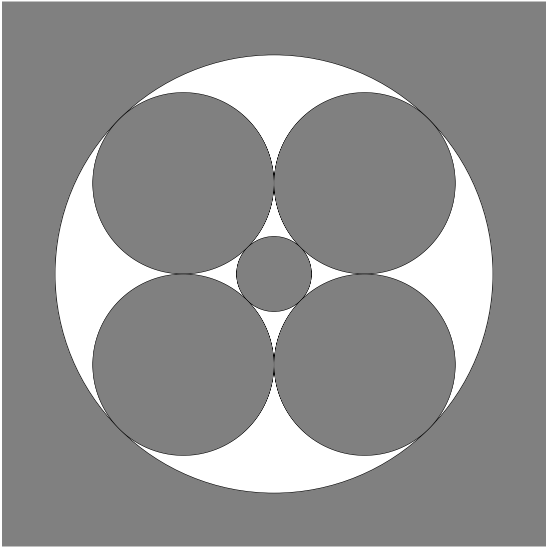

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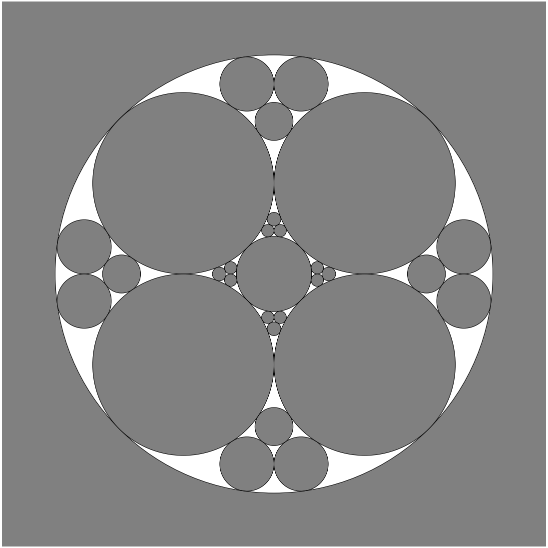

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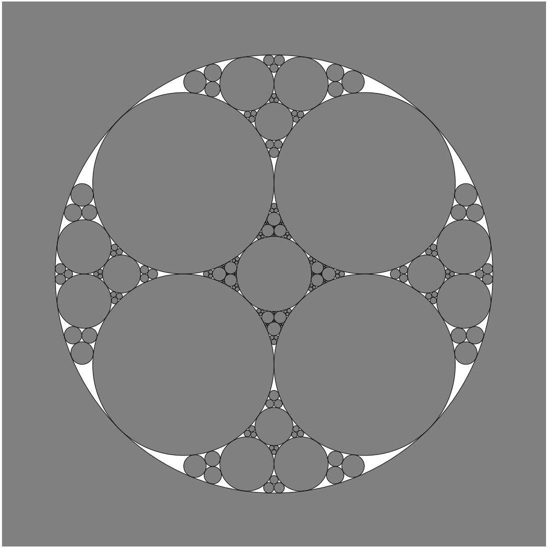

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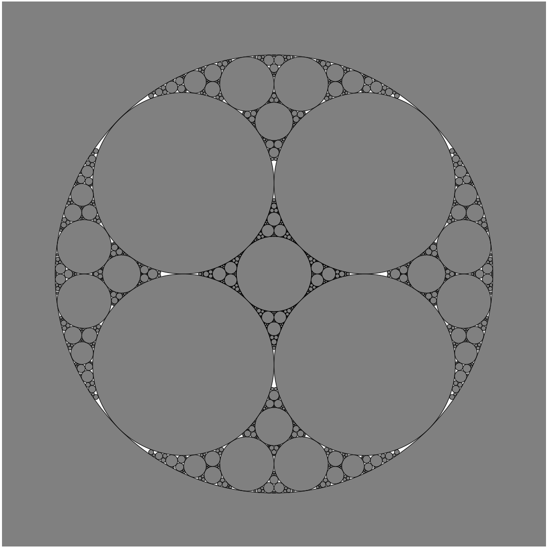

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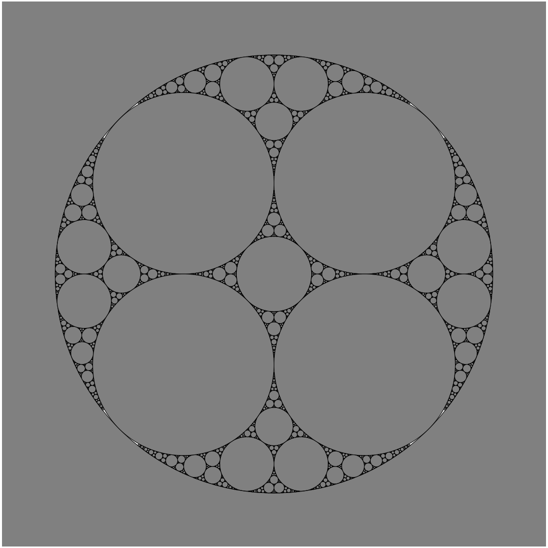

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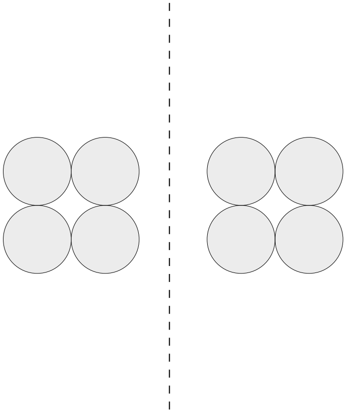
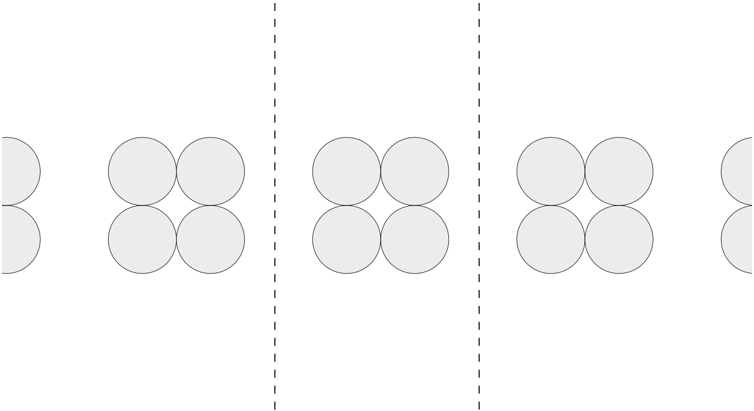


Reflection groups allow us to generate infinite packings

The geometry of sphere packings



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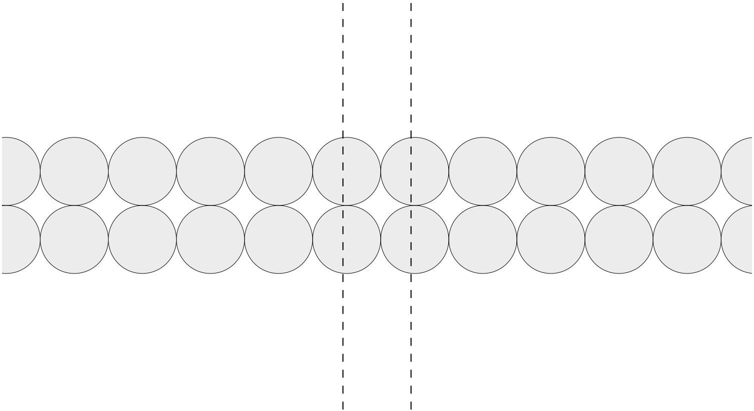

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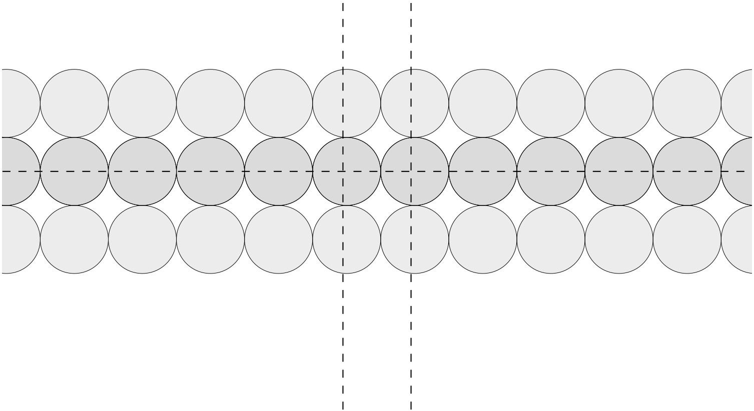

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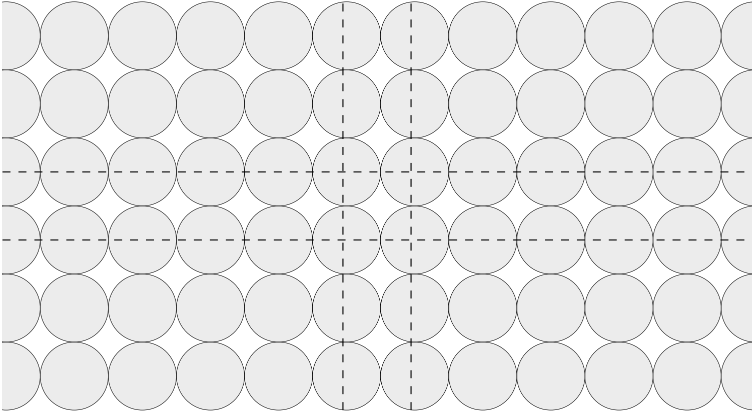
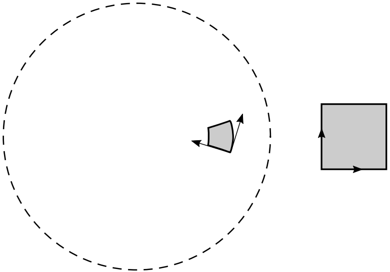


Reflection groups allow us to generate infinite packings

The geometry of sphere packings

Inversion: reflection on a spherical mirror

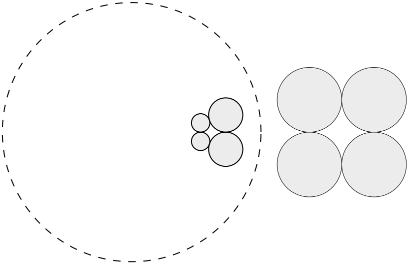
The geometry of sphere packings



Inversion

- **Preserves angles, changes volume**
- Reflects sphere packings to sphere packings
- Fixes spheres orthogonal to the mirror
- Parallel mirrors generate infinite inversions

The geometry of sphere packings



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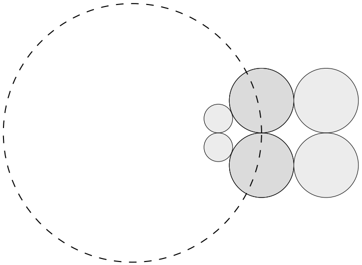

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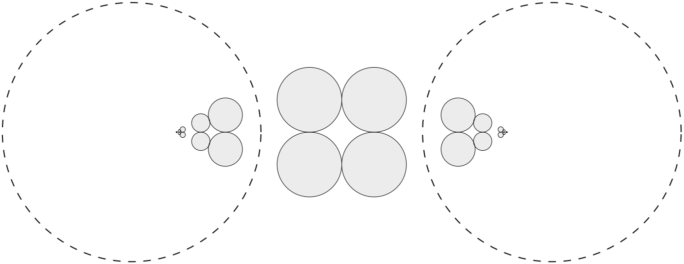

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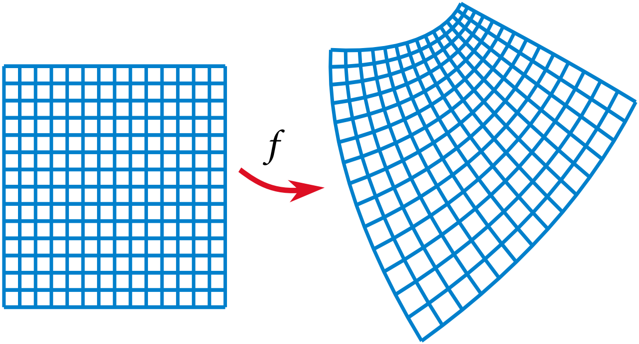
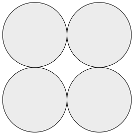
Conformal transformations



Conformal transformations, or **Möbius transformations**,
are maps $\widehat{\mathbb{R}^d} \to \widehat{\mathbb{R}^d}$ that locally preserve angles

The geometry of sphere packings



Conformal transformation
(composition of inversions)

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- Useful for constructing dense sphere packings

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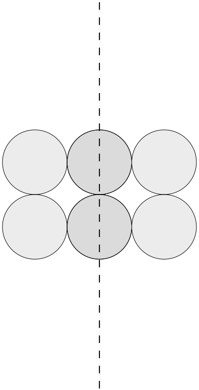

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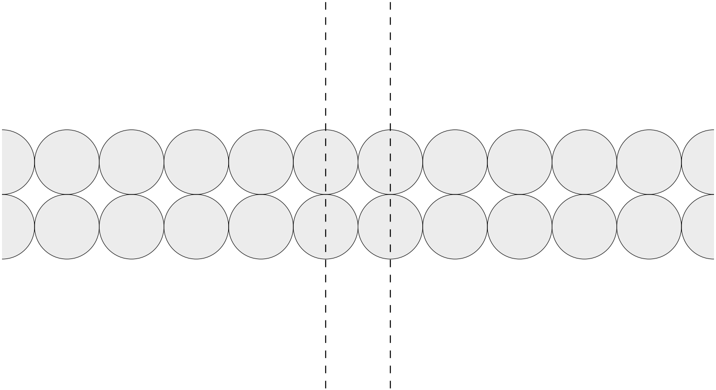

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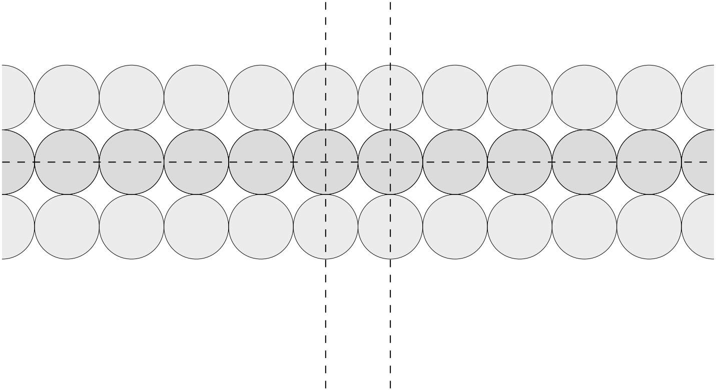

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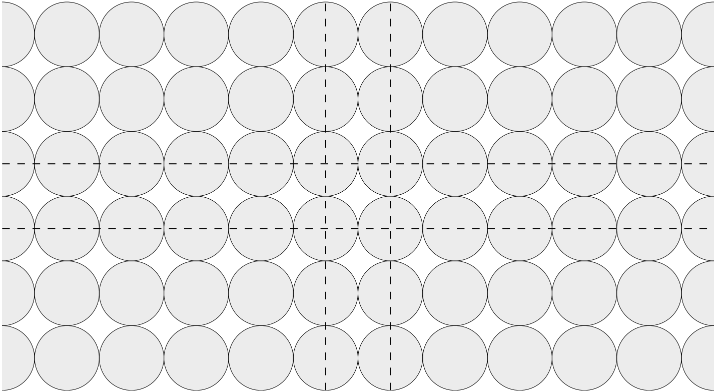

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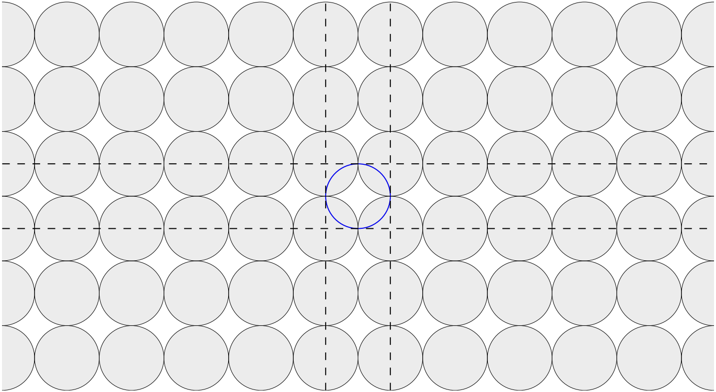

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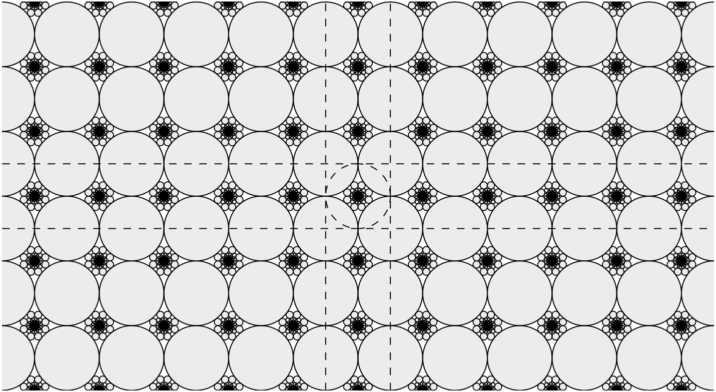

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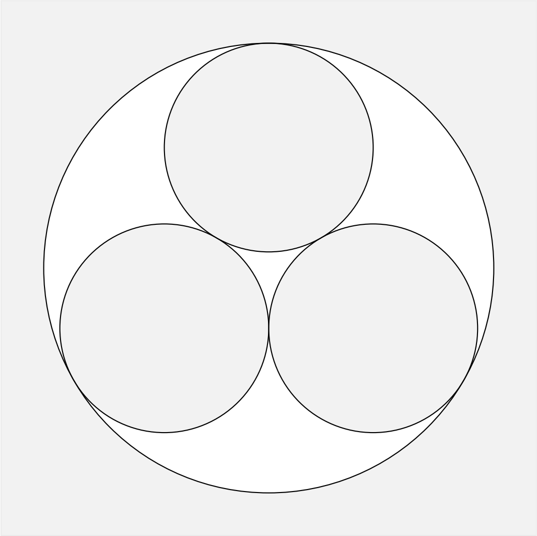
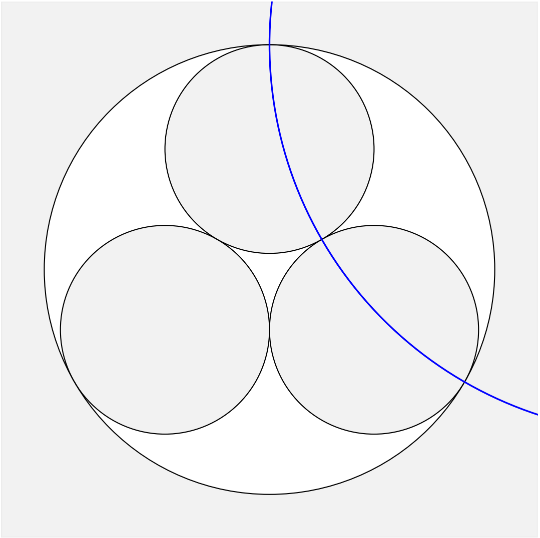
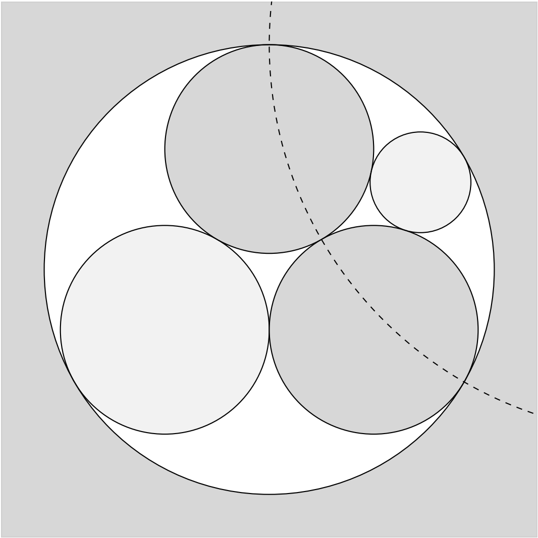


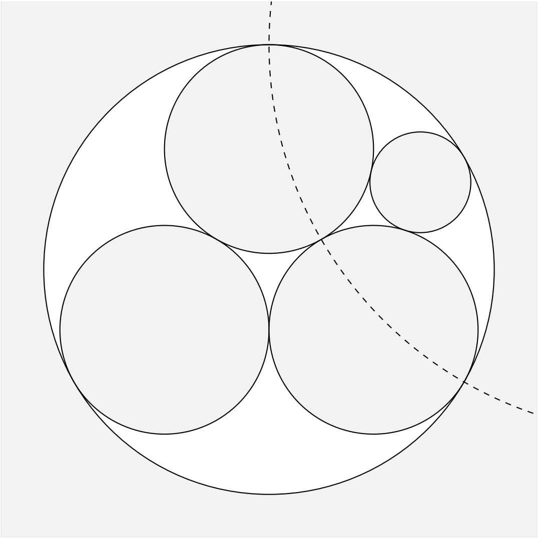
$$\mathcal{S} = \{s_1, s_2, s_3, s_4\}$$



Dual circle s_i^*: circle orthogonal to a triple $\{s_j, s_k, s_l\} \subset \mathcal{S}$



$\langle s_1^* \rangle \cdot \mathcal{S}$



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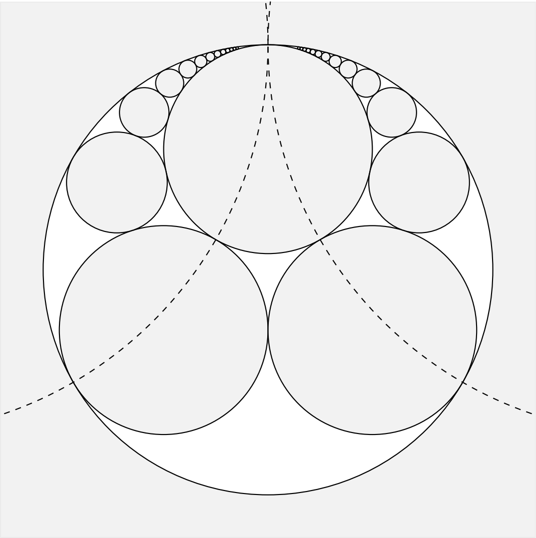
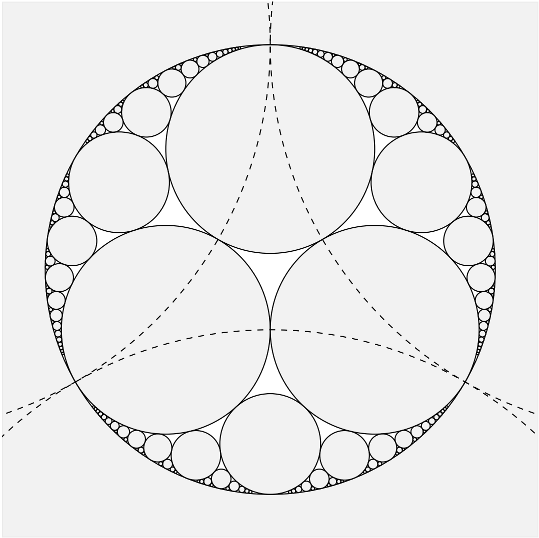
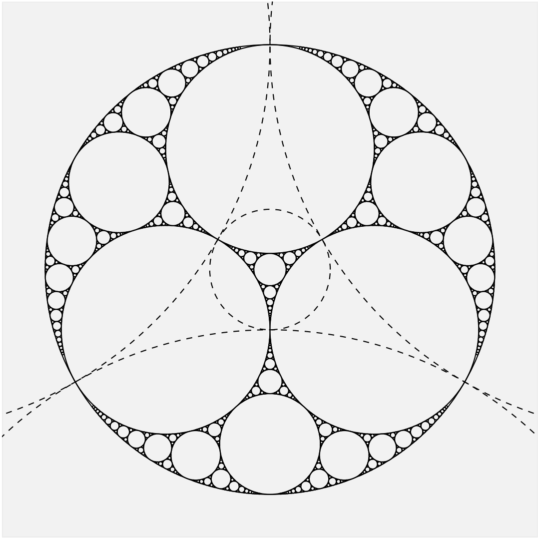


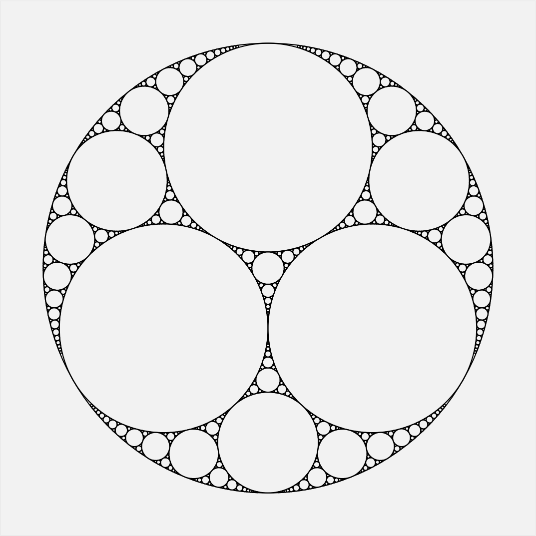
$\langle s_1^*, s_2^* \rangle \cdot \mathcal{S}$



$\langle s_1^*, s_2^*, s_3^* \rangle \cdot \mathcal{S}$



$\langle s_1^*, s_2^*, s_3^*, s_4^* \rangle \cdot \mathcal{S}$



The Apollonian Circle Packing

1. The Geometry of Sphere Packings
2. **Polytopal Sphere Packings**
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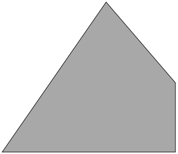
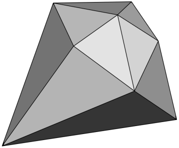
Polytopes

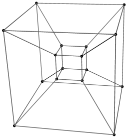
A d-**polytope** is the convex hull of $n \geq d + 1$ points of $\mathbb{R}^d$ in general position.



2-polytope



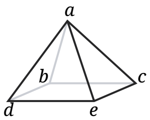
3-polytope

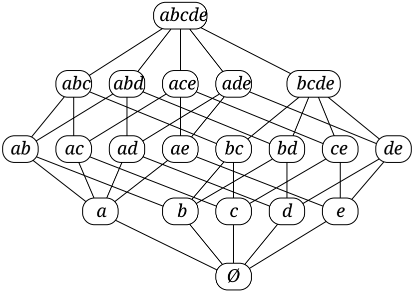


4-polytope
(Schlegel projection)

Regular polytopes

A **flag** of d-polytope $\mathcal{P}$ is a sequence of k-dimensional faces $(f_0, f_1, \ldots, f_{d-1}, f_d = \mathcal{P})$ such that $f_k \subset f_{k+1}$.





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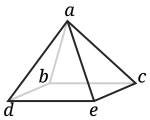
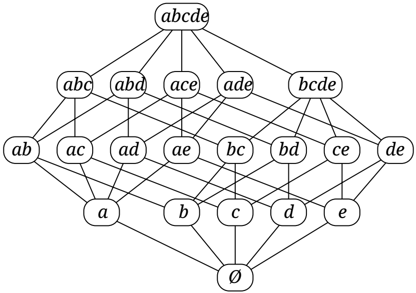


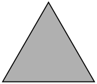


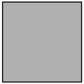
A polytope is **regular** if its symmetry group acts transitively on its flags.

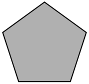
The regular polytopes in every dimension

The regular polytopes in every dimension

$d = 2$





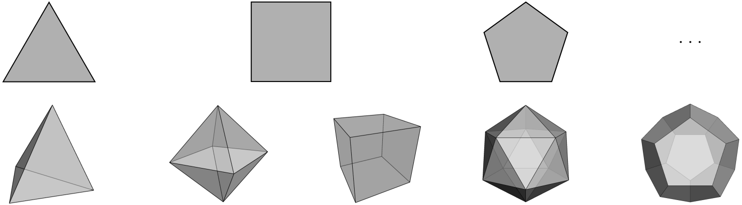


$\cdots$

The regular polytopes in every dimension

$d = 2$

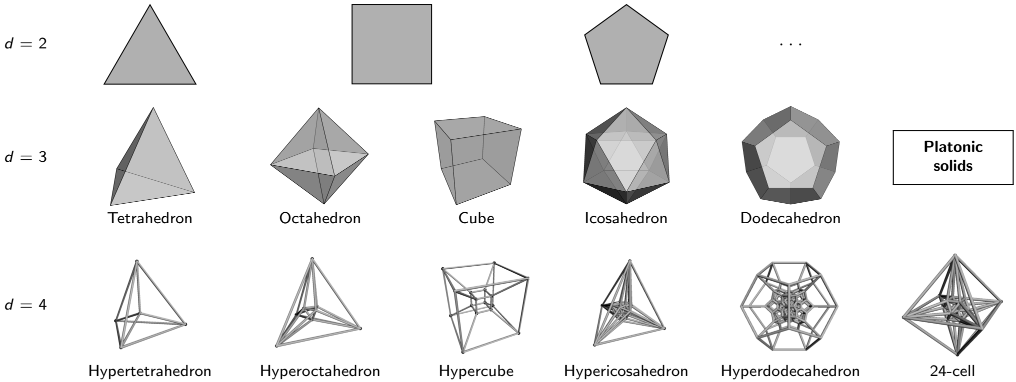
$d = 3$



Tetrahedron Octahedron Cube Icosahedron Dodecahedron

Platonic solids

The regular polytopes in every dimension



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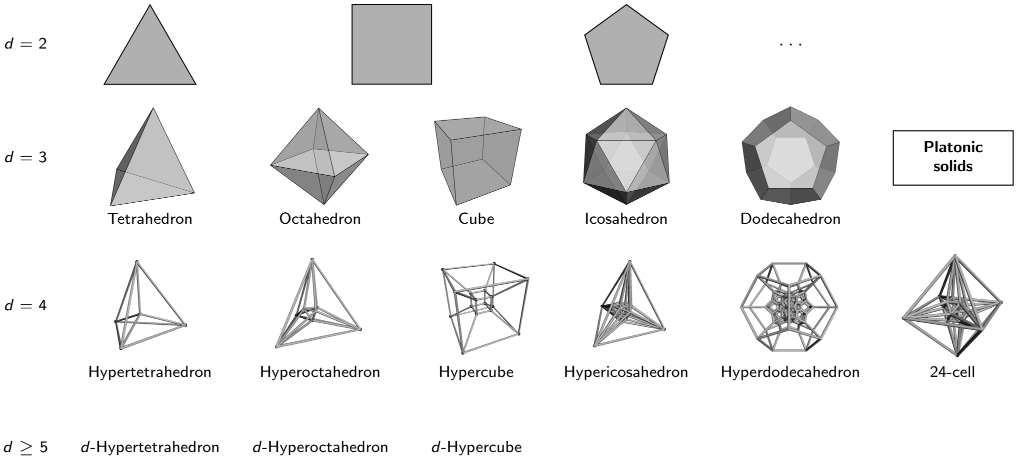
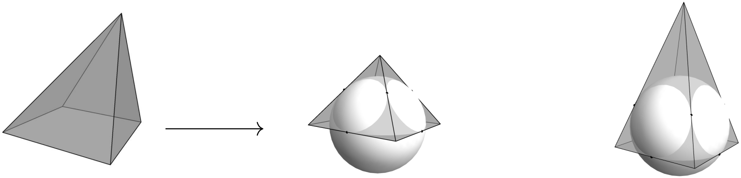


Edge-scribed polytopes

A $(d+1)$-polytope $\mathcal{P}$ is **edge-scribed** if every edge is tangent to the unit sphere $\mathbb{S}^d$

If in addition, the barycenter of $E(\mathcal{P}) \cap \mathbb{S}^d$ is the origin, then $\mathcal{P}$ is **canonical**



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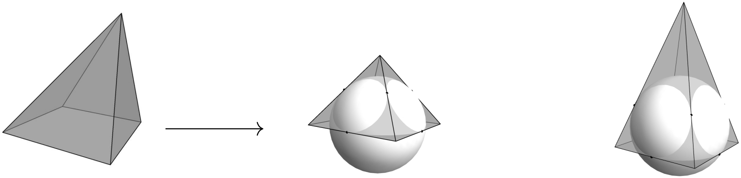

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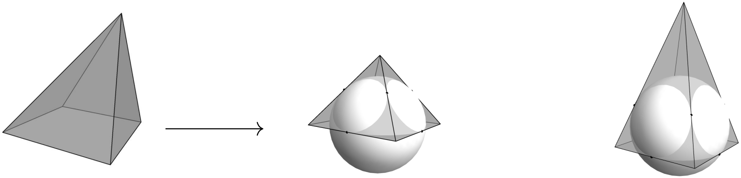

Edge-scribable **Edge-scribed** **Canonical**

- (Brightwell-Scheinerman '93) Every 3-polytope is edge-scribable
- There are non edge-scribable d-polytopes for every $d \geq 4$

Edge-scribed polytopes

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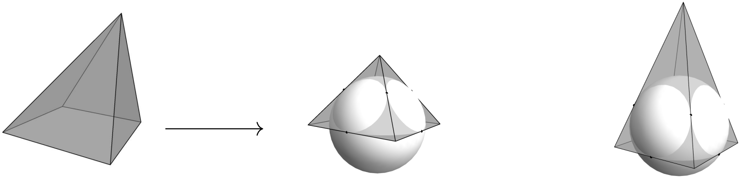
If in addition, the barycenter of $E(\mathcal{P}) \cap \mathbb{S}^d$ is the origin, then $\mathcal{P}$ is **canonical**



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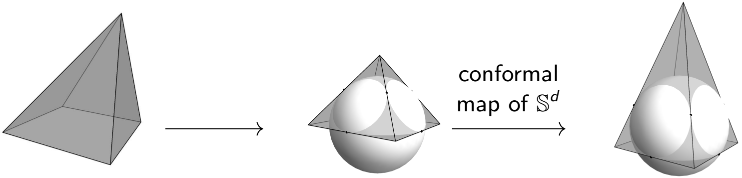

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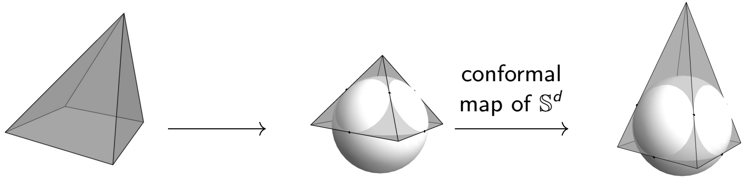

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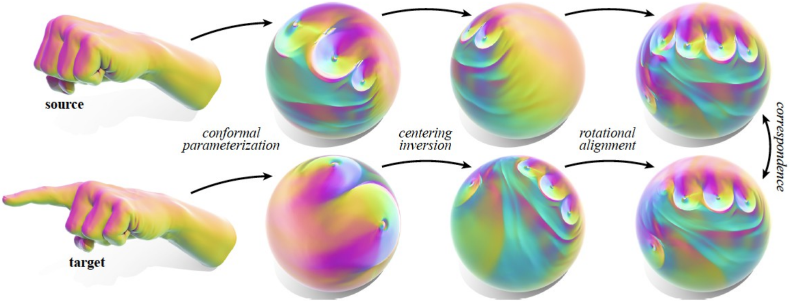

Figure of A. Baden, K. Crane, and M. Kazhdan, *Möbius Registration*, Eurographics Symposium on Geometry Processing (2018)

Polytopal sphere packings

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Arrangement projection $\beta : \{(d+1)\text{-polytopes}\} \to \{d\text{-sphere arrangements in } \widehat{\mathbb{R}^d}\}$

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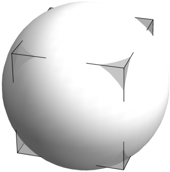

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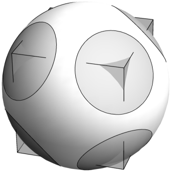

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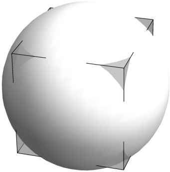

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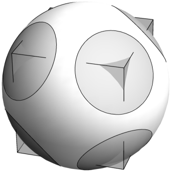

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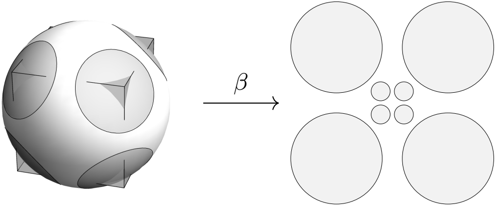

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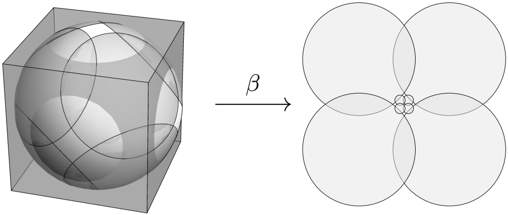

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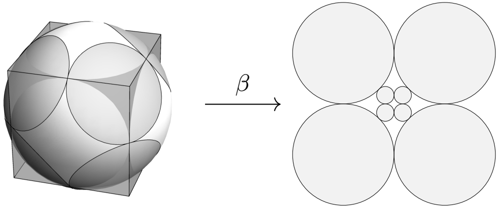

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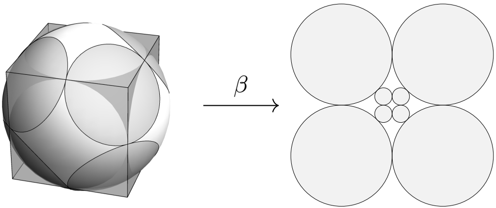
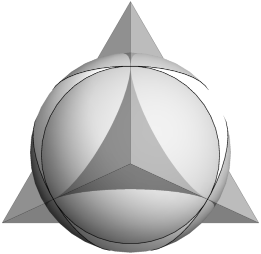
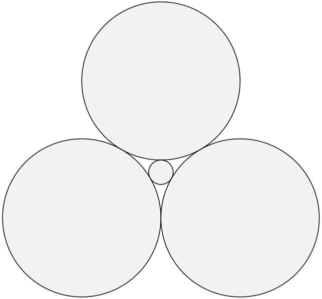


A sphere packing $\mathcal{S}_{\mathcal{P}}$ is **polytopal** if there is an edge-scribed polytope $\mathcal{P}$ such that $\mathcal{S}_{\mathcal{P}} = \beta(\mathcal{P})$, up to conformal transformations

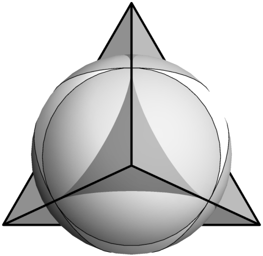
Polytopal sphere packings

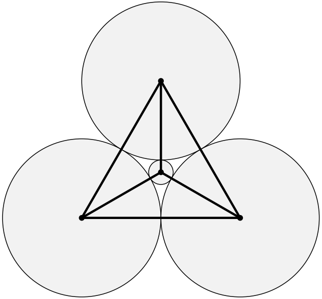




Vertices of $\mathcal{P}$	Spheres of $\mathcal{S}_{\mathcal{P}}$
Edges of $\mathcal{P}$	Tangency relations of $\mathcal{S}_{\mathcal{P}}$

Polytopal sphere packings





Vertices of $\mathcal{P}$ $\longrightarrow$ Spheres of $\mathcal{S}_{\mathcal{P}}$

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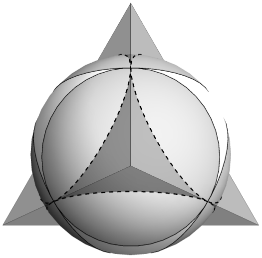
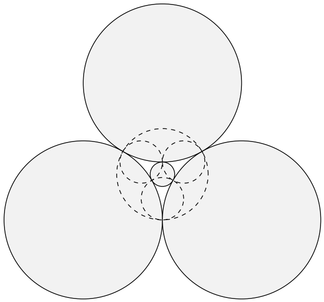


$\longrightarrow$



Vertices of $\mathcal{P}$	Spheres of $\mathcal{S}_{\mathcal{P}}$
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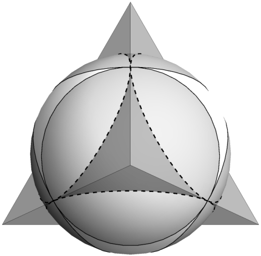
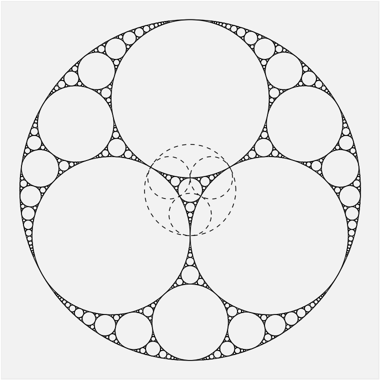
Facets of $\mathcal{P}$	Dual spheres of $\mathcal{S}_{\mathcal{P}}$

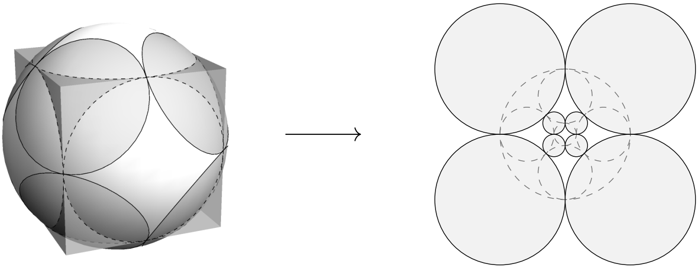
Polytopal sphere packings





In dimension 2, the union of the infinite reflections of $\mathcal{S}_{\mathcal{P}}$ through its dual spheres is a dense packing

Polytopal sphere packings



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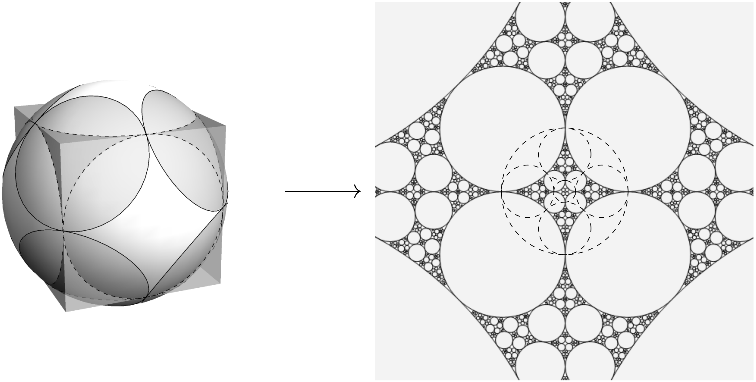

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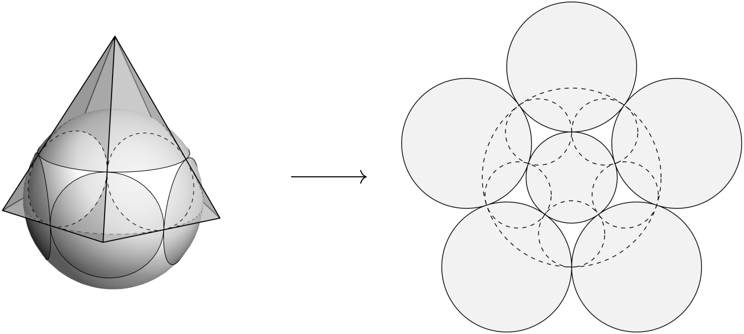

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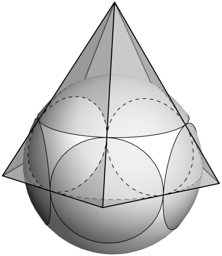

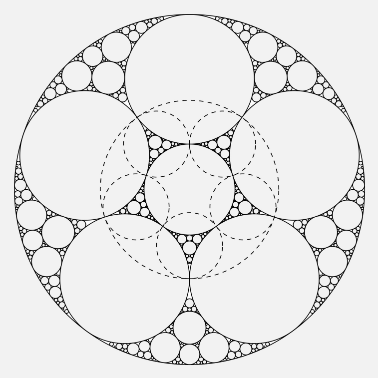

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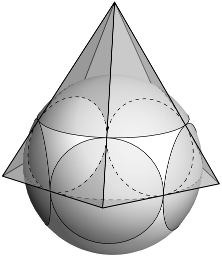
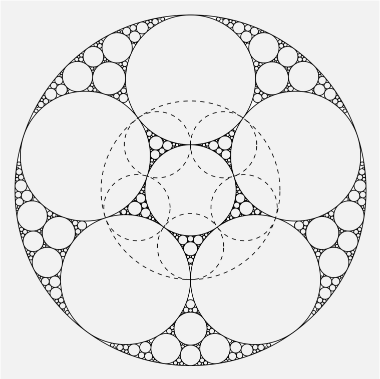
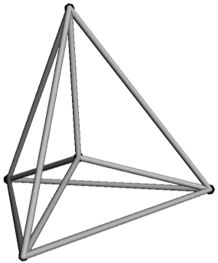




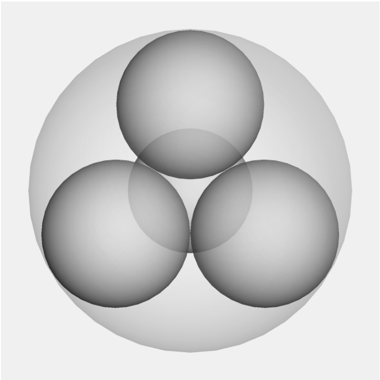
Not always true in higher dimensions!

Polytopal sphere packings



Hypertetrahedron

$\longrightarrow$



Apollonian sphere packing

Not always true in higher dimensions!

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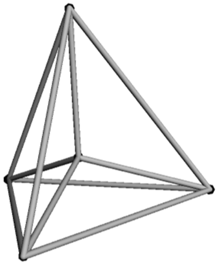

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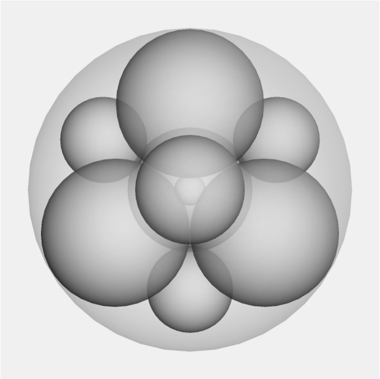

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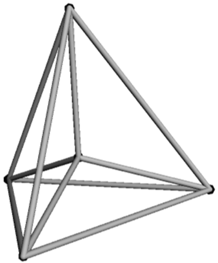

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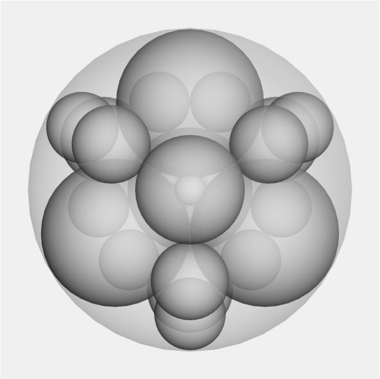

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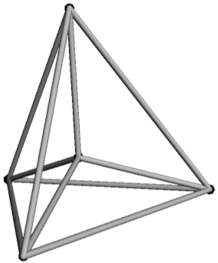

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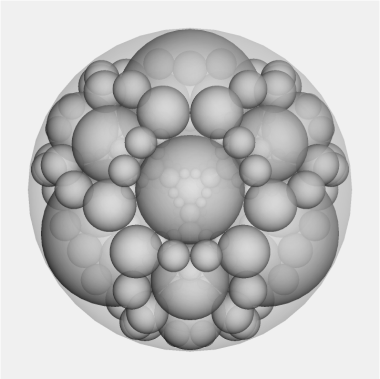

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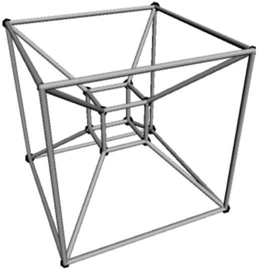
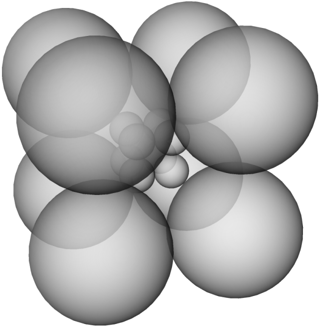


Hypercube

$\longrightarrow$



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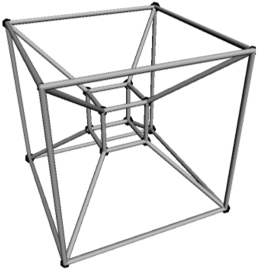

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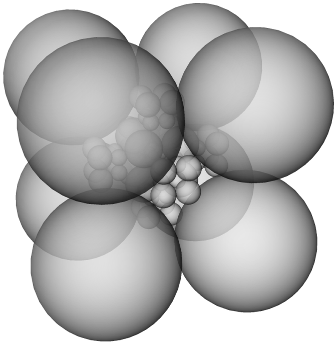

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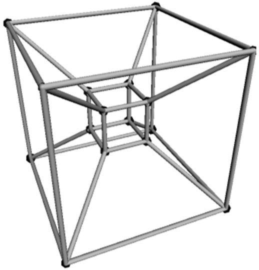

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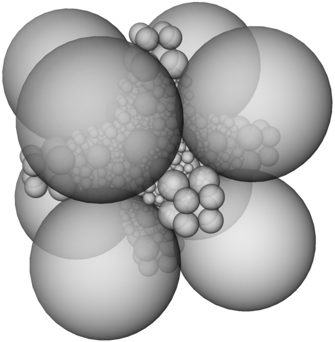
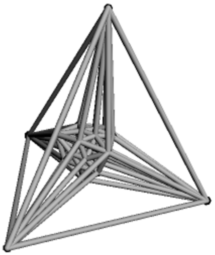


Hypercube

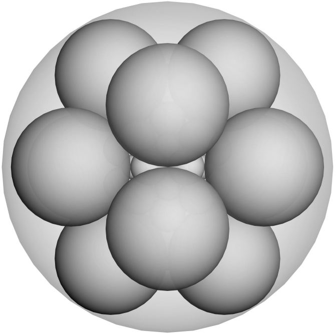
Hypercubic dense packing

Not always true in higher dimensions!

Polytopal sphere packings



Hypericosahedron



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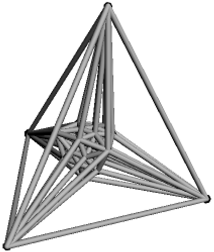
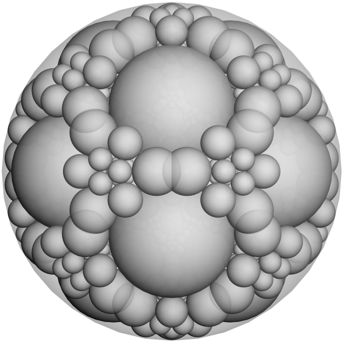


$\longrightarrow$



Hypericosahedron

Not a packing: the spheres overlap

Not always true in higher dimensions!

The crystallographic regular polytopes

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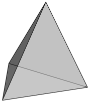
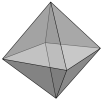
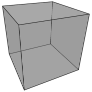
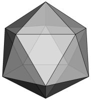
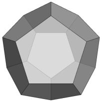
$d = 3$


Tetrahedron


Octahedron


Cube


Icosahedron


Dodecahedron

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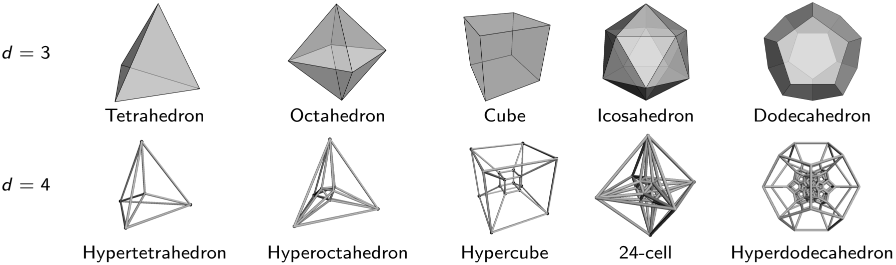


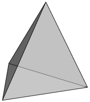
The crystallographic regular polytopes

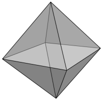
A polytope $\mathcal{P}$ is **crystallographic** if the union of the infinite inversions of $\mathcal{S}_{\mathcal{P}}$ through its dual spheres is a packing.

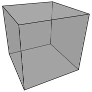
Finitiness theorem (Bogachev, Kolpakov, Kontorovich '23)
Crystallographic polytopes exist only in dimensions $3 \leq d \leq 19$.

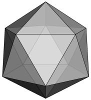
Theorem (R. '24) There are only 11 crystallographic regular polytopes.

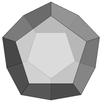
$d = 3$


Tetrahedron

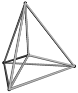

Octahedron


Cube

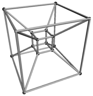

Icosahedron

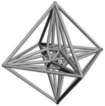

Dodecahedron

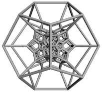
$d = 4$


Hypertetrahedron


Hyperoctahedron


Hypercube


24-cell

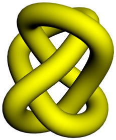
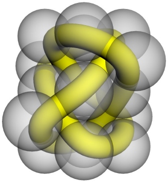
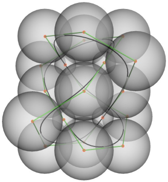
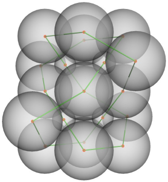

Hyperdodecahedron

$d = 6$

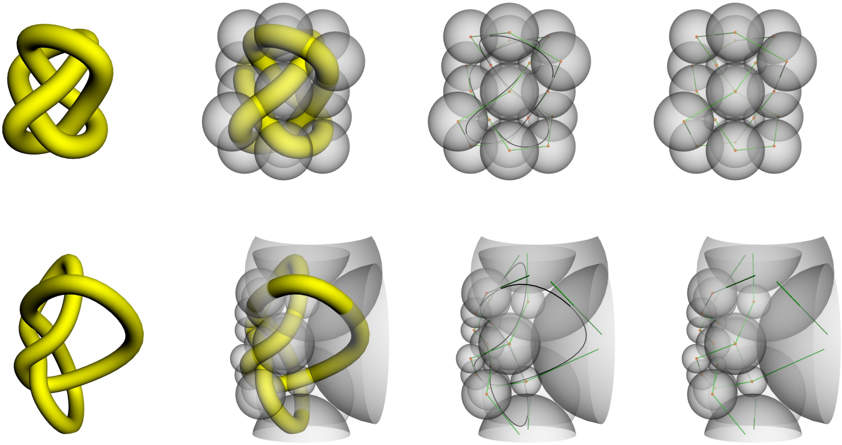
6-Hyperoctahedron

1. The Geometry of Sphere Packings
2. Polytopal Sphere Packings
3. **Tubular Surfaces and Knots**
4. Lacunary Structures and Subdivisions

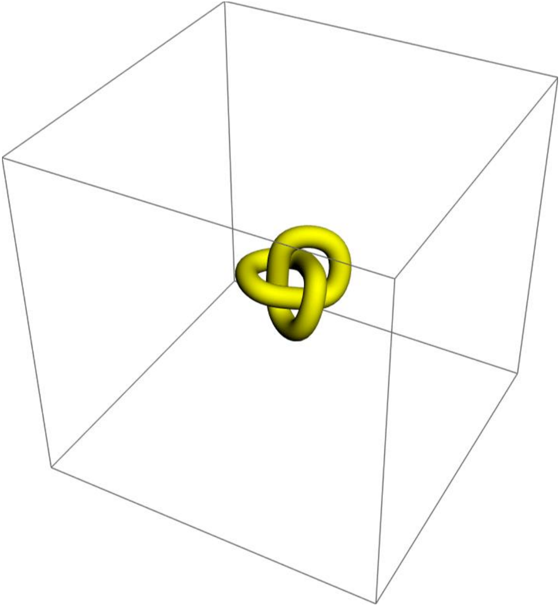
Sphere packings and deformations of tubular surfaces

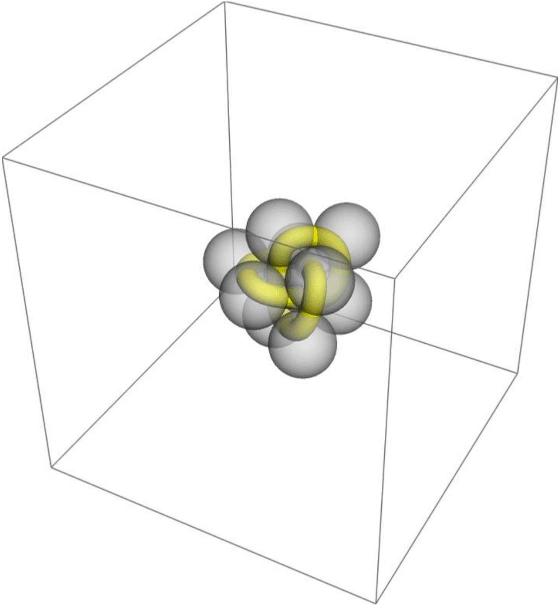
Sphere packings and deformations of tubular surfaces



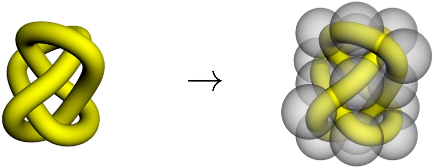
Sphere packings and deformations of tubular surfaces



Sphere packings and deformations of tubular surfaces



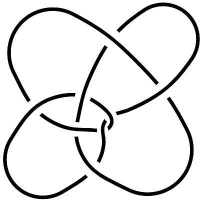
Sphere packings containing a given knot



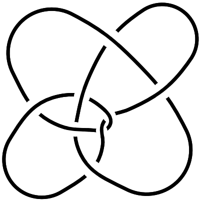
Algorithm 1 Ramírez Alfonsín-R., *Ball packings for links*,
European Journal of Combinatorics (2021)

Algorithm 2 Ramírez Alfonsín-R., *Links in orthoplicial Apollonian packings*,
European Journal of Combinatorics (2024)

Algorithm 1

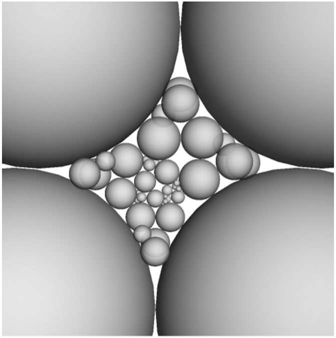


Algorithm 1



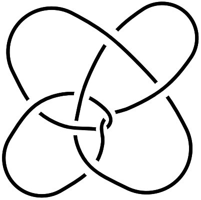
n crossings

$\to$

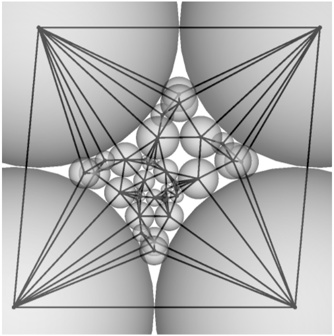


$5n$ spheres

Algorithm 1



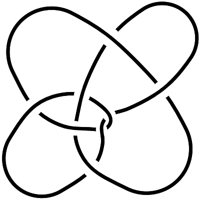
$\rightarrow$



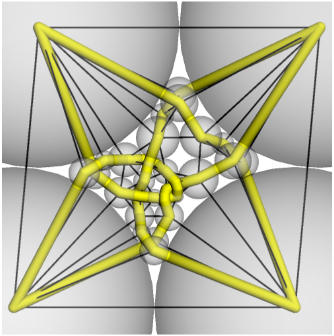
n crossings

$5n$ spheres

Algorithm 1



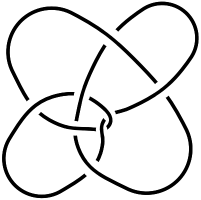
$\rightarrow$



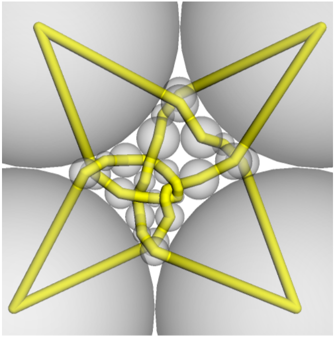
n crossings

$5n$ spheres

Algorithm 1



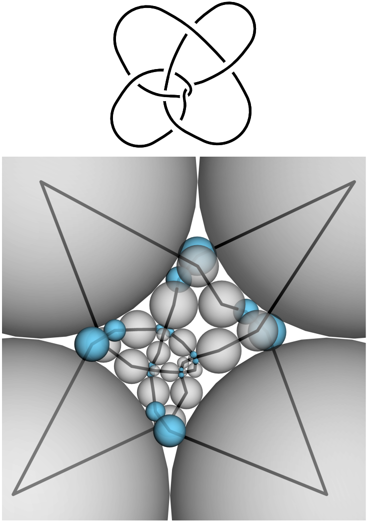
$\rightarrow$

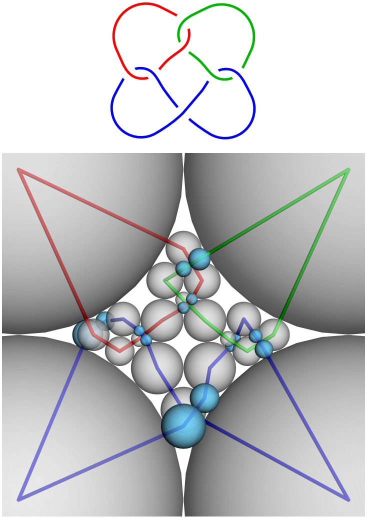


n crossings

$5n$ spheres

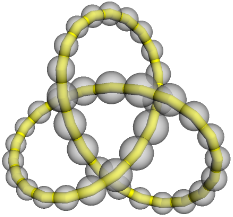
Algorithm 1



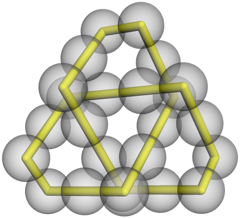


Application in geometric knot theory

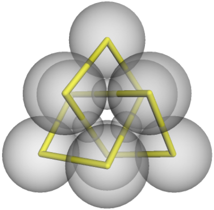
$$cr(L) := \min \#\{\text{crossings among all diagrams of } L\}$$
$$ball(L) := \min \#\{\text{spheres in a packing containing } L\}$$



40 spheres



24 spheres



12 spheres

Theorem (Ramírez-R. 21') $ball(L) \leq 5cr(L)$

Application in geometric knot theory

Theorem (Ramírez-R. '21) For any **non-trivial** and **non-splittable** link L

$$ball(L) \leq 5cr(L)$$

Application in geometric knot theory

Theorem (Ramírez-R. '21) For any **non-trivial** and **non-splittable** link L

$$ball(L) \leq 5cr(L)$$

Conjecture (Ramírez-R. '21) For any **alternating** link L

$$ball(L) = 4cr(L)$$

Application in geometric knot theory

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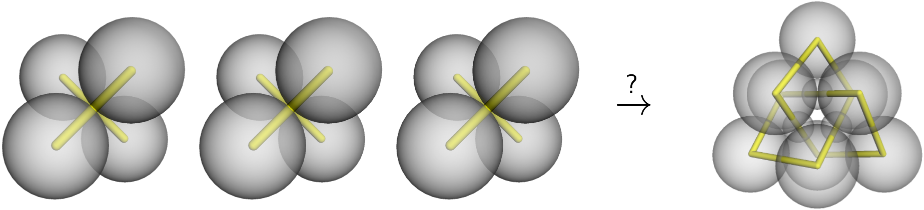


Application in geometric knot theory

Theorem (Ramírez-R. '21) For any **non-trivial** and **non-splittable** link L

$$ball(L) \leq 5cr(L)$$

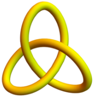
Conjecture (Ramírez-R. '21) For any **alternating** link L

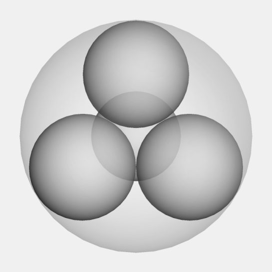
$$ball(L) = 4cr(L)$$

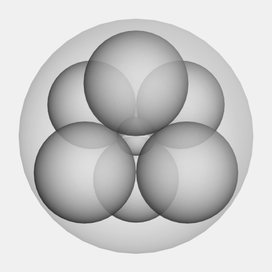
Theorem (Ramírez-R. '23) For any **rational** link L

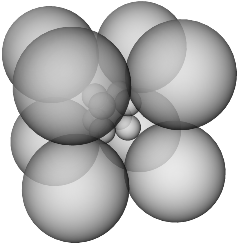
$$ball(L) \leq 4cr(L)$$

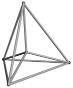
Application in geometric knot theory









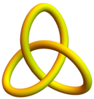


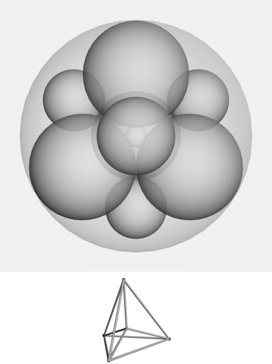
Hypertetrahedron

Hyperoctahedron

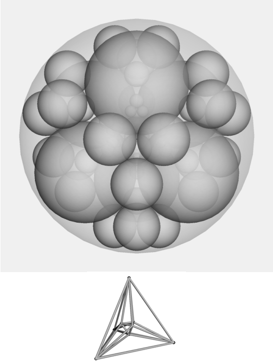
Hypercube

Application in geometric knot theory

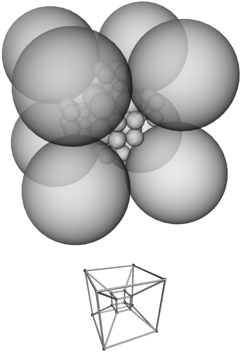




Hypertetrahedron

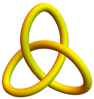


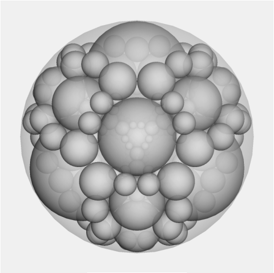
Hyperoctahedron

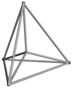


Hypercube

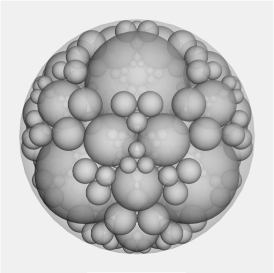
Application in geometric knot theory



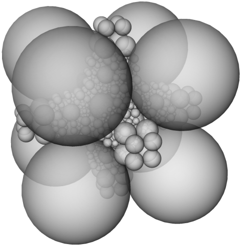


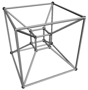


Hypertetrahedron



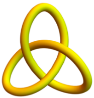
Hyperoctahedron

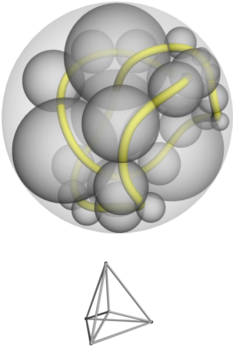




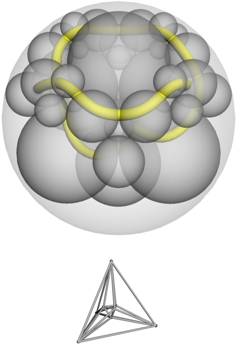
Hypercube

Application in geometric knot theory

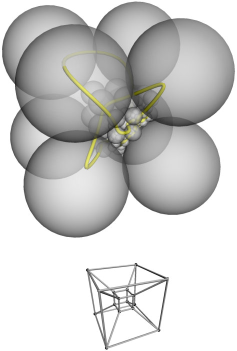




Hypertetrahedron

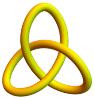


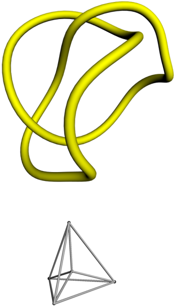
Hyperoctahedron



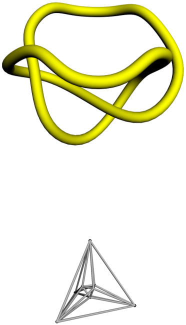
Hypercube

Application in geometric knot theory

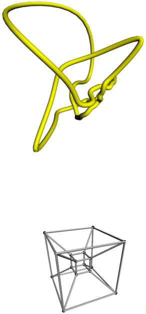




Hypertetrahedron

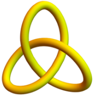


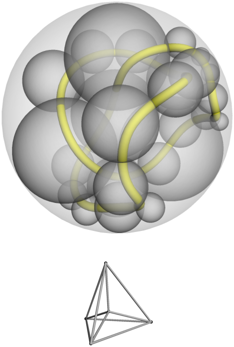
Hyperoctahedron



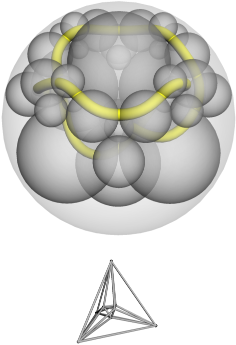
Hypercube

Application in geometric knot theory

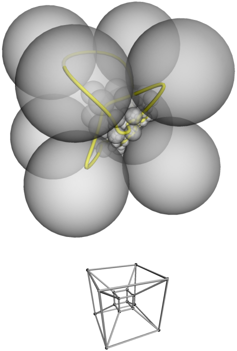




Hypertetrahedron



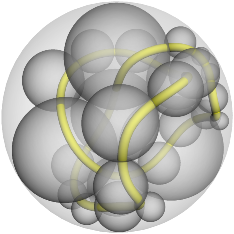
Hyperoctahedron

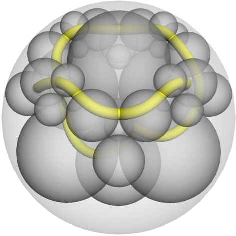


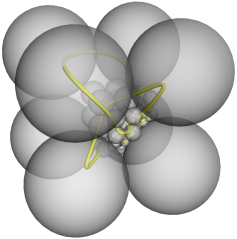
Hypercube

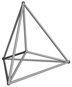
Application in geometric knot theory

Theorem (Ramírez-R. 23') Every link admits a necklace representation in the five 3D regular crystallographic packings





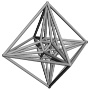




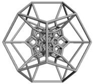
Hypertetrahedron

Hyperoctahedron

Hypercube



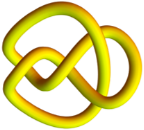
24-cell



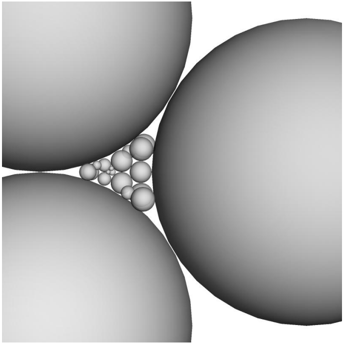
Hyperdodecahedron

Application in geometric knot theory

Theorem (Ramírez-R. 21') $ball(L) \leq 5cr(L)$



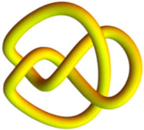
$cr(L) = 4$



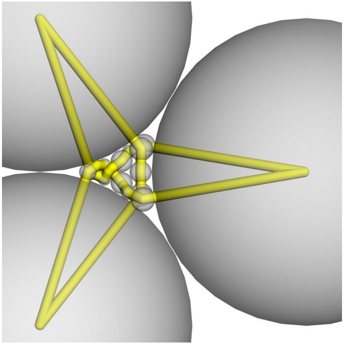
20 spheres

Application in geometric knot theory

Theorem (Ramírez-R. 21') $ball(L) \leq 5cr(L)$



$cr(L) = 4$

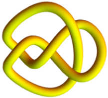


20 spheres

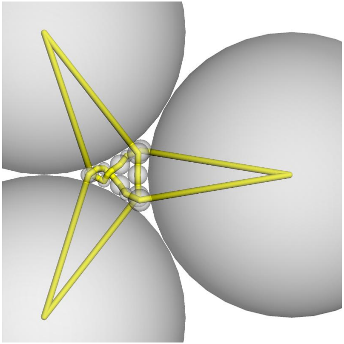
Application in geometric knot theory

Theorem (Ramírez-R. 21') $ball(L) \leq 5cr(L)$

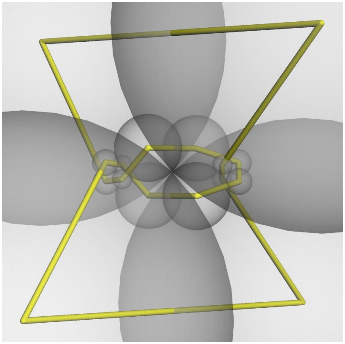
Theorem (Ramírez-R. 24') For any **rational link** L, $ball(L) \leq 4cr(L)$



$cr(L) = 4$
L rational

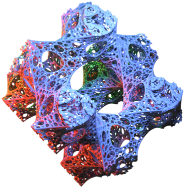


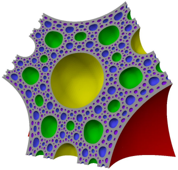
20 spheres



16 spheres contained in the hyperoctahedral packing

Lacunary structures





GT MG
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Modélisation Géométrique

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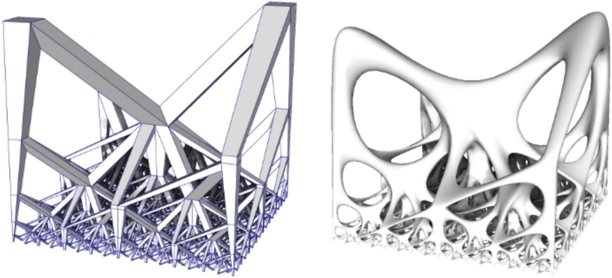
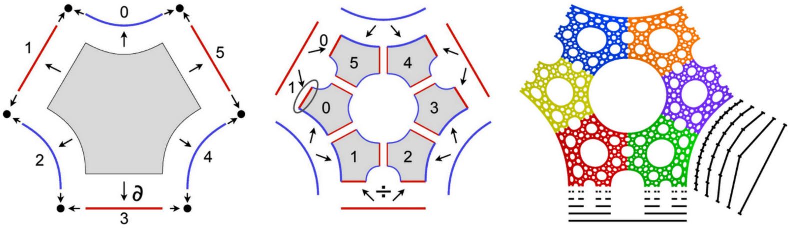
Lacunary structures



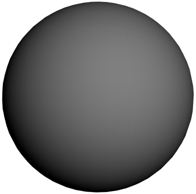
From Sokolov, Gouaty, Gentil, Mishkinis, *Boundary Controlled Iterated Function System*, Curves and Surfaces (2015), Lecture Notes in Computer Science

Lacunary structures

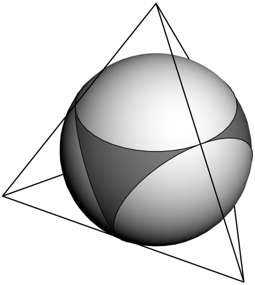


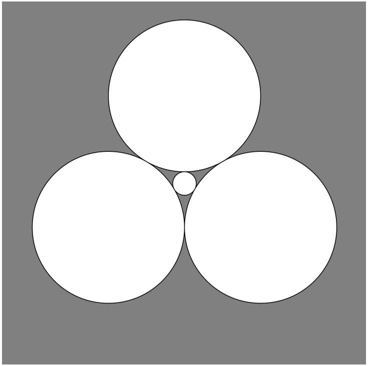
Polytopal sphere packings can be construct with the BC-IFS model.
The incidency and adjacency conditions can be expressed in terms of the combinatoric structure of the polytope

Lacunary structures



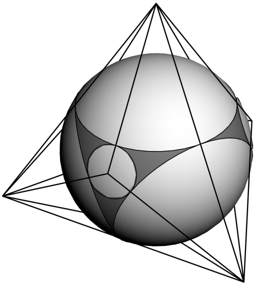
Lacunary structures

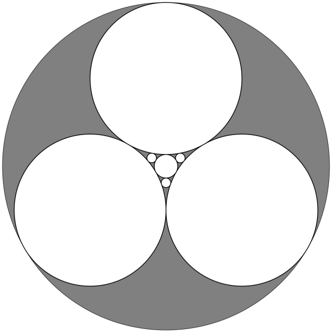




Complementary of a Polytopal Circle Packing based on a tetrahedron

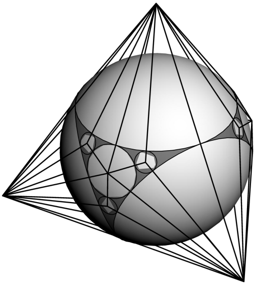
Lacunary structures

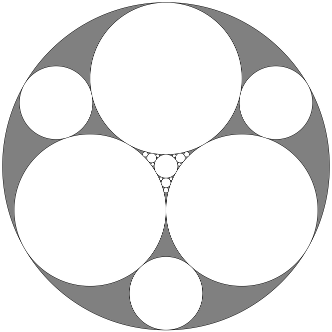




Canonical Apollonian subdivision

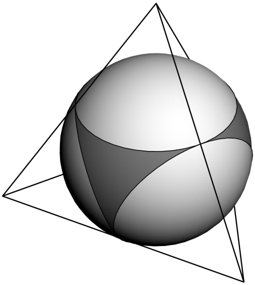
Lacunary structures

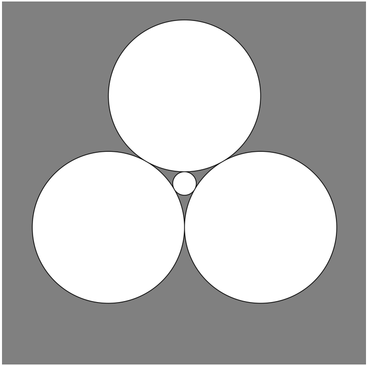




Canonical Apollonian subdivision

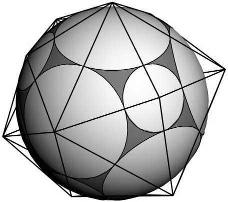
Lacunary structures

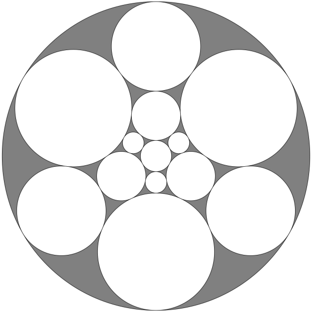




Canonical Barycentric Subdivision

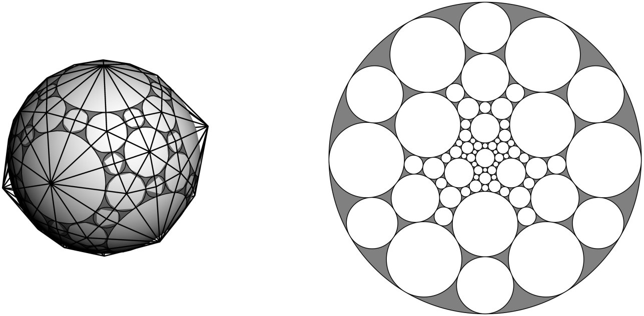
Lacunary structures





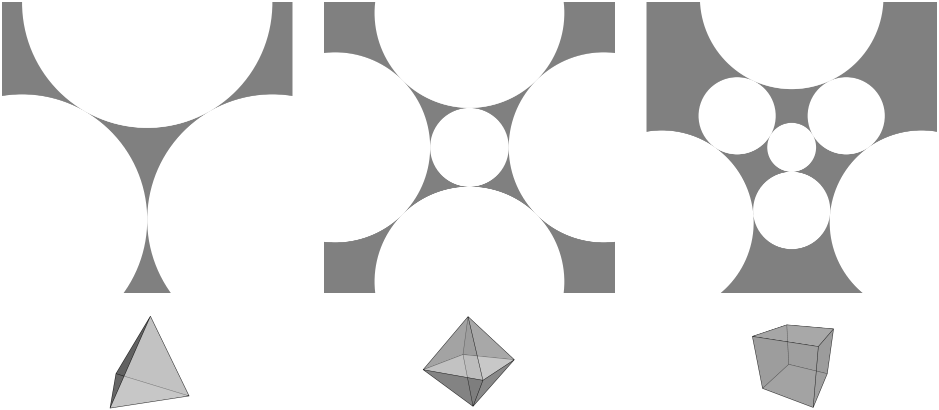
Canonical Barycentric subdivision

Lacunary structures



Canonical Barycentric subdivision

Lacunary structures



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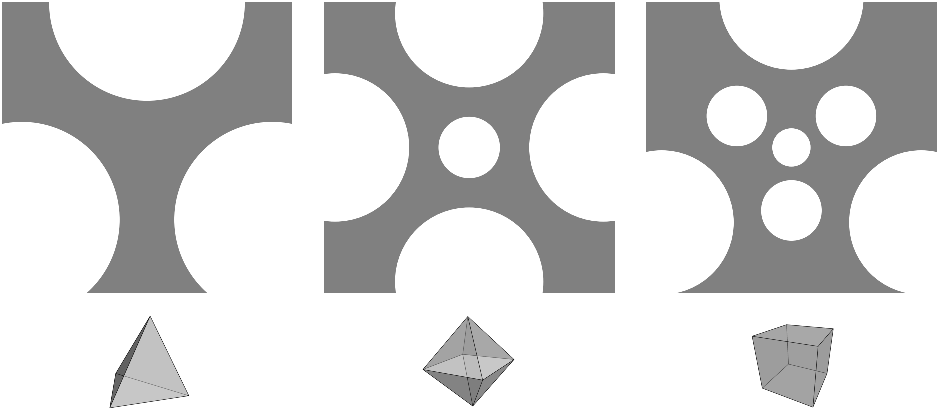

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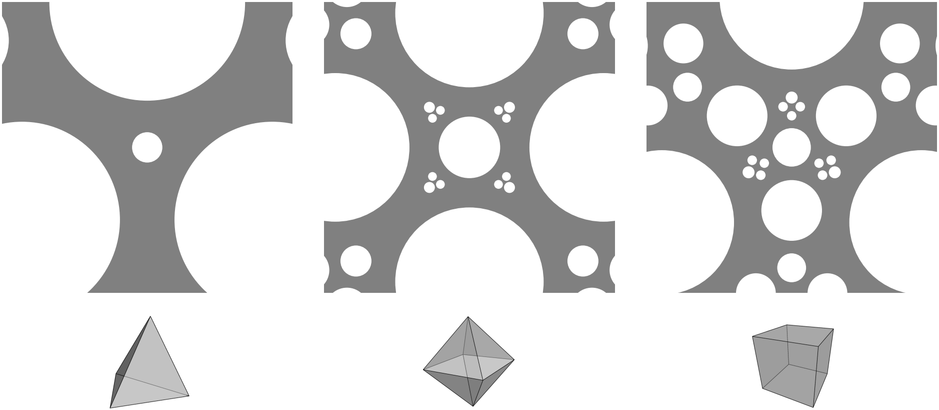

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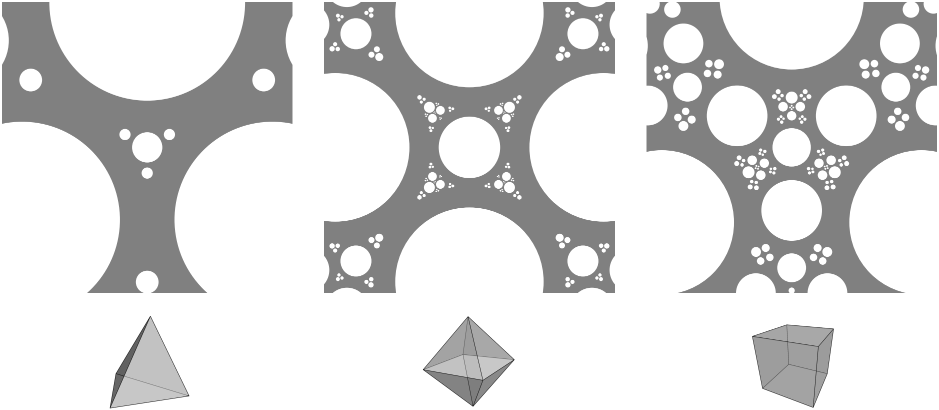

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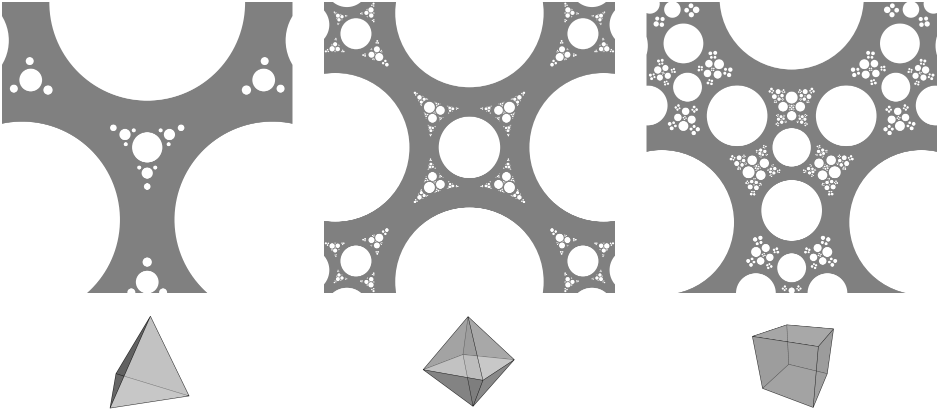

Lacunary structures

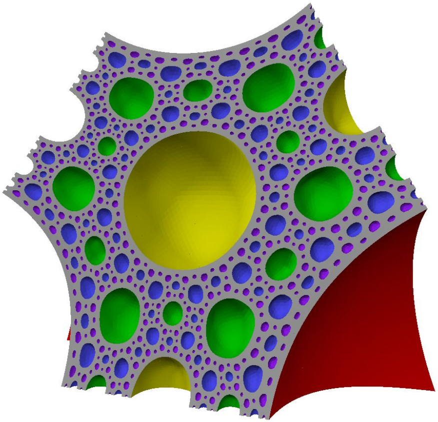

Figure of Gilles Gouaty

This lacunary structure can be obtained by taking the complementary of a Polytopal Sphere Packing of a Canonical Barycentric Subdivision of a hypertetrahedron

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Thanks !