Stability of Corrected Curvature Measures

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Joint work with J.O Lachaud, D. Coeurjolly, P. Romon, C. Labard

Geometry & Computing Days, October 21-25 2024

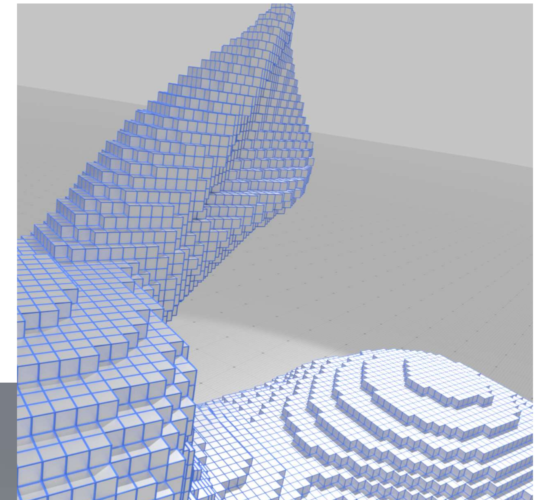
In this talk

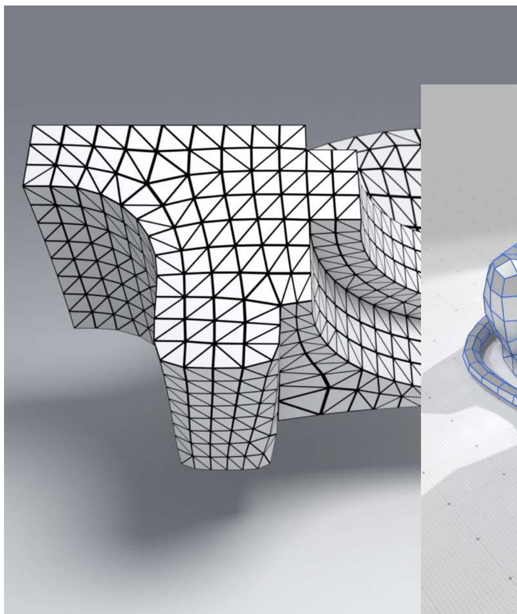
We will provide:

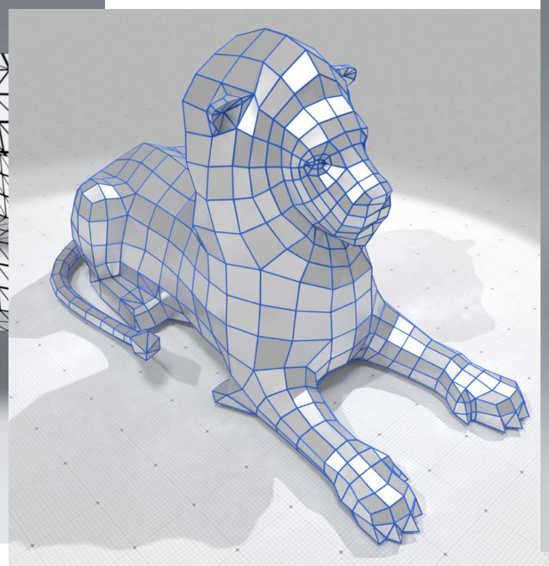
$\rightsquigarrow$ simple formulae for curvature estimations.

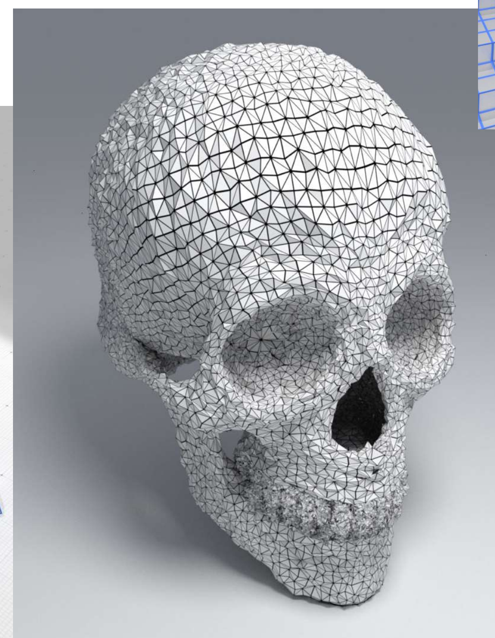
$\rightsquigarrow$ stability results

$\rightsquigarrow$ Can handle different geometries









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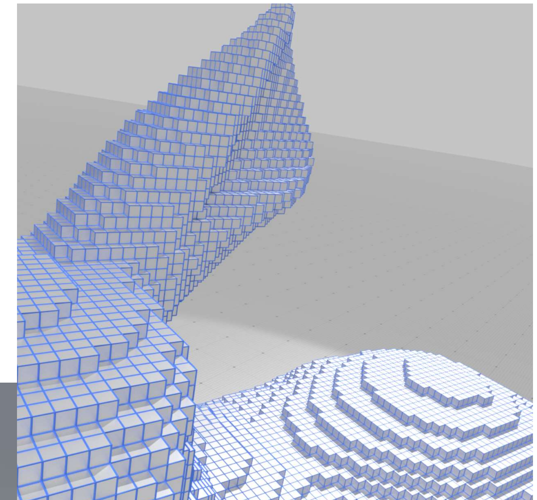
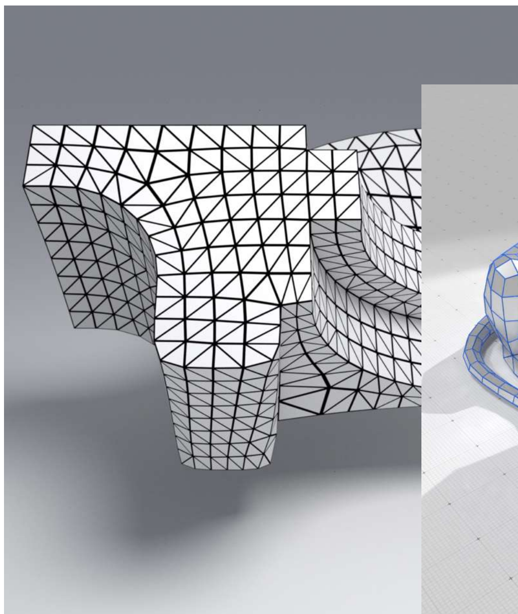
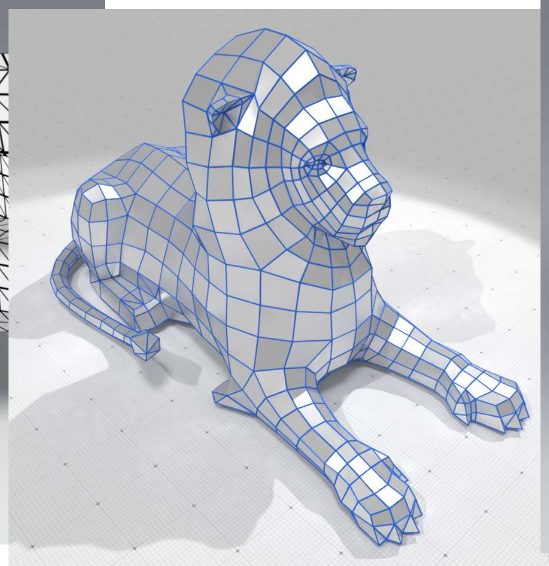
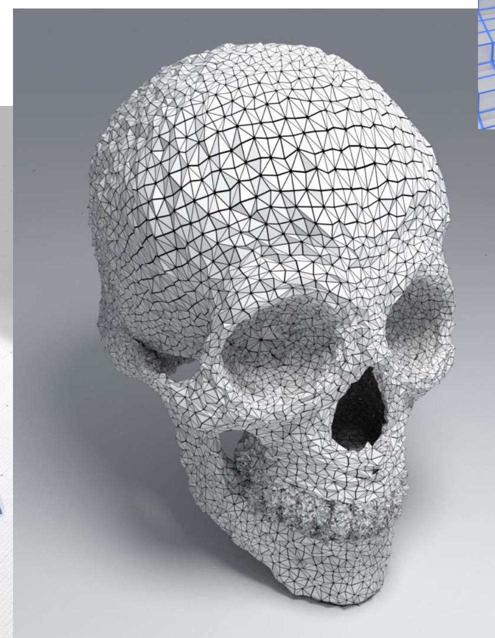








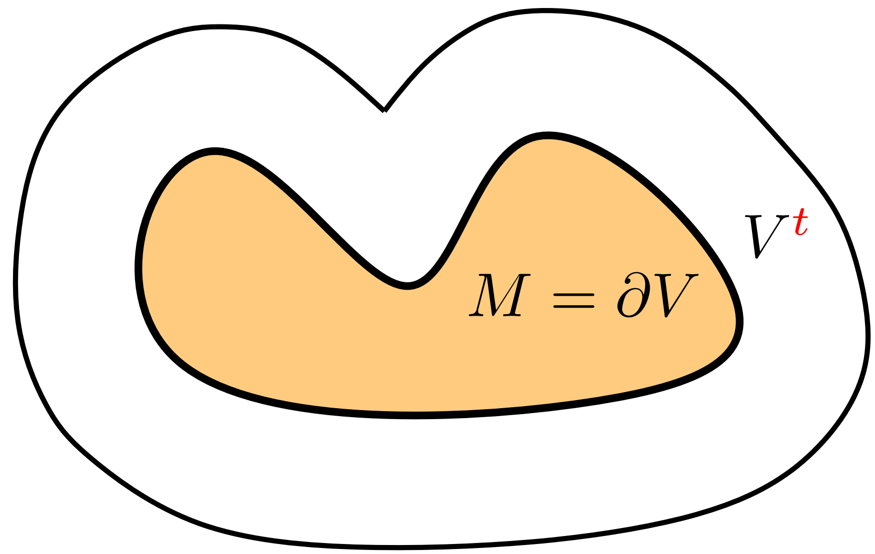
Key ingredients:

- normals "good" ⇝ curvatures good.
- normal cycle ⇝ general stability results

Curvature measures : Weyl's formula (1939)

Tube's formula: Let $M \subset \mathbb{R}^d$ be a C^2 hypersurface in $\mathbb{R}$ without boundary and $t < \mathrm{reach}(M)$. Then the volume $\mathrm{Vol}(K^t)$ is a polynomial in t:

$$\mathrm{Vol}(V^t) = \mathrm{Vol}(V) + \mathrm{Area}(M)t + \int_M H(p)\mathrm{d}p\, t^2 + \int_M G(p)\mathrm{d}p\, \frac{t^3}{3}$$



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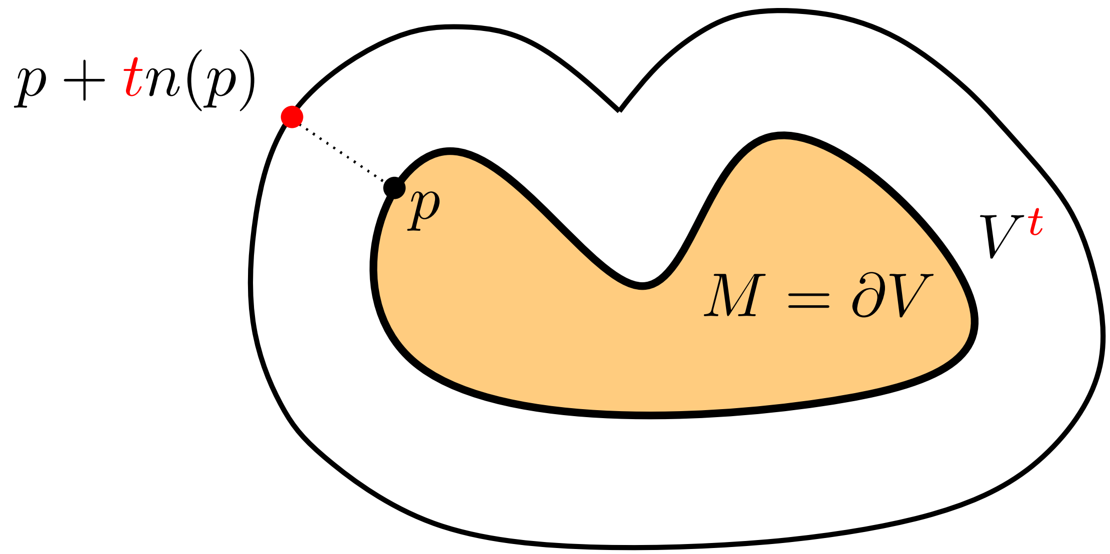
Proof

1) f is a diffeomorphism

$$f: \quad M \times]0, t[\quad \to \quad \mathbb{R}^3$$
$$(p, s) \quad \mapsto \quad p + s\; n(p).$$



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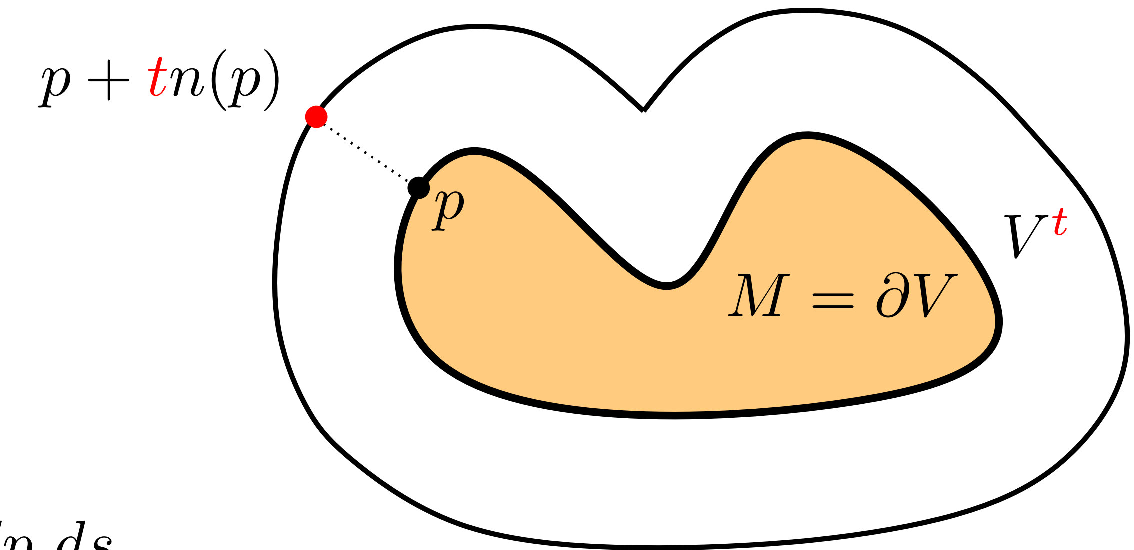

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2) Change of variable

$$\begin{aligned} \mathrm{Vol}(V^t \setminus V) &= \int_0^t \int_M \mathrm{Jac}(f) dp\, ds \\ &= \int_0^t \int_M \left[1 - s\ \kappa_1(p)\right]\left[1 - s\ \kappa_2(p)\right] dp\, ds \\ &= t\int_M dp\ + \frac{t^2}{2}\int_M H(p)\ dp\ + \frac{t^3}{3}\int_M G(p)\ dp. \end{aligned}$$

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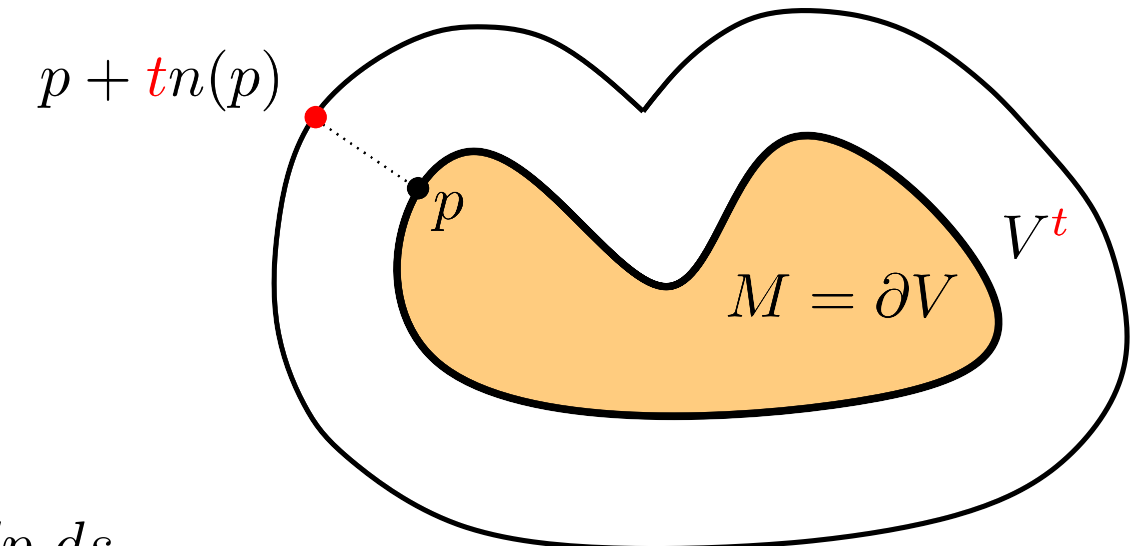

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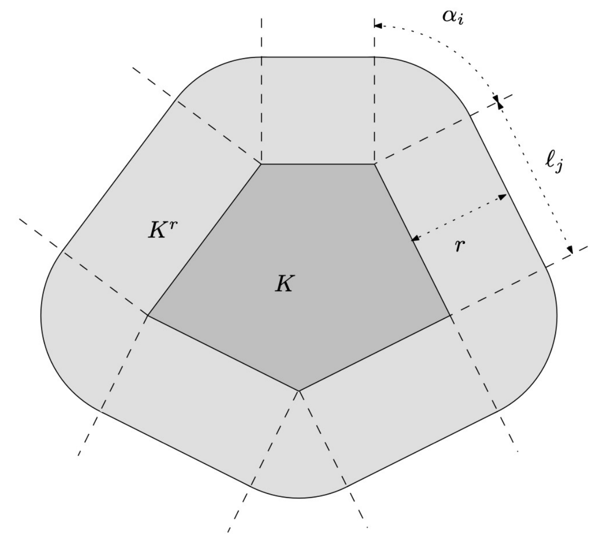

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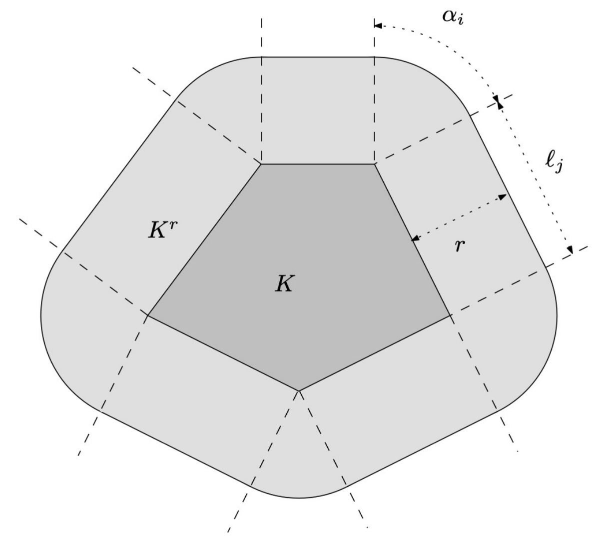

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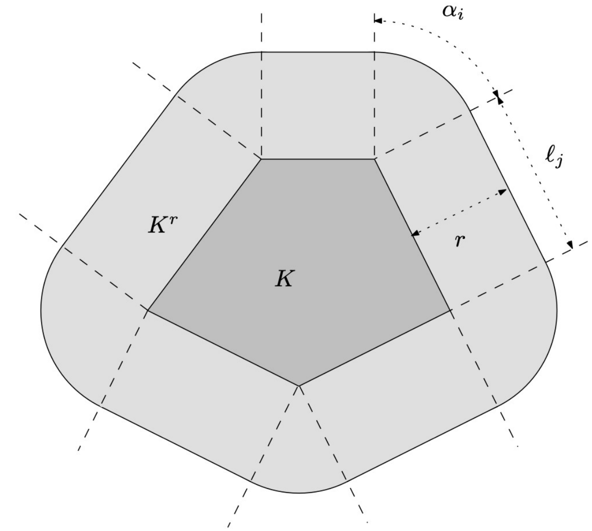

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"mean curvature" "Gauss curvature"

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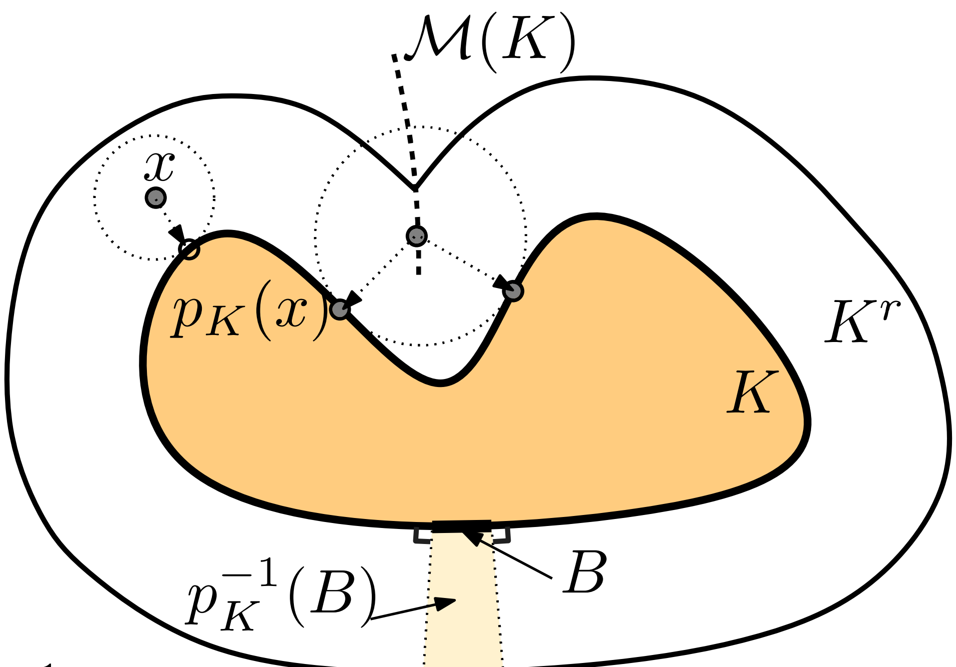
- A set K has positive reach $r > 0$ if every $x \in K^r$ has a unique closet point $p_K(x)$ on K



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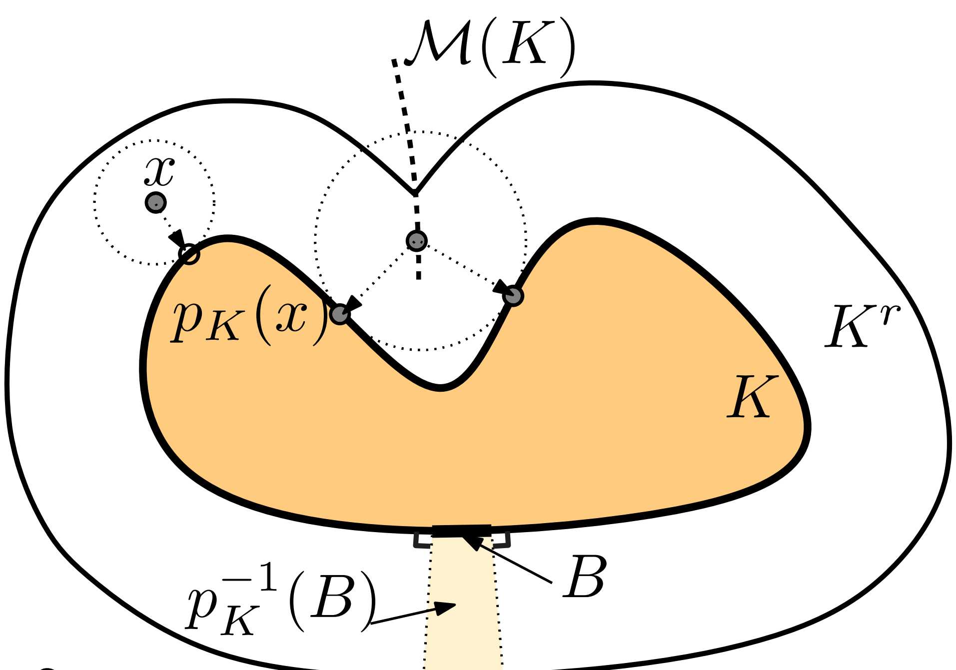

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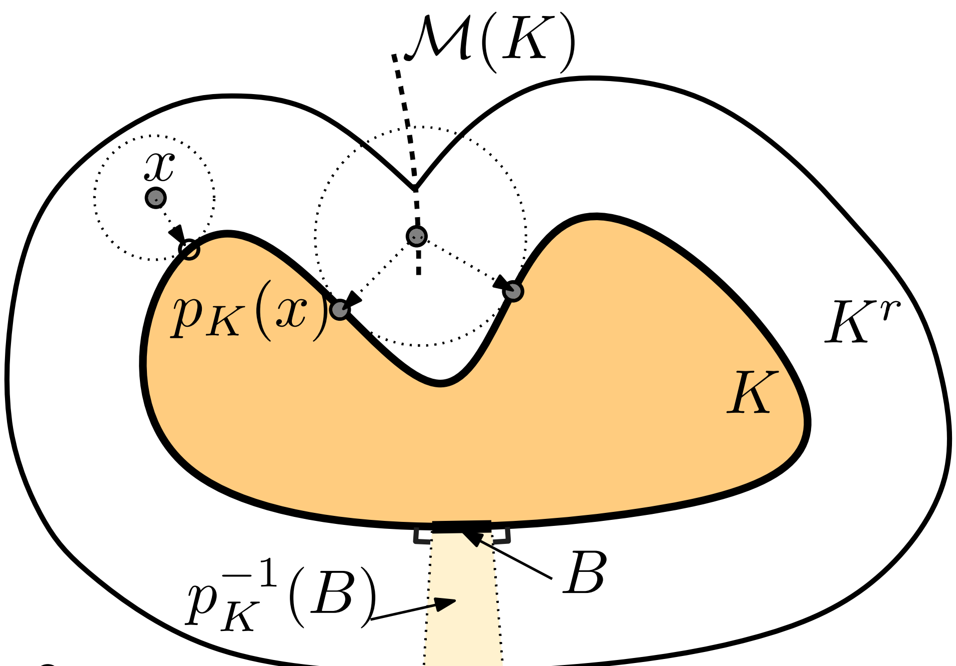

Sets with > 0 reach
- contains C^2 manifolds, convex sets
- does not contain meshes, digital shapes

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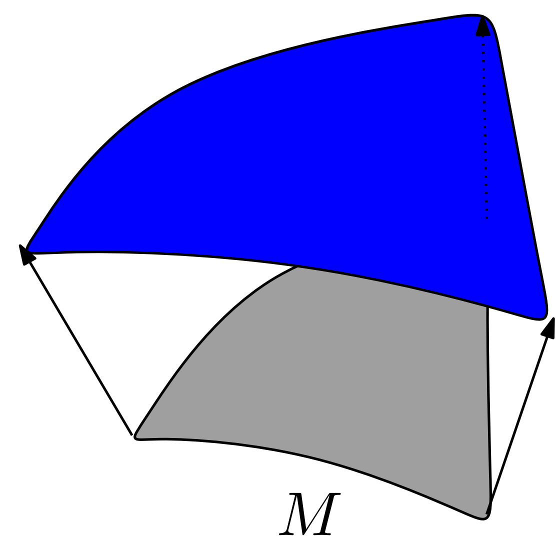

$\Rightarrow$ Does not work for digitized shapes !

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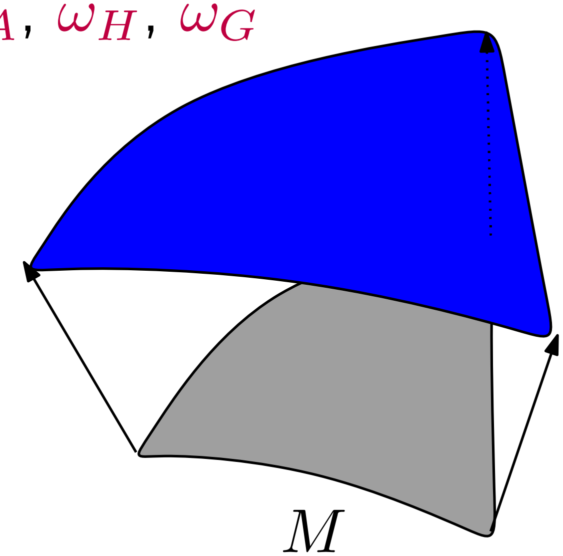

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Proposition. Let $B \subset \mathbb{R}^3$ a ball

$$\int_{\mathrm{spt}(N(M))\cap(B\times\mathbb{S}^2)} \omega_A = \mathrm{Area}(M \cap B)$$

$$\int_{\mathrm{spt}(N(M))\cap(B\times\mathbb{S}^2)} \omega_H = \int_{M\cap B} H(p)\,\mathrm{d}\,p$$

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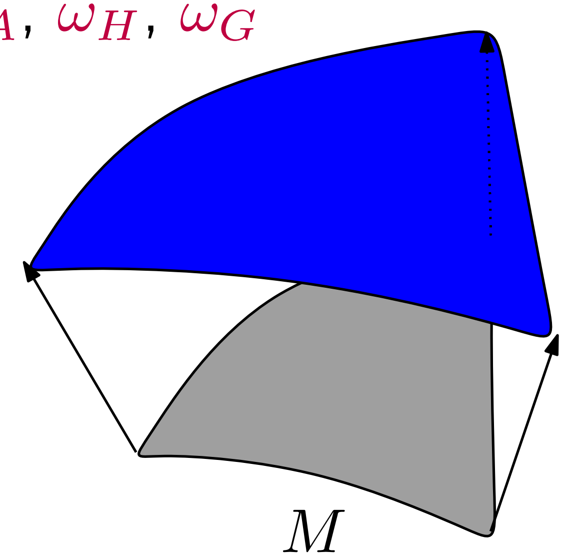

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In general: M is non smooth

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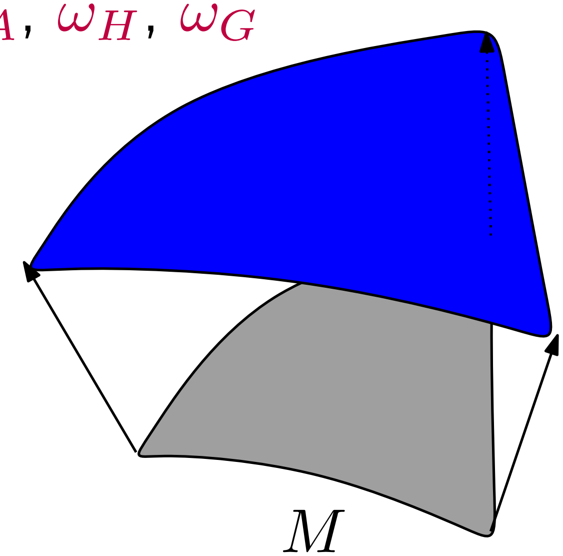
$$N(M) : \{2 - forms\} \rightarrow \mathbb{R}$$



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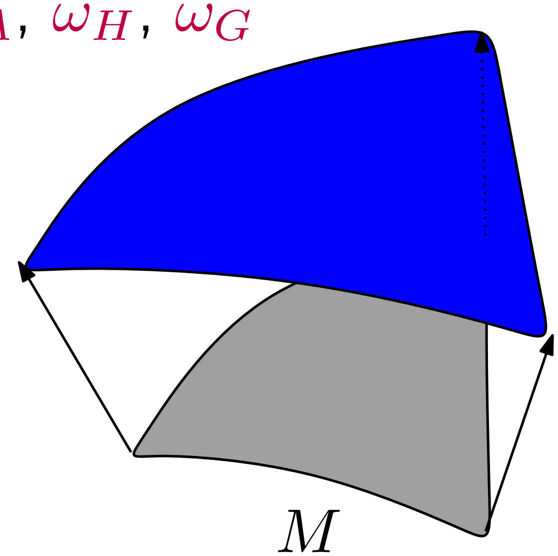
Example: $\omega \mapsto \int_{X^2 \subset \mathbb{R}^3 \times \mathbb{S}^2} \omega$



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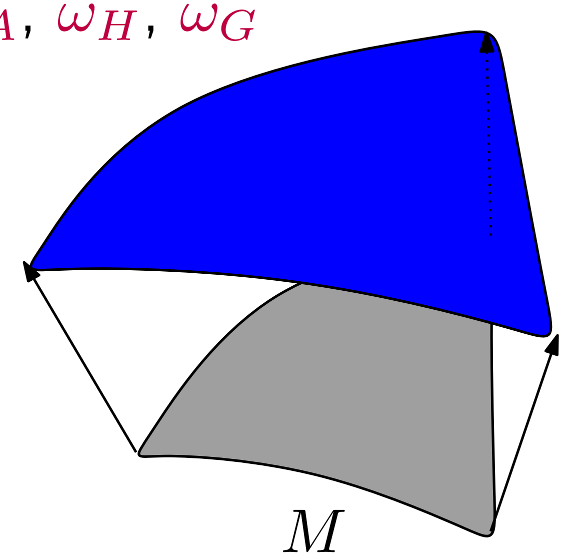

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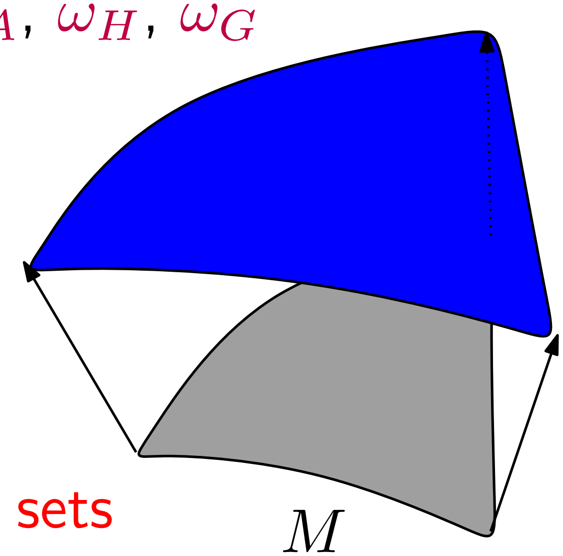

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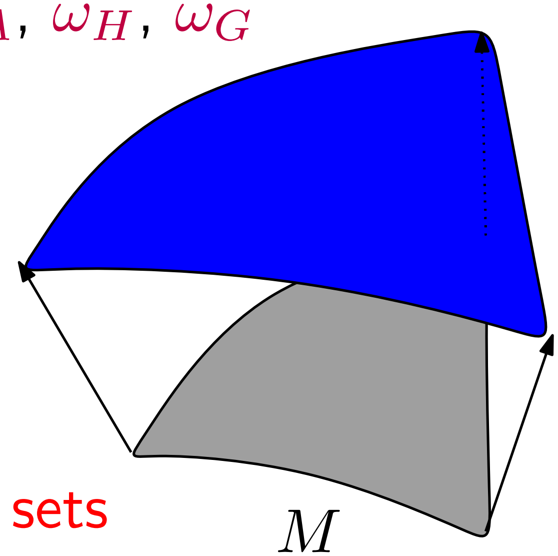

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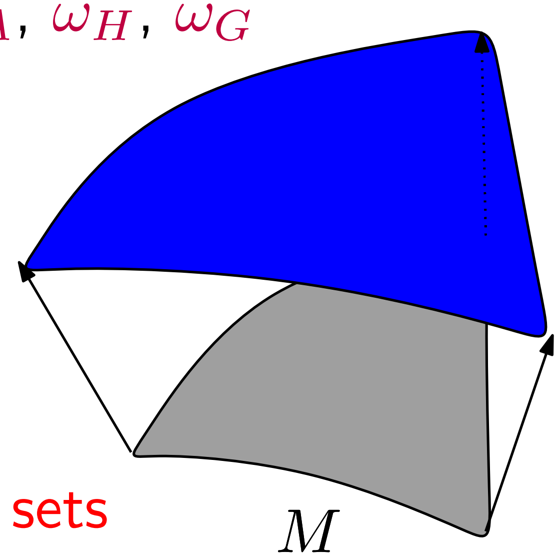

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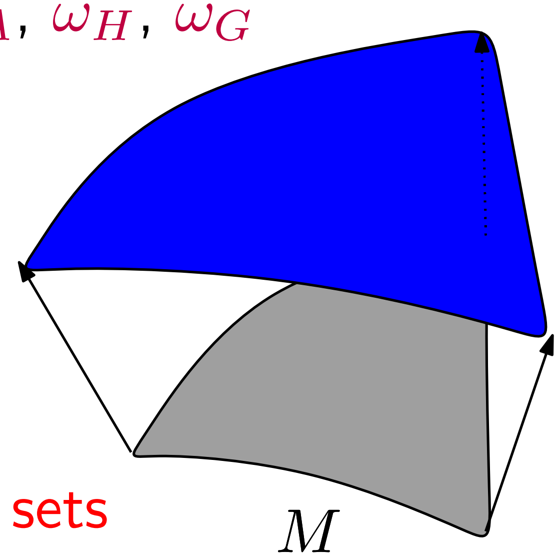

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Corrected curvature measures

Main idea:

Given a 2d non-smooth manifold M and $u : M \to \mathbb{S}^2$ normal vector field

Corrected curvature measures

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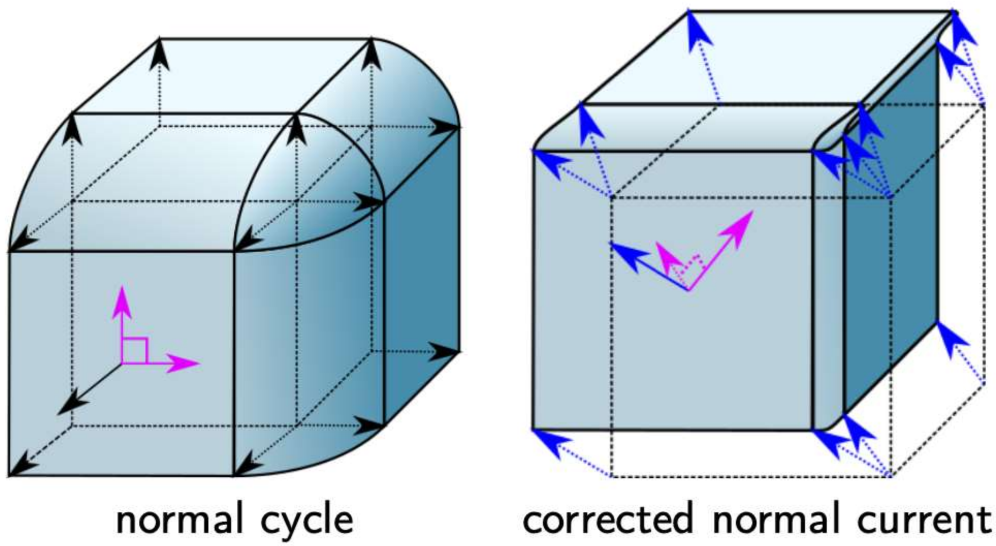
Given a 2d non-smooth manifold M and $u : M \to \mathbb{S}^2$ normal vector field

$\rightsquigarrow$ build a normal cycle $N(M,u)$ "corrected" by u.



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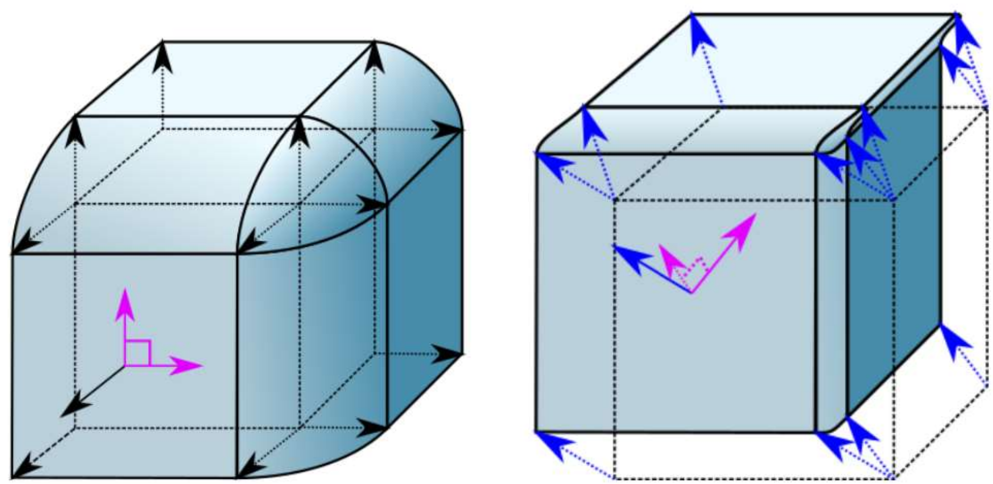

normal cycle corrected normal current

$\rightsquigarrow$ We define the "corrected" curvature measures

Area : $\mu_A(B) = \langle N(M, u)_{|B}, \omega_A \rangle$

Mean : $\mu_G(B) = \langle N(M, u)_{|B}, \omega_H \rangle$

Gauss : $\mu_G(B) = \langle N(M, u)_{|B}, \omega_G \rangle$

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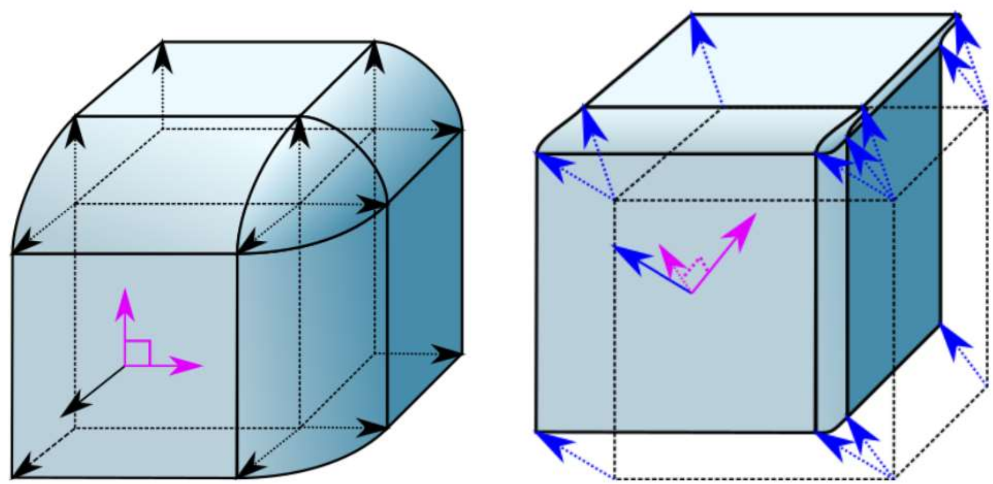

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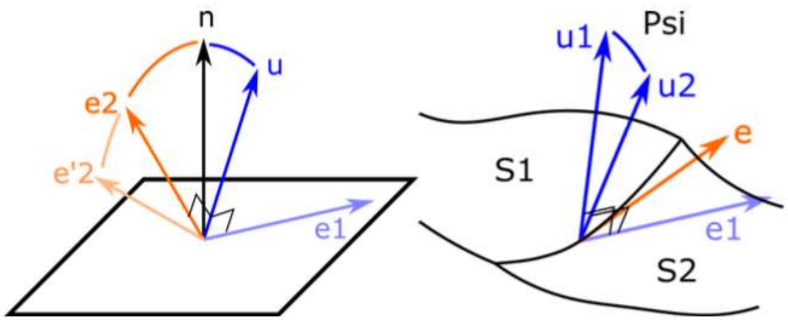
$\rightsquigarrow$ also applies to second fundamental "measures"

Corrected curvature measures

Formula:



$\mathbf{e}_1 := \mathbf{u}_1 \times \mathbf{u}_2 / \|\mathbf{u}_1 \times \mathbf{u}_2\|$

$\Psi := \angle \mathbf{u}_1, \mathbf{u}_2$

$\mathbf{e}$:= tangent to edge

Generic case: S piecewise $C^{1,1}$, $\mathbf{u}$ differentiable per face

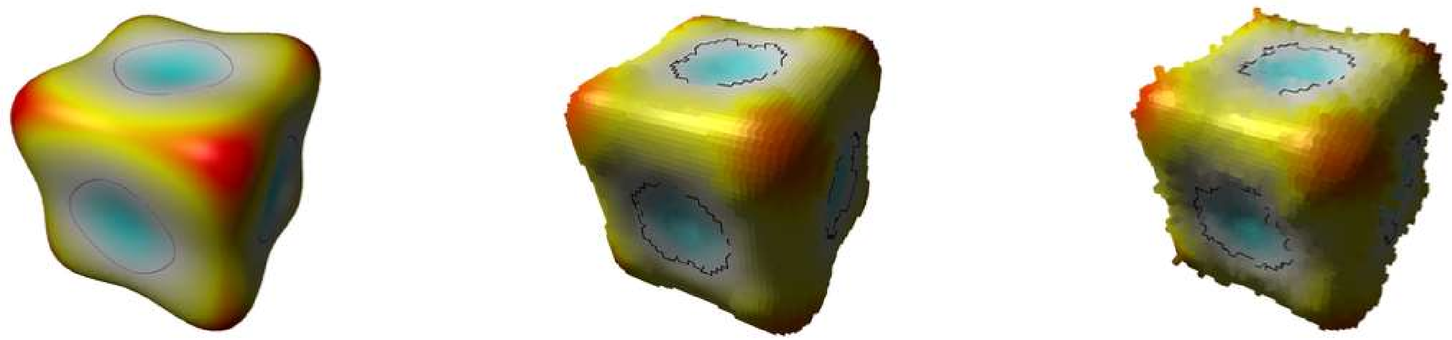
$$\mu_0^{S,\mathbf{u}}(B) = \int_{B\cap S} \langle \mathbf{u} \mid \mathbf{n} \rangle \, \mathrm{d}\mathcal{H}^2$$

B is a ball

$$\mu_1^{S,\mathbf{u}}(B) = \int_{B\cap S} \Big(\langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_2' \mid \mathbf{e}_2 \rangle + \langle \mathbf{u} \mid \mathbf{n} \rangle \, \langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_1 \mid \mathbf{e}_1 \rangle \Big) \, \mathrm{d}\mathcal{H}^2 + \sum_{i \neq j} \int_{B\cap S_{i,j}} \Psi \, \langle \mathbf{e} \mid \mathbf{e}_1 \rangle \, \mathrm{d}\mathcal{H}^1$$

$$\mu_2^{S,\mathbf{u}}(B) = \int_{B\cap S} \langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_1 \mid \mathbf{e}_1 \rangle \, \langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_2' \mid \mathbf{e}_2 \rangle - \langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_1 \mid \mathbf{e}_2 \rangle \, \langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_2' \mid \mathbf{e}_1 \rangle \, \mathrm{d}\mathcal{H}^2 - \sum_{i \neq j} \int_{B\cap S_{i,j}} \tan \frac{\Psi}{2} \langle \mathbf{u}_j + \mathbf{u}_i \mid \mathrm{d}\mathbf{e}_1 \cdot \mathbf{e} \rangle \mathrm{d}\mathcal{H}^1 + \sum_{p \in B \cap \mathrm{Vtx}(S)} \mathrm{AArea}(NC(p, \mathbf{u})).$$

Corrected curvature measures



X digital surface S noisy digital surface S

Theorem [Lachaud, Romon T, DCG 22]
Let X be a compact surface of $\mathbb{R}^3$ of class C^2, of normal vector $\mathbf{n}$, bounding a volume V, and $S = \cup_i S_i$ be a piecewise $C^{1,1}$ surface bounding a volume W, $\mathbf{u}$ a corrected normal vector field on S.

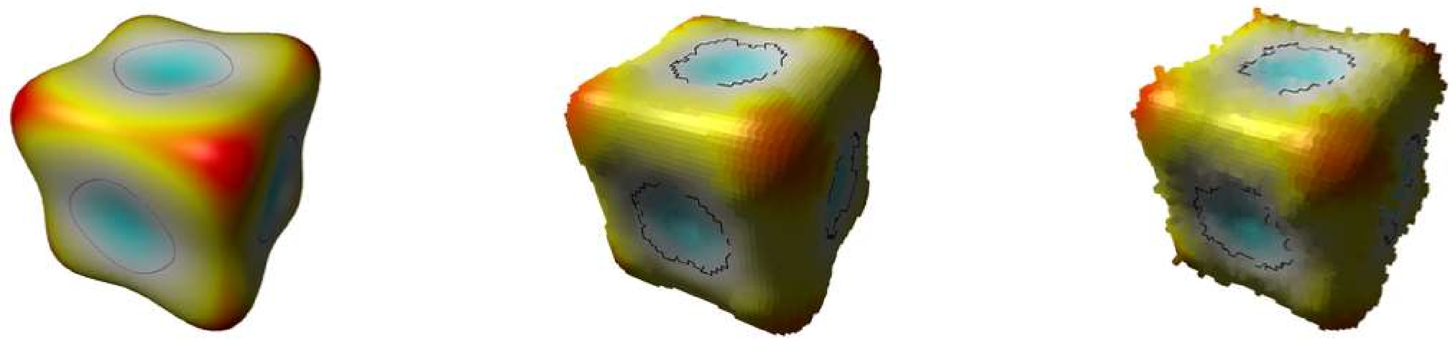
- $\epsilon := d_{\mathrm{H}}(S, X) < \mathrm{reach}(X)$ *is the position error.*
- $\eta := \sup_{p \in S} \|\mathbf{u}(p) - \mathbf{n}(\pi_X(p))\|$ *is the normal error,*

Then the corrected curvature measures of $(S, \mathbf{u})$ are close to the curvature measures of X. More precisely, for any connected union $B = \cup_{i \in I} S_i$ of faces S_i of S, one has

$$|\mu_k^{S,\mathbf{u}}(B) - \mu_k^X(\pi_X(B))| \leq K(L_{\mathbf{u}}, B \cap S, comb(B \cap S))(\epsilon + \eta),$$

where $L_{\mathbf{u}}$:= max. Lipschitz cst. of $\mathbf{u}$ per face and variation of $\mathbf{u}$ across edges.

Corrected curvature measures



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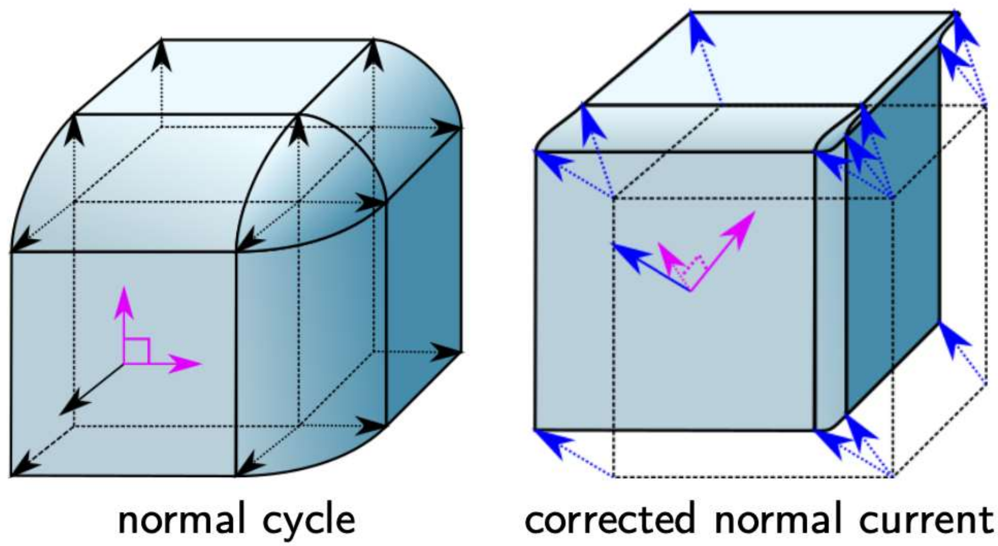
$\rightsquigarrow$ also CV result for pointwise curvatures on digitzed shapes

Corrected curvature measures on meshes

To make faster computation on meshes/voxelised shapes:

- Remark 1: If u is continuous, no term above vertices.



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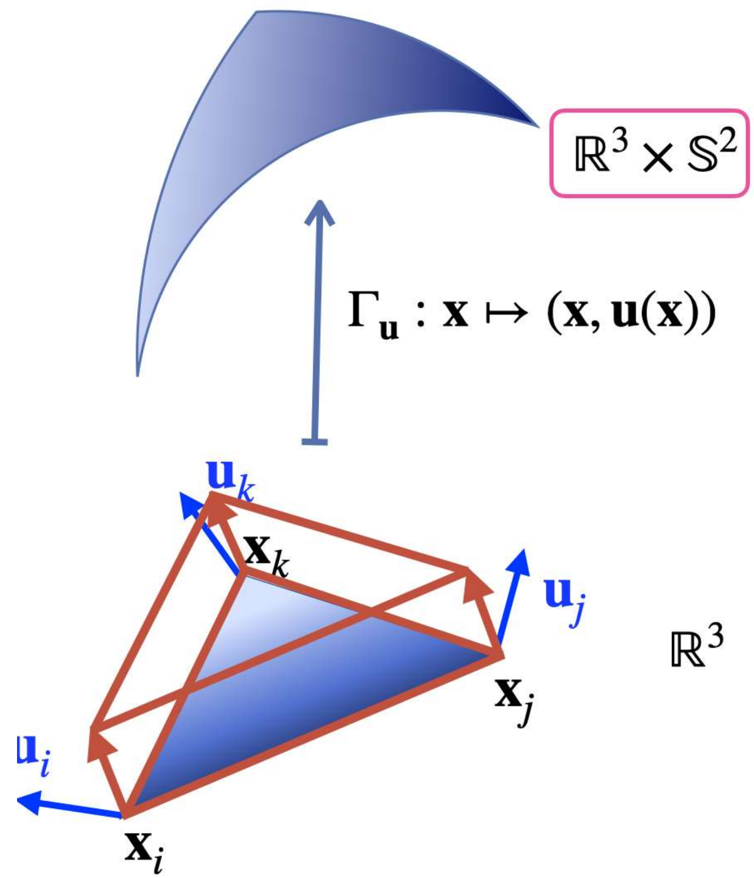
Lipschitz-Killing differential form (area)

Area measure $\mu_{\mathbf{u}}^{(0)}(\tau) = \int_\tau \Gamma_{\mathbf{u}}^* \omega^{(0)}$

$$= \int_0^1 \int_0^{1-t} \det\left(\mathbf{u}, \frac{\partial \mathbf{x}}{\partial s}, \frac{\partial \mathbf{x}}{\partial t}\right) \mathrm{d}s\mathrm{d}t$$

$$= \frac{1}{2} \langle \bar{\mathbf{u}} \mid (\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i) \rangle$$

with $\bar{\mathbf{u}} := (\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k)/3$



[Lachaud, Coeurjolly, Romon T, SGP 20]

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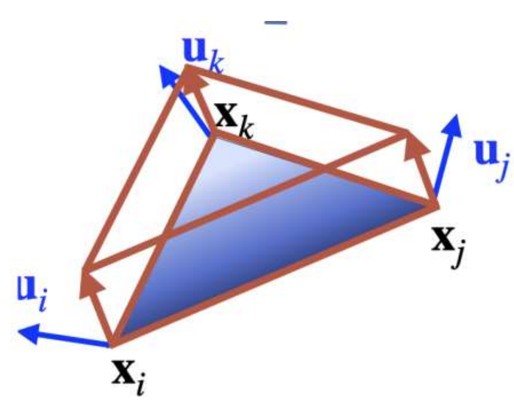
Area measure $\mu_{\mathbf{u}}^{(0)}(\tau) = \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i)\rangle$

Mean curvature measure

$$\mu_{\mathbf{u}}^{(1)}(\tau) = \int_\tau \Gamma_{\mathbf{u}}^* \omega^{(1)}$$
$$= \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{u}_k - \mathbf{u}_j) \times \mathbf{x}_i + (\mathbf{u}_i - \mathbf{u}_k) \times \mathbf{x}_j + (\mathbf{u}_j - \mathbf{u}_i) \times \mathbf{x}_k\rangle$$

Gaussian curvature measure

$$\mu_{\mathbf{u}}^{(2)}(\tau) = \int_\tau \Gamma_{\mathbf{u}}^* \omega^{(2)}$$
$$= \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{u}_j - \mathbf{u}_i) \times (\mathbf{u}_k - \mathbf{u}_i)\rangle$$



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Stability theorem for measures

Let S a compact surface of $\mathbb{R}^3$, C^2 smooth, without boundary

Let M a compact mesh without boundary, with $\mathbf{u}$ linearly interpolated

$$\varepsilon := d_H(S, M) < \mathrm{reach(S)}/2 \quad \text{“position error”}$$

$$\eta := \sup_{\mathbf{x} \in M} \|\mathbf{u}(\mathbf{x}) - \mathbf{n}(\pi_S(\mathbf{x}))\| \quad \text{“normal error”}$$

Then

$$\left| \mu^k_{M,\mathbf{u}}(B) - \mu^k_S(\pi_S(B)) \right| \leq K(\varepsilon + \eta) \quad \text{(for all measures k)}$$

where B is union of triangles of M, and K depends on $\mathrm{Area}(B)$, $\mathrm{Length}(\partial B)$, Lipschitz constant of $\mathbf{u}$, max curvature of S.

[Lachaud, Coeurjolly, Romon T, SGP 20]

Corrected curv. meas. on oriented point sets

Key idea:
measures do not need consistent mesh topology

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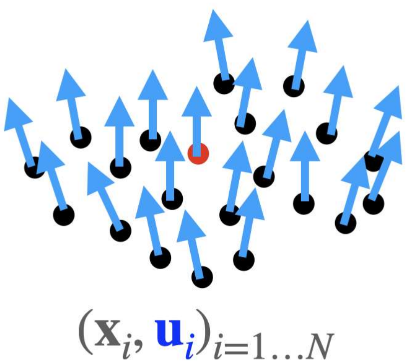

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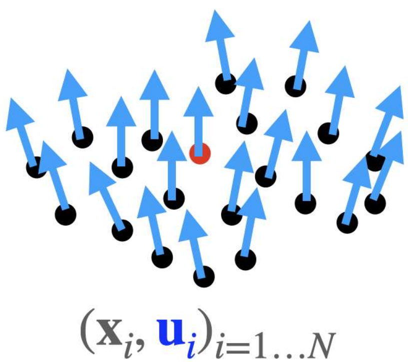
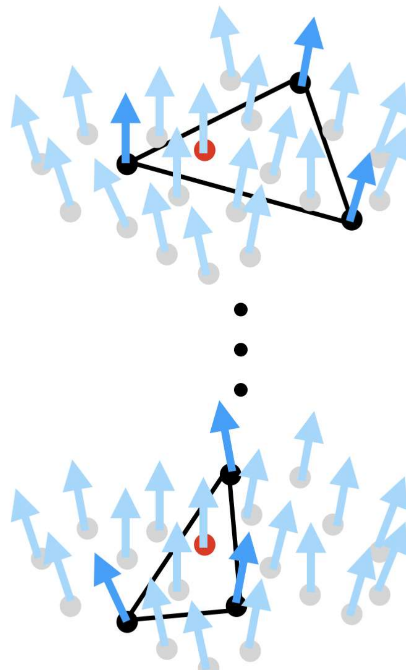


Local neighborhood



Random triangles

Corrected curv. meas. on oriented point sets

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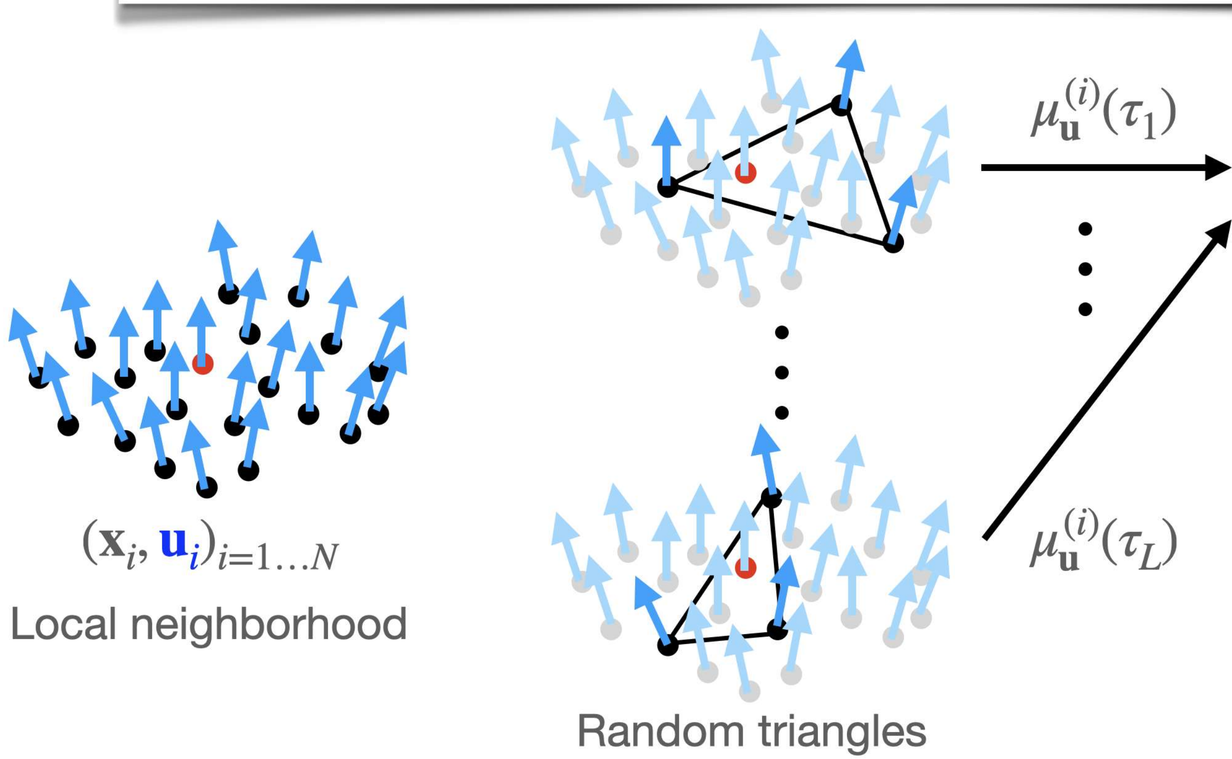

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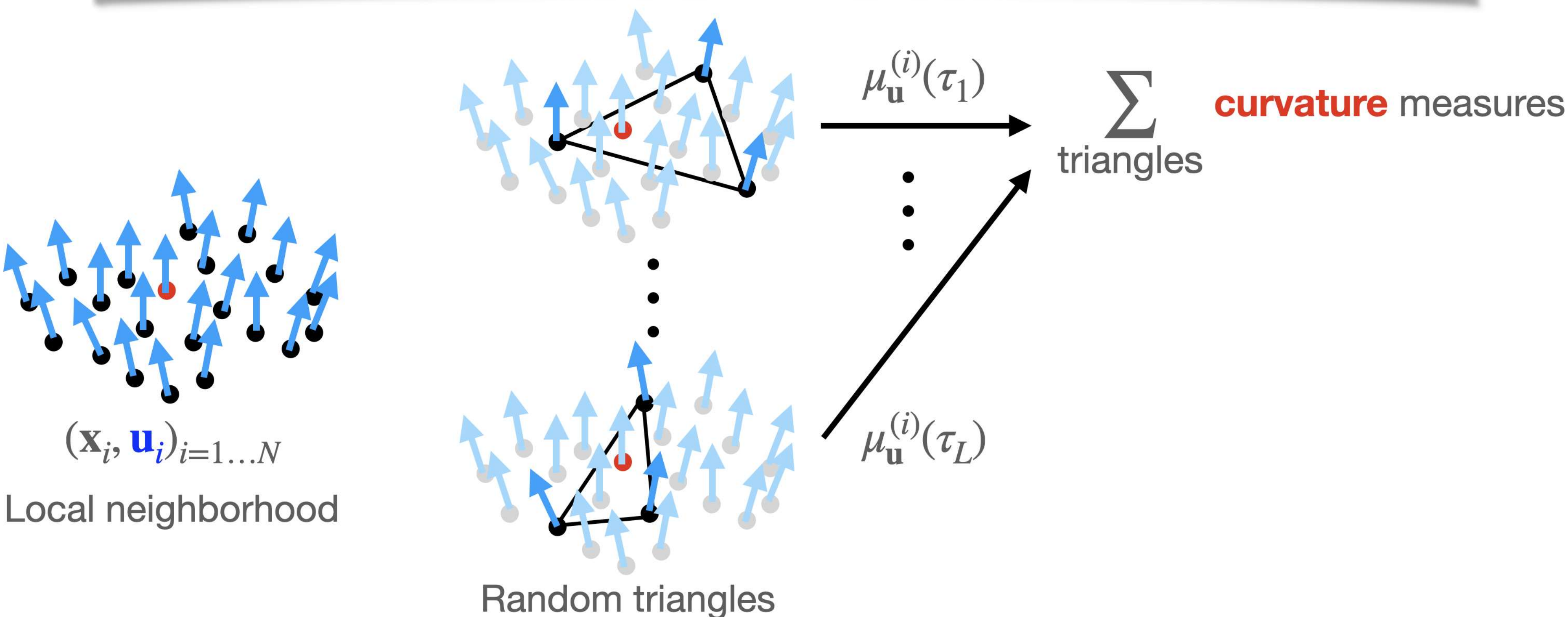

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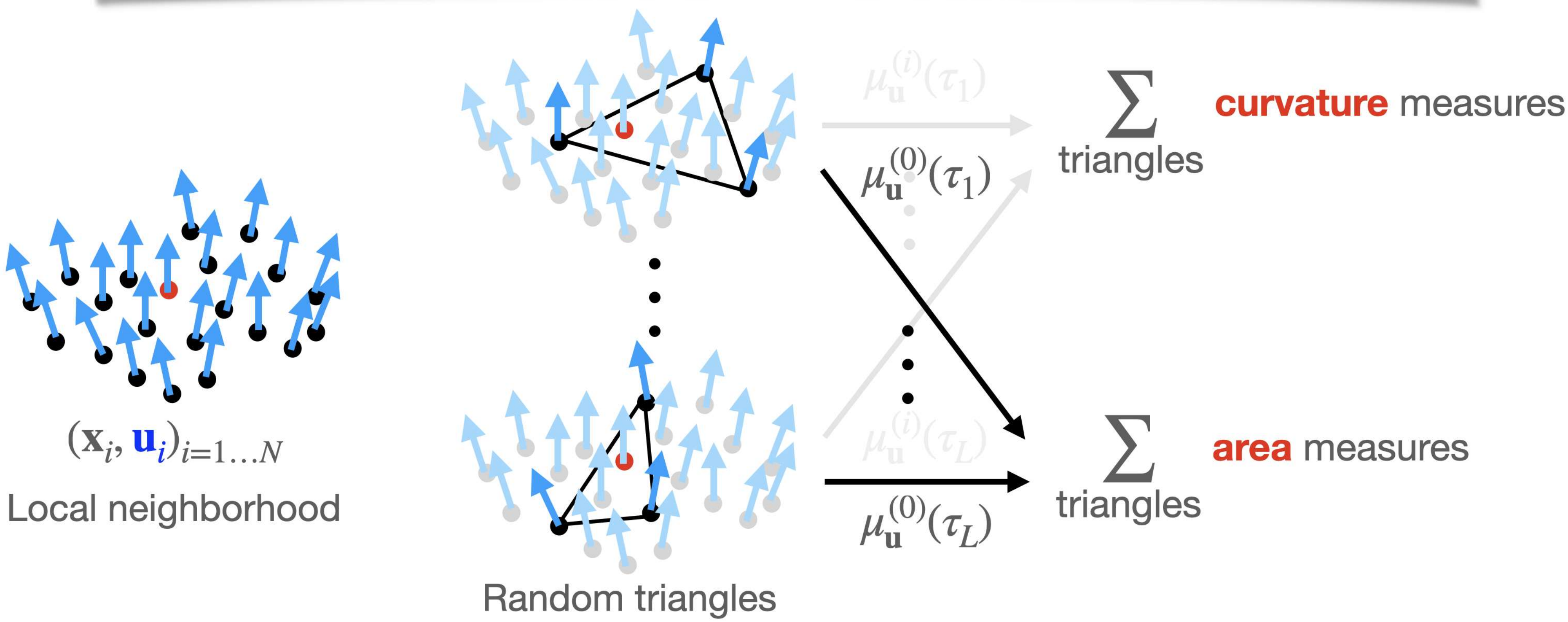

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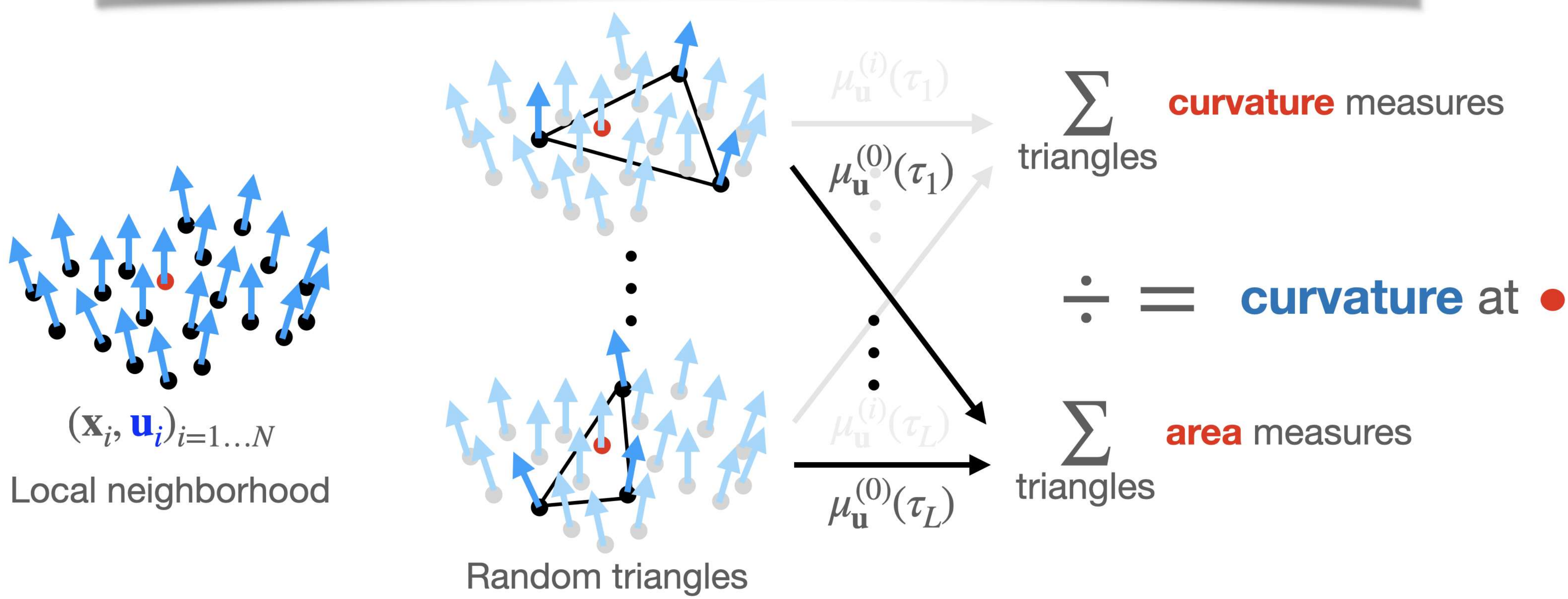
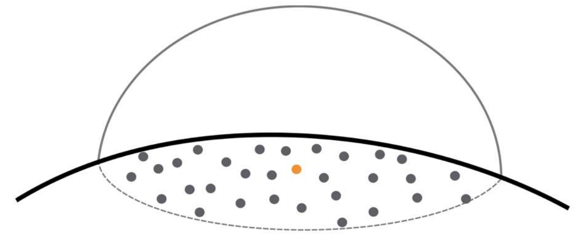
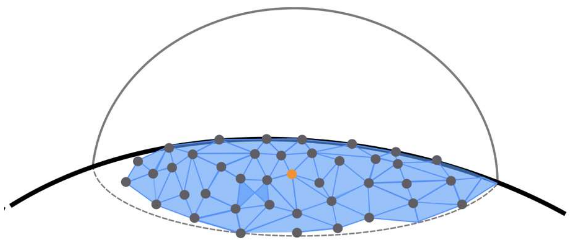


Corrected curv. meas. on oriented point sets

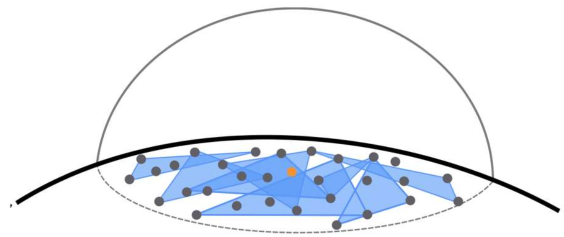
Selection of triangles

- Neighbours of $\mathbf{x}$: either K nearest or within Ball($\mathbf{x}$, δ)
- Choose a strategy to build L triangles within

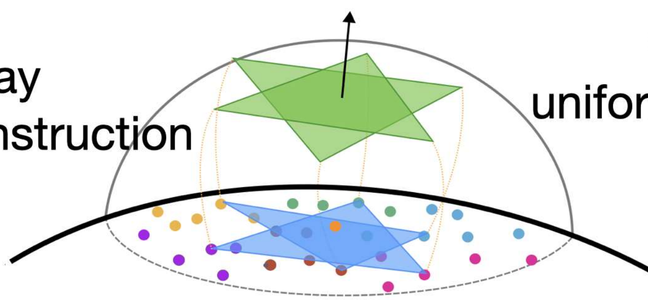




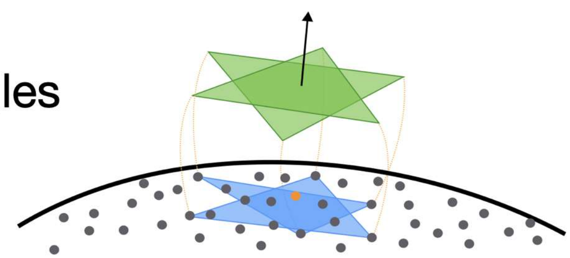
CNC-Delaunay
local Delaunay reconstruction



CNC-Uniform
uniform random triangles



CNC-AvgHexagram
2 triangles with average nearest points



CNC-Hexagram
2 triangles with nearest points

[Lachaud, Coeurjolly, Romon T, Labart, SGP 23]

Results

Example: mean curvature with Avg-Hexagram

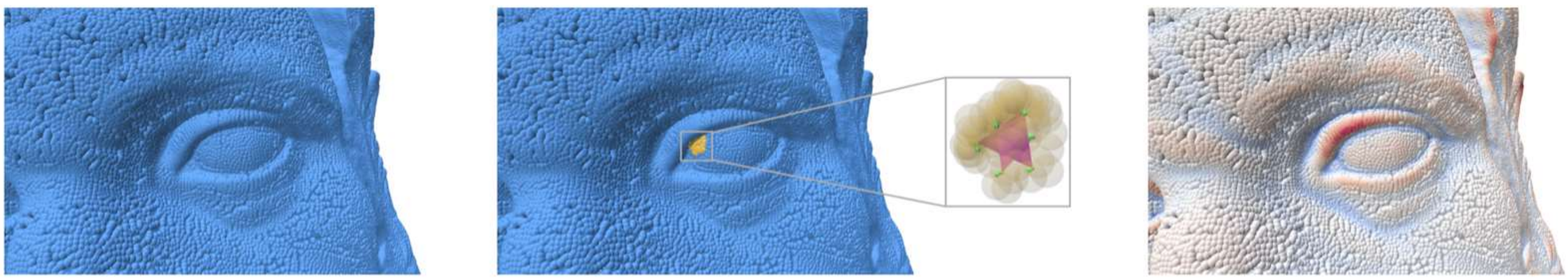
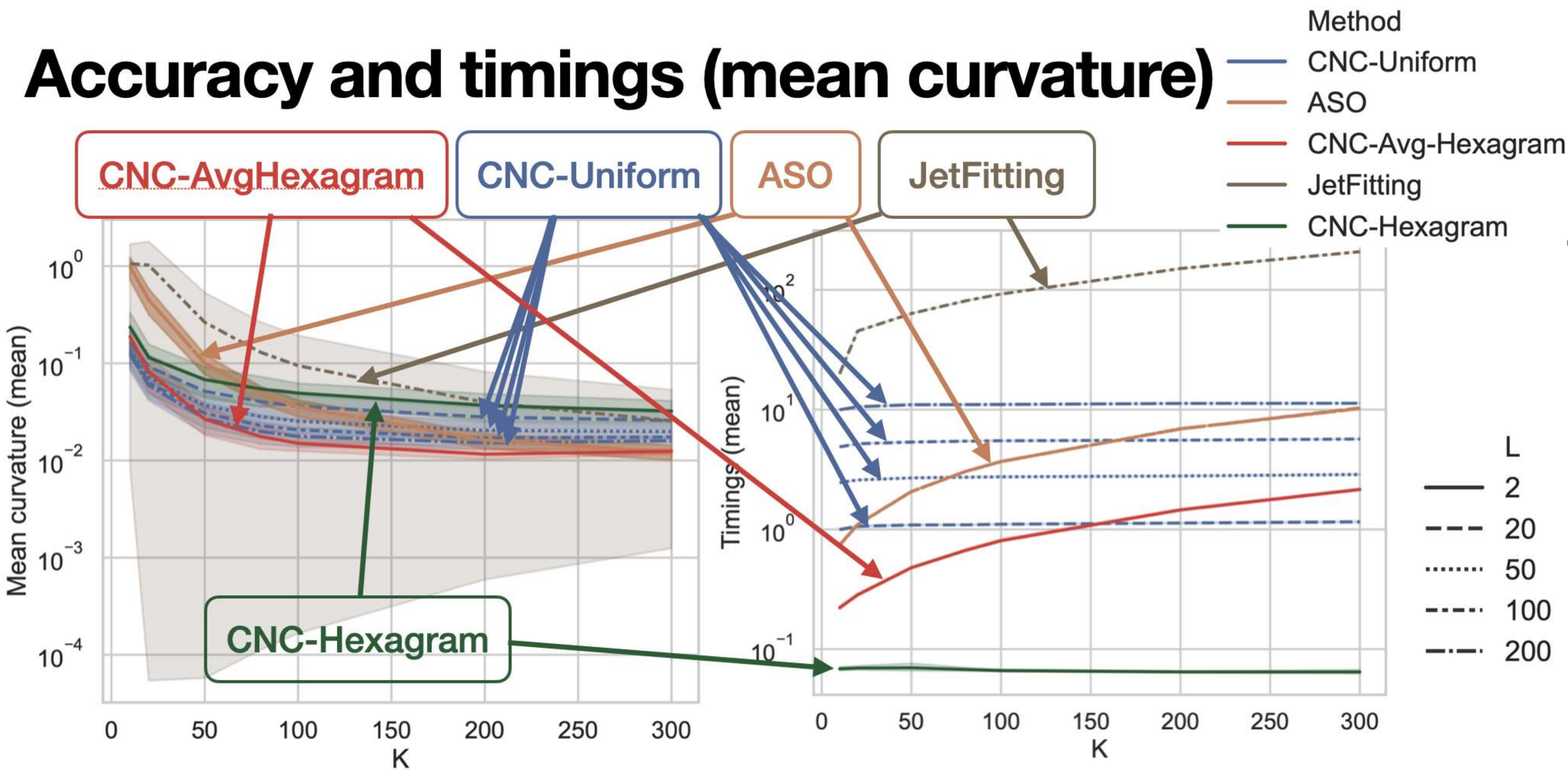


Figure 1: *Our new technique uses corrected curvature measures on (quasi-)random triangles to estimate differential quantities on point clouds: stable and accurate estimations (mean curvature here) are achieved with few neighbors (50) and triangles (2).*

Results

κ_1 κ_2 e_1 e_2

Results



- **Goursat** shape : $N \in \{10000, 25000, 50000, 75000, 100000\}$, $\sigma_\epsilon, \sigma_\xi \in \{0, 0.1, 0.2\}$

Conclusion

- Handle different geometries : digital, meshes, point sets
- Theoretical stability in the presence of noise
- For point sets, local computations, without reconstruction, parallelizable.
- Fast and accurate compared to state-of-the-art

Thanks !

https://github.com/JacquesOlivierLachaud/PointCloudCurvCNC